Timelike Surfaces in the Lorentz-Minkowski Space with Prescribed Gaussian Curvature and Gauss Map

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Abstract. In this paper we obtain a Lelieuvre-type representation for timelike surfaces with prescribed Gauss map in the 3-dimensional Lorentz-Minkowski space $\mathbf{L}^3$. As a main result we classify the complete timelike surfaces with positive constant Gaussian curvature in $\mathbf{L}^3$ in terms of harmonic diffeomorphisms between simply connected Lorentz surfaces and the universal covering of the de Sitter Space.

1 Introduction

The existence and uniqueness of a surface in $\mathbf{R}^3$ (and also in the rest of Riemannian space forms) with prescribed Gauss map and a given conformal structure have been subject of a wide study (see [11], [6], [2] and references therein). This study has also been extended to spacelike surfaces in $\mathbf{L}^3$. Specifically, Kobayashi [8] and Akutagawa and Nishikawa [1] obtained Lorentzian versions of the classical Enneper-Weierstrass and Kenmotsu representations for maximal and constant mean curvature surfaces respectively. On the other hand, Martínez, Milán and the third author [3] have rencently obtained a representation for spacelike surfaces in $\mathbf{L}^3$ using the Gauss map and the conformal structure given by the second fundamental form.

Regarding to timelike surfaces in $\mathbf{L}^3$, Magid [9] found a Weierstrass representation for any such a surface in terms of its Gauss map and mean curvature, showing that the language of complex analysis used when the surface can be considered as a Riemann surface is a useful device but not essential. This will also become plain in Section 4 of this paper.

On the other hand, we observe that the study of surfaces with constant Gaussian curvature in $\mathbf{L}^3$ is being developed thanks to its relationship with some classical equations as sinh-Gordon, cosh-Gordon, sine-Laplace, and its associated Bäcklund and Darboux transformations (see, for instance, [10], [14]).

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Thus, one can find new surfaces with constant Gaussian curvature from a known one. Nevertheless, these results are local (in essence) and few is known about completeness of the new surfaces.

Our main goal makes usage of another well known relationship between surfaces with constant Gaussian curvature and harmonic maps [7]. Following the ideas of Lelieuvre [5], we give a representation for timelike surfaces with positive Gaussian curvature in terms of its Gauss map using the conformal structure induced by its second fundamental form. Thus, we classify the (geodesically) complete timelike surfaces with positive constant Gaussian curvature by showing the existence of a bijective correspondence between these immersions and the harmonic diffeomorphisms from Lorentz surfaces into the universal covering of the de Sitter space.

The paper is organized as follows. After a brief section of preliminaries, we study in Section 3 complete revolution timelike surfaces with positive constant Gaussian curvature.

We devote Section 4 to the timelike surfaces with positive Gaussian curvature. Thus, we find a Lelieuvre-type representation for such surfaces with prescribed Gauss map (Theorem 2). Moreover, we study under what assumptions a differentiable map $N : M \longrightarrow \mathbf{S}^2_1$ from a simply connected Lorentz surface M into the de Sitter space determines a timelike immersion $X : M \longrightarrow \mathbf{L}^3$ with Gauss map N and positive Gaussian curvature (Theorem 3). As a result of these theorems, we obtain some immediate consequences for the case of positive constant Gaussian curvature (Corollaries 5 and 6). Next we obtain the classification of the complete simply connected timelike immersions with positive constant Gaussian curvature (Theorem 7).

Finally, Section 5 is devoted to study the case of negative Gaussian curvature. Under that hypothesis, the second fundamental form II defines a Riemannian metric on the surface, so we will consider it as a Riemann surface with the conformal structure induced by II. Following a similar sketch to the one in Section 4, we get a Lelieuvre-type representation for such surfaces with prescribed Gauss map (Theorem 9) and study under what assumptions a differentiable map $N : M \longrightarrow \mathbf{S}^2_1$ from a simply connected Riemann surface M determines a timelike immersion $X : M \longrightarrow \mathbf{L}^3$ with Gauss map N and negative Gaussian curvature (Theorem 10). To end up, we particularize these results to the case of negative constant Gaussian curvature (Corollaries 11 and 12).

2 Preliminaries

Let $\mathbf{L}^3$ be the 3-dimensional *Lorentz-Minkowski space*, that is, the real vector space $\mathbf{R}^3$ endowed with the Lorentzian metric tensor $\langle , \rangle$ given by

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) are the canonical coordinates of $\mathbf{R}^3$. Associated to that metric, one has the cross product of two vectors $u, v \in \mathbf{L}^3$ given by

$$u \times v = (-u_2 v_3 + u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1),$$

which is the unique vector such that $\langle u \times v, w \rangle = \det(u, v, w)$ for all $w \in \mathbf{L}^3$, where det denotes the usual determinant in $\mathbf{R}^3$.

An immersion $X : M \longrightarrow \mathbf{L}^3$ from a connected surface M into $\mathbf{L}^3$ is said to be a *timelike surface* if the induced metric via ψ is a Lorentzian metric on M, which will be also denoted by $\langle , \rangle$ or by I. Throughout this paper we will assume that M is orientable, and so we can choose a unit normal vector field N globally defined on M which can be seen as a map from M into the de Sitter Space $\mathbf{S}_1^2 = \{x \in \mathbf{L}^3 : \langle x, x\rangle = 1\}$. We will refer to N as the *Gauss map* of the immersion.

We will denote by $II = -\langle dN, dX\rangle$ the second fundamental form of the immersion X. Then the Gaussian curvature of M is given by $K = \det(II)/\det(I)$. In particular, note that K is positive (resp. negative) if, and only if, II is a Lorentzian (resp. Riemannian) metric on M.

3 Complete revolution surfaces with positive constant Gaussian curvature.

In this section we center our attention in the rotation timelike surfaces in $\mathbf{L}^3$ with positive constant Gaussian curvature with respect to a timelike axis. Up to a homothety, we can assume that its Gaussian curvature is one.

Let $\psi(r,\theta) = (r, f(r)\cos\theta, f(r)\sin\theta)$ be a rotation immersion in $\mathbf{L}^3$, where f is a positive function. Then, ψ is a timelike immersion with constant Gaussian curvature $K = 1$ if and only if

$$f'(r)^2 < 1 \quad \text{and} \quad f''(r) = f(r)(f'(r)^2 - 1)^2$$

In addition, the induced metric is given by $I = (f'(r)^2 - 1)\, dr^2 + f(r)^2\, d\theta^2$.

Thus, in order to find (geodesically) complete immersions, we consider the initial value problem

$$\begin{cases} f''(r) = f(r)(f'(r)^2 - 1)^2 \\ f(0) = c_1 \\ f'(0) = c_2 \end{cases} \tag{1}$$

Let $f : (\alpha, \omega) \longrightarrow \mathbf{R}$ be the maximal solution to (1) for arbitrary real numbers c_1, c_2 satisfying $c_1 \geq 1$ and $c_1^2 \leq \frac{1}{1-c_2^2} < 1 + c_1^2$. Then, it is easy to check that f satisfies the ordinary differential equation

$$f'(r)^2 = 1 - \frac{1}{c_0 + f(r)^2} \tag{2}$$

for $c_0 = \frac{1}{1-c_2^2} - c_1^2 \in [0, 1)$.

Now, we observe that $f(r) \geq \sqrt{1-c_0}$ for all $r \in (\alpha, \omega)$. In fact, the inequality holds at $r = 0$ because $f(0) = c_1 \geq 1 \geq \sqrt{1-c_0}$. In addition, if $r_0 \in (\alpha, \omega)$ satisfies $f(r_0) = \sqrt{1-c_0}$ then, from (2), $f'(r_0) = 0$ and, from (1), $f(r) > \sqrt{1-c_0}$ in a punctured neighbourhood of r_0 because f is strictly convex there.

On the other hand, from (2), $f'(r)^2 < 1$, that is, $f(r) \leq c_1 + |r|$ for all $r \in (\alpha, \omega)$. Hence, a standard argument shows that $(\alpha, \omega) = \mathbf{R}$, that is, f is well defined in the whole real line.

Thereby, the revolution surface ψ associated with the above f is a timelike immersion with constant Gaussian curvature $K = 1$ which is well defined for all $(r,\theta) \in \mathbf{R}^2$. To show completeness of ψ it is sufficient to prove that $1/f(r)^2$ is bounded and the associated Riemannian metric $I_R = (1 - f'(r)^2)\, dr^2 + f(r)^2\, d\theta^2$ is complete (see [12] and [13]).

Since $f(r) \geq \sqrt{1-c_0}$ the first assertion is clear. For the second one we consider the new parameter ρ such that $d\rho = \sqrt{(1-f'(r)^2)}\, dr$, then

$$\rho(r) \;=\; cte + \int_0^r \frac{1}{\sqrt{c_0 + f(s)^2}}\, ds \qquad \text{and} \qquad \frac{1}{\sqrt{c_0+f(s)^2}} \geq \frac{1}{\sqrt{c_0+(c_1+|s|)^2}}.$$

Hence, ρ takes values in the whole real line and $I_R \geq d\rho^2 + (1-c_0)\, d\theta^2$, with $(\rho,\theta) \in \mathbf{R}^2$, that is, I_R is complete.

Theorem 1 *There exist infinitely many complete timelike revolution surfaces in* $\mathbf{L}^3$ *with positive constant Gaussian curvature.*

From the above result it is clear the interest of the classification of the complete timelike immersions in $\mathbf{L}^3$ with positive constant Gaussian curvature as we will study in the following section.

4 Timelike Surfaces with Positive Gaussian Curvature

Throughout this section the concept of Lorentz surface plays an important role. We refer the reader to [15] to revise this subject.

Let $X : M \longrightarrow \mathbf{L}^3$ be a timelike surface with positive Gaussian curvature K in $\mathbf{L}^3$, or equivalently, a timelike surface whose second fundamental form II is a Lorentzian metric on M. Then we can take local II-null coordinates (u,v) such that the first and second fundamental forms are given by

$$\begin{aligned} &I = E\, du^2 + 2F\, dudv + G\, dv^2 \\ &II = 2f\, dudv \end{aligned} \tag{3}$$

where f is a real positive function. Any other II-null coordinates $(\widetilde{u},\widetilde{v})$ are related to (u,v) by $\widetilde{u} = \widetilde{u}(u)$ and $\widetilde{v} = \widetilde{v}(v)$, being $\widetilde{u}'(u) > 0$, $\widetilde{v}'(v) > 0$. Let us recall that the 2-forms $E\, du^2$ and $G\, dv^2$ do not depend on the chosen II-null coordinates. Therefore, they are globally well-defined on M and we will refer to those as the $(2,0)$-part and $(0,2)$-part of the metric, respectively.

Now, the Weingarten equations become

$$\begin{aligned} N_u &= \tfrac{\partial N}{\partial u} = \tfrac{f}{EG-F^2}(F X_u - E X_v) \\ N_v &= \tfrac{\partial N}{\partial v} = \tfrac{f}{EG-F^2}(-G X_u + F X_v). \end{aligned} \tag{4}$$

From (3), the Gaussian curvature and the Gauss map of M can be written as

$$K = \frac{-f^2}{EG-F^2} \qquad \text{and} \qquad N = \frac{X_u \times X_v}{\sqrt{F^2 - EG}}$$

respectively. Hence, thanks to the Weingarten equations (4) we obtain

$$X_u = \frac{-1}{\sqrt{K}}\, N \times N_u, \qquad X_v = \frac{1}{\sqrt{K}}\, N \times N_v, \tag{5}$$

that is, the immersion can be recovered in terms of its Gaussian curvature and its Gauss map. Even more, the Gaussian curvature is determined, up to multiplications by positive constants, by the Gauss map. In fact, using (5) we obtain from $X_{uv} - X_{vu} = 0$ that

$$N \times (K_u N_u + K_v N_v - 4K N_{uv}) = 0, \tag{6}$$

and so

$$(\log K)_u N_u + (\log K)_v N_v = 4(N_{uv} - \langle N_{uv}, N\rangle N). \tag{7}$$

By taking the inner product of both terms in (7) with X_u and X_v respectively and using again (5), one easily gets

$$-(\log K)_v = 4\,\frac{\det(N, N_{uv}, N_u)}{\det(N, N_u, N_v)}, \qquad -(\log K)_u = 4\,\frac{\det(N, N_{uv}, N_v)}{\det(N, N_u, N_v)}$$

as we had announced.

In the following theorem we summarize all the comments above jointly with more geometric information about the timelike surface:

Theorem 2 *Let $X : M \longrightarrow \mathbf{L}^3$ be a timelike immersion with positive Gaussian curvature K, N its Gauss map and (u, v) II-null coordinates. Then X can be recovered in terms of K and N as*

$$X_u = \frac{-1}{\sqrt{K}}\, N \times N_u, \qquad X_v = \frac{1}{\sqrt{K}}\, N \times N_v, \tag{8}$$

and the Gaussian curvature is determined, up to multiplications by positive constants, by the Gauss map in the following way:

$$-(\log K)_u = 4\,\frac{\det(N, N_{uv}, N_v)}{\det(N, N_u, N_v)}, \qquad -(\log K)_v = 4\,\frac{\det(N, N_{uv}, N_u)}{\det(N, N_u, N_v)}. \tag{9}$$

Moreover, the first and second fundamental forms of the immersion are given by

$$\begin{aligned} I &= \tfrac{1}{K}\left(-\langle N_u, N_u\rangle\, du^2 + 2\langle N_u, N_v\rangle\, dudv - \langle N_v, N_v\rangle\, dv^2\right) \\ II &= \tfrac{2}{\sqrt{K}} \det(N, N_u, N_v)\, dudv \end{aligned} \tag{10}$$

and its mean curvature by

$$H = \sqrt{K}\, \frac{\langle N_u, N_v\rangle}{\det(N, N_u, N_v)}. \tag{11}$$

Proof: The expressions for the derivatives of X and $\log K$ have already been obtained. As regard to (10), it follows easily from (5). Finally, (11) can be obtained using (10) and the fact that $H = -Ff/(EG - F^2)$, $K = -f^2/(EG - F^2)$. ∎

In the following Theorem we show under what assumptions a map $N : M \longrightarrow \mathbf{S}^2_1$ from a simply connected Lorentz surface M determines a timelike immersion $X : M \longrightarrow \mathbf{L}^3$ with Gauss map N.

Theorem 3 *Let M be a simply connected Lorentz surface and $N : M \longrightarrow \mathbf{S}^2_1$ a differentiable map. Then, there exists a timelike immersion $X : M \longrightarrow \mathbf{L}^3$ with Gauss map N and such that the structure given by its second fundamental form is the one of M if and only if*

$$\det(N, N_u, N_v) > 0 \tag{12}$$

and

$$\left(\frac{\det(N, N_{uv}, N_v)}{\det(N, N_u, N_v)}\right)_v = \left(\frac{\det(N, N_{uv}, N_u)}{\det(N, N_u, N_v)}\right)_u \tag{13}$$

for any II-null coordinates (u, v).

In this case, the immersion is unique (up to similarity transformations of $\mathbf{L}^3$) and can be calculated as in Theorem 2.

Proof: If $X : M \longrightarrow \mathbf{L}^3$ is a timelike immersion with Gauss map N such that II is a Lorentzian metric on M, the direct implication follows from Theorem 2. The converse implication is a straightforward computation, bearing in mind that (12) says that II is a Lorentzian metric on M, (13) is the necessary condition for the existence of K and both of them are the integrability conditions of the immersion. Note that the integration is single-valued because M is simply connected.

Finally, if X and Y are two timelike immersions as above with Gaussian curvatures K_X and K_Y respectively, then from (9) we have that $K_X = \lambda K_Y$ for a positive real constant λ, and so it follows from (8) that $Y = \sqrt{\lambda}X + V$, $V \in \mathbf{L}^3$. ∎

Thanks to the above theorem, we are able to get the following uniqueness result for timelike immersions with positive Gaussian curvature. The concept of conformal equivalence between Lorentz surfaces herein coincides, in essence, with the corresponding one between Riemann surfaces, and can be revised, for example, in [15]. Anyway, it must worth pointing out that while in the case of simply connected Riemann surfaces there exist three classes of equivalence (the complex plane $\mathbf{C}$, the unit disk $\mathbf{D}$ and the Riemann sphere $\mathbf{S}^2$), in the case of simply connected Lorentz surfaces there exist infinitely many ones.

Proposition 4 *Let M be a simply connected Lorentz surface and $X_1, X_2 : M \longrightarrow \mathbf{L}^3$ two timelike immersions with positive Gaussian curvatures K_1, K_2 and Gauss maps $N_1, N_2 : M \longrightarrow \mathbf{S}^2_1$, respectively. Then the following conditions are equivalents:*

(i) There exist a conformal equivalence φ on M and an isometry f of $\mathbf{L}^3$ such that $f \circ X_1 = X_2 \circ \varphi$.

(ii) There exist a conformal equivalence φ on M and an isometry i of $\mathbf{S}^2_1$ such that $i \circ N_1 = N_2 \circ \varphi$ and $K_1 = K_2 \circ \varphi$.

Proof: First, let us suppose that $f \circ X_1 = X_2 \circ \varphi$ where f and φ are as in the statement. Since every isometry preserves the Gaussian curvature, it follows that $K_1 = K_2 \circ \varphi$. Moreover, by taking i as the restriction of df to $\mathbf{S}^2_1$ it is easy to check that $i \circ N_1 = N_2 \circ \varphi$.

Conversely, given an isometry i of $\mathbf{S}^2_1$ we can consider f' as the extension of i to an isometry of $\mathbf{L}^3$. If we take $X'_1 = f' \circ X_1$, *(ii)* says that $N'_1 = N_2 \circ \varphi$ and $K'_1 = K_2 \circ, \varphi$, where N'_1 and K'_1 stand for the Gauss map and the Gaussian curvature of X'_1. Now, Theorem 2 assures that $d(X'_1 - X_2 \circ \varphi) = 0$ and so $X'_1 = X_2 \circ \varphi + V$, $V \in \mathbf{L}^3$, which ends up the proof. ∎

4.1 Positive Constant Gaussian Curvature Surfaces

Let $X : M \longrightarrow \mathbf{L}^3$ be a timelike immersion with positive constant Gaussian curvature and let us consider II-null coordinates (u, v). Then (6) says that $N \times N_{uv} = 0$, and so the Gauss map N is harmonic into $\mathbf{S}^2_1$. Conversely if N is harmonic it follows from (9) that K is constant. So we have (see also [7]):

Corollary 5 *Let $X : M \longrightarrow \mathbf{L}^3$ be a timelike immersion with positive constant Gaussian curvature K. Then K is constant if, and only if, its Gauss map is harmonic for the second fundamental form.*

In the case when M is a simply connected Lorentz surface, Theorem 3 allows us to obtain the following:

Corollary 6 *Given a harmonic local diffeomorphism $N : M \longrightarrow \mathbf{S}^2_1$ preserving the orientation from a simply connected Lorentz surface M and a positive constant K, there exists a unique timelike immersion (up to translations) with positive constant Gaussian curvature K, Gauss map N, and such that the structure induced by the second fundamental form is the one given on M.*

Proof: In fact, note that the harmonicity of N implies that (13) holds. On the other hand, since N is a local diffeomorphism preserving the orientation, it follows that (12) also holds. ∎

To finish this section, we are going to establish a classification result for complete timelike surfaces with positive constant Gaussian curvature K in terms of suitable harmonic maps.

Let us denote by $\mathcal{B}$ the set of harmonic diffeomorphisms preserving the orientation from any simply connected Lorentz surface onto the universal covering $\widetilde{\mathbf{S}^2_1}$ of $\mathbf{S}^2_1$, where two harmonic diffeomorphisms $h_i : M_i \longrightarrow \widetilde{\mathbf{S}^2_1}$, $i = 1, 2$, will be identified if there exist a conformal equivalence $\varphi : M_1 \longrightarrow M_2$ and an isometry i of $\widetilde{\mathbf{S}^2_1}$ such that $h_1 = i \circ h_2 \circ \varphi$.

On the other hand, $\mathcal{A}_K$ will stand for the set of complete timelike immersions with constant Gaussian curvature $K > 0$ from any simply connected Lorentz surface into $\mathbf{L}^3$, where we will identify two immersions $X_i : M_i \longrightarrow \mathbf{L}^3$, $i = 1, 2$, if there exist a conformal equivalence $\varphi : (M_1, II) \longrightarrow (M_2, II)$ and an isometry f of $\mathbf{L}^3$ such that $X_2 \circ \varphi = f \circ X_1$ (i.e., if X_1 and X_2 are *congruent*).

Then we have the following:

Theorem 7 *There exists a bijective correspondence between $\mathcal{A}_K$ and $\mathcal{B}$ for all $K > 0$.*

Proof: We will suppose, without loss of generality, that $K = 1$.

Let us take $X \in \mathcal{A}_1$, $X : M \longrightarrow \mathbf{L}^3$, and (u, v) local II-null coordinates on M. Then the $(2, 0)$ and $(0, 2)$ parts of the metric I verify that $(I^{(2,0)})_v = 0 = (I^{(0,2)})_u$, which implies that the identity map $Id_M : (M, II) \longrightarrow (M, I)$ is a harmonic diffeomorphism.

Since (M, I) is complete, simply connected and has Gaussian curvature $K = 1$, there exists an isometry $i : (M, I) \longrightarrow \widetilde{\mathbf{S}^2_1}$. Hence, $i \circ Id_M$ is a harmonic diffemorphism uniquely determined up to isometries of $\widetilde{\mathbf{S}^2_1}$. We will denote by $\Phi(X) = [i \circ Id_M]$ the class of $i \circ Id_M$ in $\mathcal{B}$. Then we define $\overline{\Phi} : \mathcal{A}_1 \longrightarrow \mathcal{B}$ given by $\overline{\Phi}([X]) = \Phi(X)$.

First, let us see that, indeed, $\overline{\Phi}$ is a map. Let $X_i : M_i \longrightarrow \mathbf{L}^3$, $i = 1, 2$, be two congruent complete timelike immersions, that is, there exist a conformal equivalence $\varphi : (M_1, II) \longrightarrow (M_2, II)$ and an isometry f of $\mathbf{L}^3$ so that $X_2 \circ \varphi = f \circ X_1$. If $i : (M_2, I) \longrightarrow \widetilde{\mathbf{S}^2_1}$ is an isometry then $i \circ f \circ Id_{M_1} = i \circ Id_{M_2} \circ \varphi$ (where p and $X_i(p)$ are identified, $i = 1, 2$). Hence, $\Phi(X_1) = \Phi(X_2)$.

To see that $\overline{\Phi}$ is injective, let us take $X_i : M_i \longrightarrow \mathbf{L}^3$, $i = 1, 2$, two complete timelike immersions such that $\Phi(X_1) = \Phi(X_2)$. Then there exist a conformal equivalence $\varphi : (M_1, II) \longrightarrow (M_2, II)$ and an isometry $i : (M_1, I) \longrightarrow (M_2, I)$ such that $Id_{M_1} = i^{-1} \circ Id_{M_2} \circ \varphi$. Consequently $i = \varphi$ and X_1,

$X_2 \circ i$ are two timelike immersions from (M_1, I) into $\mathbf{L}^3$ with the same induced metric, the same conformal structure for the second fundamental form and the same Gaussian curvature. Therefore, Gauss' egregium theorem assures that X_1 and $X_2 \circ i$ have the same second fundamental form, and so they coincide up to an isometry of $\mathbf{L}^3$, whence $[X_1] = [X_2]$.

Finally, let us check that $\overline{\Phi}$ is onto. To see that, let us take a harmonic diffeomorphism $h : M \longrightarrow \widetilde{\mathbf{S}_1^2}$ preserving the orientation from a simply connected Lorentz surface M onto $\widetilde{\mathbf{S}_1^2}$, and let us consider the composition map $\widetilde{h} = \pi \circ h : M \longrightarrow \mathbf{S}_1^2$, where $\pi : \widetilde{\mathbf{S}_1^2} \longrightarrow \mathbf{S}_1^2$ is the canonical projection. Observe that $\widetilde{h}$ is also harmonic and so, from Theorem 3, there exists a timelike immersion $Y : M \longrightarrow \mathbf{S}_1^2$ with Gauss map $\widetilde{h}$ and constant Gaussian curvature $K = 1$. On the other hand, note that the identity map $Id_M : (M, II_Y) \longrightarrow (M, I_Y)$ is harmonic, (where, for instance, I_Y stands for the first fundamental form of Y) and, since M is simply connected and (M, I_Y) has constant Gaussian curvature $K = 1$, there exists an isometric immersion $i : (M, I_Y) \longrightarrow \widetilde{\mathbf{S}_1^2}$. Thus, as the composition map $\widetilde{i} = \pi \circ i : (M, I_Y) \longrightarrow \mathbf{S}_1^2$ is also an isometric immersion, Theorem 3 says that there exists a timelike immersion $X : M \longrightarrow \mathbf{L}^3$ with Gauss map $\widetilde{N} = \widetilde{i} \circ Id_M : (M, II_Y) \longrightarrow \mathbf{S}_1^2$ and constant Gaussian curvature 1.

Observe that the third fundamental form of X, III_X, is nothing but the pullback by $\widetilde{i}$ of the standard metric of $\mathbf{S}_1^2$, whence $III_X = I_Y$. Therefore, from (10),

$$-\langle \widetilde{h}_u, \widetilde{h}_u \rangle = \langle \widetilde{N}_u, \widetilde{N}_u \rangle, \qquad \langle \widetilde{h}_u, \widetilde{h}_v \rangle = \langle \widetilde{N}_u, \widetilde{N}_v \rangle, \qquad -\langle \widetilde{h}_v, \widetilde{h}_v \rangle = \langle \widetilde{N}_v, \widetilde{N}_v \rangle$$

for any II-null coordinates (u, v). Thus, we have that $I_X = III_Y$, and also that $II_X = II_Y$, because $\det(\widetilde{h}, \widetilde{h}_u, \widetilde{h}_v) = \det(\widetilde{N}, \widetilde{N}_u, \widetilde{N}_v)$. Finally, since $I_X = \langle d\widetilde{h}, d\widetilde{h} \rangle = \langle dh, dh \rangle$ is the pullback by h of the standard metric of $\widetilde{\mathbf{S}_1^2}$, it follows that X is complete, that is, $X \in \mathcal{A}_1$, which jointly with $\overline{\Phi}(X) = [h]$ ends up the proof. ∎

Remark 8 Every harmonic diffeomorphism into $\mathbf{S}_1^2$ can be lifted to one into $\widetilde{\mathbf{S}_1^2}$ and vice versa. Moreover, every Lorentz surface is locally conformally equivalent to a domain of $\mathbf{L}^2$. Therefore, the study of Gu [4] on the harmonic maps from $\mathbf{L}^2$ into $\mathbf{S}_1^2$ and the above results can be used to the construction of complete and non complete timelike immersions with positive constant Gaussian curvature.

5 Timelike Surfaces with Negative Gaussian Curvature

Let $X : M \longrightarrow \mathbf{L}^3$ be a timelike surface with negative Gaussian curvature K in $\mathbf{L}^3$, or equivalently, a timelike surface whose second fundamental form is a Riemannian metric on M. Thus, from now on M will be considered as a Riemann surface with the conformal structure induced by II.

If $z = u + iv$ is a conformal parameter, the first and second fundamental forms are given by

$$I = E\,du^2 + 2F\,dudv + G\,dv^2$$
$$II = e(du^2 + dv^2)$$

where e is a positive real function.

Then, the Weingarten equations become

$$N_u = \frac{e}{EG - F^2}\,(-GX_u + FX_v) \qquad N_v = \frac{e}{EG - F^2}\,(FX_u - EX_v) \tag{14}$$

whereas the Gaussian curvature and the Gauss map of M are given by

$$K = \frac{e^2}{EG - F^2} \qquad \text{and} \qquad N = \frac{X_u \times X_v}{\sqrt{F^2 - EG}}$$

respectively.

Now, using the Weingarten equations (14) we get

$$X_u = \frac{-1}{\sqrt{-K}}\, N \times N_v, \qquad X_v = \frac{1}{\sqrt{-K}}\, N \times N_u,$$

or equivalently

$$X_z = \frac{-i}{\sqrt{-K}}\, N \times N_z, \tag{15}$$

so that the immersion can be recovered in terms of the Gaussian curvature and the Gauss map.

As in the case of positive Gaussian curvature, the Gauss map determines, up to multiplications by positive constants, the Gaussian curvature of the surface. In fact, using (15) one gets from $X_{z\overline{z}} - X_{\overline{z}z} = 0$ that

$$N \times (K_z N_z + K_{\overline{z}} N_{\overline{z}} - 4K N_{z\overline{z}}) = 0, \tag{16}$$

whence

$$(\log K)_z N_z + (\log K)_{\overline{z}} N_{\overline{z}} = 4(N_{z\overline{z}} - \langle N_{z\overline{z}}, N\rangle N). \tag{17}$$

Now, by taking the inner product of both terms in (17) by X_z and using again (15), we finally have

$$-(\log K)_z = 4\, \frac{\det(N, N_{\overline{z}}, N_{z\overline{z}})}{\det(N, N_z, N_{\overline{z}})}.$$

We summarize all the comments above jointly with more geometric information about the surface in the following theorem:

Theorem 9 *Let $X : M \longrightarrow \mathbf{L}^3$ be a timelike immersion with negative Gaussian curvature K, N its Gauss map and $z = u + iv$ a local conformal parameter on M. Then X can be recovered in terms of K and N as*

$$X_z = \frac{-i}{\sqrt{-K}}\, N \times N_z, \tag{18}$$

and the Gaussian curvature is determined, up to multiplications by positive constants, by the Gauss map in the following way:

$$-(\log K)_z = 4\, \frac{\det(N, N_{\overline{z}}, N_{z\overline{z}})}{\det(N, N_z, N_{\overline{z}})}. \tag{19}$$

Moreover, the first and second fundamental forms of the immersion are given by

$$\begin{aligned} I &= \tfrac{1}{K}\left(-\langle N_z, N_z\rangle\, dz^2 + 2\langle N_z, N_{\overline{z}}\rangle\, dz d\overline{z} - \langle N_{\overline{z}}, N_{\overline{z}}\rangle\, d\overline{z}^2\right) \\ II &= \tfrac{-2i}{\sqrt{-K}} \det(N, N_z, N_{\overline{z}})\, dz d\overline{z} \end{aligned} \tag{20}$$

and its mean curvature by

$$H = 2i\sqrt{-K}\, \frac{\langle N_z, N_{\overline{z}}\rangle}{\det(N, N_z, N_{\overline{z}})}. \tag{21}$$

Proof: The expressions for X_z and $(\log K)_z$ have already been obtained. Concerning to (20), it follows easily from (15). Finally, (21) can be obtain using (20) and that $H = e(E+G)/(EG-F^2)$, $K = e^2/(EG-F^2)$. ∎

In the following Theorem we show under what assumptions a map $N : M \longrightarrow \mathbf{S}^2_1$ from a simply connected Riemann surface M determines a timelike immersion $X : M \longrightarrow \mathbf{L}^3$ with Gauss map N. Note that M must be conformally equivalent to the complex plane $\mathbf{C}$ or the unit disk $\mathbf{D}$ because M is a non compact, simply connected surface and, so, a global conformal coordinate is available.

Theorem 10 *Let M be a simply connected Riemann surface and $N : M \longrightarrow \mathbf{S}^2_1$ a differentiable map. Then, there exists a timelike immersion $X : M \longrightarrow \mathbf{L}^3$ with Gauss map N and such that the structure given by its second fundamental form is the one of M if and only if*

$$-i\det(N, N_z, N_{\overline{z}}) > 0 \tag{22}$$

$$-\left(\frac{\det(N, N_{\overline{z}}, N_{z\overline{z}})}{\det(N, N_z, N_{\overline{z}})}\right)_{\overline{z}} = \left(\frac{\det(N, N_z, N_{z\overline{z}})}{\det(N, N_z, N_{\overline{z}})}\right)_z \tag{23}$$

for a global conformal coordinate z.

In this case, the immersion is unique (up to similarity transformations of $\mathbf{L}^3$) and can be calculated as in Theorem 9.

Proof: If $X : M \longrightarrow \mathbf{L}^3$ is a timelike immersion with Gauss map N such that II is a Riemann metric on M, the direct implication follows from Theorem 9. The converse implication is a straightforward computation, bearing in mind that (22) says that II is a Riemannian metric on M and (23) is the integrability condition for K.

To end up, if X and Y are two timelike immersions as above with Gaussian curvatures K_X and K_Y respectively, then from (19) we have that $K_X = \lambda K_Y$ for a positive real constant λ, and hence, using (18), we have that $Y = \sqrt{\lambda}X + V$, $V \in \mathbf{L}^3$. ∎

5.1 Negative Constant Gaussian Curvature Surfaces

Let $X : M \longrightarrow \mathbf{L}^3$ be a timelike immersion with negative constant Gaussian curvature. If we consider on M a local conformal parameter $z = u + iv$, (16) says that $N \times N_{z\overline{z}} = 0$, and so the Gauss map N is harmonic into $\mathbf{S}^2_1$. Conversely, if N is harmonic we get from (19) that K is constant. Thus, we can state (see [7]):

Corollary 11 *Let $X : M \longrightarrow \mathbf{L}^3$ be a timelike immersion with negative Gaussian curvature K. Then K is constant if, and only if, its Gauss map is harmonic for the second fundamental form.*

In the case when M is a simply connected Riemann surface, Theorem 10 allows us to obtain the following:

Corollary 12 *Given a harmonic local diffeomorphism $N : M \longrightarrow \mathbf{S}^2_1$ preserving the orientation from a simply connected Riemann surface M and a negative constant K, there exists a unique timelike immersion (up to translations) with negative constant Gaussian curvature K, Gauss map N and such that the structure induced by the second fundamental form is the one given on M.*

Proof: It results immediately by observing that the harmonicity of N implies that (23) holds. On the other hand, since N is a local diffeomorphism preserving the orientation, it follows that (22) also holds. ∎

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