The space of solutions to the Hessian one equation in the finitely punctured plane

José A. Gálvez$^a$, Antonio Martínez$^b$ and Pablo Mira$^c$

$^a$, $^b$ Departamento de Geometría y Topología, Universidad de Granada, E-18071 Granada, Spain.
e-mail: jagalvez@ugr.es ; amartine@ugr.es

$^c$ Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, E-30203 Cartagena, Murcia, Spain.
e-mail: pablo.mira@upct.es

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Abstract

We construct the space of solutions to the elliptic Monge-Ampère equation $\det(D^2\phi) = 1$ in the plane $\mathbb{R}^2$ with $n$ points removed. We show that, modulo equiaffine transformations and for $n > 1$, this space can be seen as an open subset of $\mathbb{R}^{3n-4}$, where the coordinates are described by the conformal equivalence classes of once punctured bounded domains in $\mathbb{C}$ of connectivity $n - 1$. This approach actually provides a constructive procedure that recovers all such solutions to the Monge-Ampère equation, and generalizes a theorem by K. Jörgens.

Rédumé

Nous construisons l’espace des solutions de l’équation elliptique de Monge-Ampère $\det(D^2\phi) = 1$ définies dans le plan $\mathbb{R}^2$ sauf en $n$ points. Nous montrons que, pour $n > 1$ et modulo les transformations équiaﬃnes, cet espace peut être identifié à un ouvert de $\mathbb{R}^{3n-4}$, dont les coordonnées de l’espace sont décrites par des classes d’équivalences conformes de domaines bornés de $\mathbb{C}$ perforés en un seul point et de connectivité $n – 1$. Ce point de vue permet de construire et de retrouver toutes ces solutions de l’équation de Monge-Ampère et généralise un théorème de K. Jörgens.
1 Introduction

A celebrated result by K. Jörgens [Jor1] states that all solutions to the elliptic Monge-Ampère equation
\[ \phi_{xx}\phi_{yy} - \phi_{xy}^2 = 1 \] (1.1)
which are globally defined on \( \mathbb{R}^2 \) are quadratic polynomials. This theorem has motivated a large amount of works, which essentially follow two research lines. One of them deals with the extension of the result to more general classes of Monge-Ampère equations (see for instance [BCGJ, Cal, CaLi2, ChYa, ChWa, Pog, TrWa1, TrWa2]). The other one was initiated by K. Jörgens in 1955, and concerns the validity of the theorem in large proper domains of \( \mathbb{R}^2 \), and the study of the new solutions that may arise. Some results of this type can be found in [Jor2, CaLi1, FMM1, FMM2, TrWa1]. In particular, in [Jor2] all the solutions to (1.1) globally defined in \( \mathbb{R}^2 \setminus \{(0,0)\} \) were obtained.

In the present work we follow this last direction, and plan to determine all the solutions to (1.1) in \( \mathbb{R}^2 \) with \( n \) points removed. In this way, our result generalizes the two mentioned theorems by Jörgens, and suggests a new line of inquiry into the theory of Monge-Ampère equations.

For stating such a general description, some remarks have to be made. First, note that if \( \phi \) is a solution to (1.1) in a finitely punctured plane, the punctures correspond then to isolated singularities of \( \phi \). We shall assume without loss of generality that isolated singularities are non-removable, i.e. \( \phi \) cannot be \( C^1 \)-extended across the singularity. A well-known criterion to determine whether an isolated singularity of a solution to (1.1) is removable is found in [Jor2]. We shall provide a different one in Section 3, in terms of the underlying conformal structure of the solution \( \phi \).

Another remark concerns non-unicity of the solutions to (1.1). Let \( \mathcal{E} \) be the group of equiaffine transformations of \( \mathbb{R}^3 \) whose differentials fix the \( x_3 \)-direction, i.e. the set of maps \( \Phi : \mathbb{R}^3 \to \mathbb{R}^3 \) of the form
\[
\Phi \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad a_{11}a_{22} - a_{21}a_{12} = 1. \tag{1.2}
\]
Then, if \( \phi(x, y) \) solves (1.1) and \( \Phi \in \mathcal{E} \), the function \( \phi'(x', y') \) given by \( \Phi(x, y, \phi(x, y)) = (x', y', \phi'(x', y')) \) is a new solution to (1.1) in terms of the variables \( x', y' \). Therefore, we shall be interested in obtaining the classification result for solutions to (1.1) modulo equiaffine transformations. In other words, two solutions to (1.1) differing only by an element of \( \mathcal{E} \) will be considered to be equivalent.

Keeping these facts in mind, the classification by Jörgens of the solutions to (1.1) in the once punctured plane is formulated as follows: the unique solutions to (1.1) in \( \mathbb{R}^2 \) and in \( \mathbb{R}^2 \setminus \{(0,0)\} \) are respectively, up to equiaffine transformations, the quadratic polynomial \( \phi(x, y) = (x^2 + y^2) / 2 \) and the rotational example
\[
\phi(x, y) = \frac{1}{2} \left( \sqrt{x^2 + y^2} \sqrt{1 + x^2 + y^2} + \sinh^{-1} \left( \sqrt{x^2 + y^2} \right) \right).
\]
So, the following question arises naturally: given $n > 1$, is there a unique (up to equiaffine transformations) solution to (1.1) defined in an $n$-times punctured plane, and so that the punctures are non-removable isolated singularities? In Section 3 we shall analyze to what extent the answer to this question is yes.

The paper is organized as follows. In Section 2 we review the theory of solutions to (1.1), and its equivalence to the theory of locally convex embedded parabolic affine spheres in the affine 3-space $\mathbb{R}^3$. In particular, we shall recall a conformal representation available for these surfaces, and indicate some basic properties of parabolic affine spheres in terms of the holomorphic data. The main result in this Section is the construction of solutions to (1.1) in a punctured plane $\mathbb{R}^2 \setminus \{p_1, \ldots, p_n\}$ so that the $p_j$’s are non-removable isolated singularities of such solutions. The construction relies on the conformal representation of parabolic affine spheres, and the existence of two classical holomorphic functions with the fundamental property of mapping bijectively a once punctured bounded domain in $\mathbb{C}$ with $n$ boundary components onto $\mathbb{C}$ with $n$ vertical (resp. horizontal) slits.

Section 3 contains the main theorem of this work, which characterizes the examples constructed in Section 2 as the only solutions (up to equiaffine transformations) to (1.1) that are globally defined over a finitely punctured plane. This theorem provides an explicit one-to-one correspondence between the conformal equivalence classes of $\mathbb{C}$ with $n$ disks removed, and the space of solutions to (1.1) in a finitely punctured plane with exactly $n$ non-removable isolated singularities, modulo equiaffine transformations. Particularly, for $n > 1$, this quotient space can be naturally identified with an open subset of $\mathbb{R}^{3n-4}$.

Finally, in Section 4 we show that our main theorem for the particular choice $n = 1$ recovers the Jörgens one, and we determine explicitly in terms of theta functions all the solutions to (1.1) in the twice punctured plane. This indicates that all the solutions we have constructed are highly non-trivial.

## 2 Construction of the canonical examples

Let $\phi(x, y) : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a solution to (1.1) on a planar domain $D$. We shall assume without loss of generality that $\phi_{xx}$ is always positive. Then the subset $S_\phi = \{(x, y, \phi(x, y)) : (x, y) \in D\} \subseteq \mathbb{R}^3$ describes an embedded locally convex parabolic affine sphere in the affine 3-space $\mathbb{R}^3$ with affine normal vector $\xi = (0, 0, 1)$ and affine metric $ds^2 = \phi_{xx} \, dx^2 + 2\phi_{xy} \, dx \, dy + \phi_{yy} \, dy^2$ (see for instance [FMM1, FMM2] for more information). Conversely, any embedded locally convex parabolic affine sphere with affine normal vector $\xi = (0, 0, 1)$ is locally the graph over a domain in the $x, y$-plane of a solution to (1.1). Along this paper, by a parabolic affine sphere we will always mean a locally convex one.

The affine metric $ds^2$ of $S_\phi$ induces a Riemann surface structure on $S_\phi$. We shall call this Riemann surface structure the underlying conformal structure of $\phi(x, y)$.

The solutions to (1.1) can be explicitly recovered in terms of holomorphic data with respect to this underlying conformal structure. We expose here the global version of
this fact in [FMM2], for the general case of immersed parabolic affine spheres.

**Theorem 1** Let \( \Sigma \) be a simply connected Riemann surface, and let \( F, G : \Sigma \to \mathbb{C} \) be two holomorphic functions for which \( dG \neq 0 \) and \( |dF| < |dG| \) hold everywhere. Then the map \( \psi : \Sigma \to \mathbb{R}^3 \) given by

\[
\psi = \frac{1}{2} \left( G + F, \frac{1}{4} \{ |G|^2 - |F|^2 + 2 \text{Re}(GF) \} - \text{Re} \int FdG \right) \tag{2.1}
\]

is a parabolic affine sphere with affine normal vector \( \xi = (0, 0, 1) \) and affine metric \( ds^2 = \frac{1}{4} (|dG|^2 - |dF|^2) \).

Conversely, any parabolic affine sphere can be recovered in this way with respect to the conformal structure induced by its affine metric.

In order to construct examples of solutions to (1.1) in the \( n \)-times punctured plane, we need to recall first a classical description of two special holomorphic functions, which we will denote by \( p(z), q(z) \).

Let \( \Omega \subset \mathbb{C} \) be a bounded domain whose boundary is made up by \( n \geq 1 \) analytic Jordan curves \( C_1, \ldots, C_n \), where we will let \( C_n \) be the exterior component, and take \( z_0 \in \Omega \). Let \( G(z, z_0) \) be the Green function of \( \Omega \) with respect to \( z_0 \), given by \( G(z, z_0) = D(z, z_0) - \log |z - z_0| \), where \( D \) is the solution to the Dirichlet problem in \( \Omega \) with boundary conditions \( D(\zeta) = \log |\zeta - z_0|, \zeta \in C_k \). Then the map

\[
u(z) = \frac{\partial G(z, z_0)}{\partial z_0} + \frac{\partial G(z, z_0)}{\partial \bar{z}_0} \tag{2.2}
\]

is harmonic in \( \Omega \setminus \{z_0\} \), and vanishes on \( \partial \Omega \). Now let \( \omega_j \) denote the *harmonic measure* of \( C_j \) with respect to \( \Omega \), characterized by the conditions

\[
\begin{cases}
\Delta \omega_j = 0 & \text{on } \Omega, \\
\omega_j(C_j) = 1, \\
\omega_j(C_k) = 0 & \text{if } k \neq j.
\end{cases}
\]

We shall denote by \( \alpha_{j,k} \) the period of the conjugate harmonic function of \( \omega_j \) along \( C_k \):

\[
\alpha_{j,k} = \int_{C_k} \frac{\partial \omega_j}{\partial n} \, ds,
\]

where \( n \) is the unit normal to the curve \( C_k \). Let finally \( A_k \in \mathbb{R} \) denote the period of the conjugate harmonic function of \( u(z) \) along \( C_k \). Then, if \( \lambda_1, \ldots, \lambda_{n-1} \) are real numbers, the harmonic function

\[
u + \lambda_1 \omega_1 + \cdots + \lambda_{n-1} \omega_{n-1} : \Omega \setminus \{z_0\} \to \mathbb{R} \tag{2.3}
\]

has a well defined conjugate harmonic function on \( \Omega \setminus \{z_0\} \) if and only if the \( \lambda_i \)'s solve the following linear system:
\[
\begin{align*}
\lambda_1 \alpha_{1,1} + \cdots + \lambda_{n-1} \alpha_{n-1,1} &= -A_1 \\
\vdots \\
\lambda_1 \alpha_{1,n-1} + \cdots + \lambda_{n-1} \alpha_{n-1,n-1} &= -A_{n-1}
\end{align*}
\]

This system is classically known to have a unique solution (see [Ahl] for instance), and thus there is a holomorphic function \( p(z) : \Omega \setminus \{z_0\} \to \mathbb{C} \) whose real part is the above harmonic function, i.e. \( \Re p(z) = u(z) + \sum_{i=1}^{n-1} \lambda_i \omega_i(z) \). We must remark that \( p(z) \) is well defined up to pure imaginary additive constants.

In addition, by the construction of \( u(z) \) it becomes clear that \( p(z) \) has a pole of order one at \( z_0 \), with residue 1. Moreover, \( \Re p(z) = \lambda_k \in \mathbb{R} \) along \( C_k \) for every \( k \in \{1, \ldots, n\} \).

Actually, \( p(z) \) is characterized by these conditions: it is the unique (up to pure imaginary additive constants) holomorphic map in \( \Omega \setminus \{z_0\} \) with a simple pole of residue 1 at \( z_0 \) whose real part is constant along each boundary component in \( \partial \Omega \). Finally, \( p(z) \) is a conformal equivalence between \( \Omega \setminus \{z_0\} \) and its image, which is a vertical slit domain in \( \mathbb{C} \) [Ahl].

The analogous process to the above one, but this time starting with \( v(z) = \partial G(z, z_0) / \partial z_0 - \partial G(z, z_0) / \partial \overline{z}_0 \), produces a holomorphic function \( q(z) : \Omega \setminus \{z_0\} \to \mathbb{C} \) with a simple pole of residue 1 at \( z_0 \), and such that its imaginary part \( \Im q(z) \) is constant along each boundary curve \( C_k \). Again, \( q(z) \) is defined up to real additive constants, and it is the unique holomorphic map in \( \Omega \setminus \{z_0\} \) satisfying these properties. Further, it maps \( \Omega \setminus \{z_0\} \) bijectively onto a horizontal slit domain in \( \mathbb{C} \).

**Theorem 2** For any \( n \in \mathbb{N} \), there exists a regular solution \( \phi(x, y) \) to (1.1) globally defined in \( \mathbb{R}^2 \) minus \( n \) points, and such that the punctures are non-removable isolated singularities of \( \phi(x, y) \).

**Proof:** We start with a complex domain \( \Omega \) as above, with a distinguished point \( z_0 \in \Omega \). Let then \( p(z), q(z) \) be the holomorphic functions constructed following the previous procedure. Now, define

\[
G(z) = p(z) + q(z), \quad F(z) = p(z) - q(z).
\]

It then follows readily that \( (G + F)|_{C_k} \) is constant for all \( k \in \{1, \ldots, n\} \). Differentiating now this expression we infer that \( |dF/dG| = 1 \) at every point in \( \partial \Omega \). On the other hand, since \( p(z), q(z) \) have simple poles of equal residues at \( z_0 \), we obtain that \( |dF/dG|(z_0) = 0 \). Hence, \( |dF/dG| < 1 \) in all \( \Omega \). In addition, as \( p, q \) are holomorphic bijections between \( \Omega \setminus \{z_0\} \) and a certain slit domain in \( \mathbb{C} \), we get that \( dF, dG \) cannot vanish simultaneously. Particularly, \( dG \) never vanishes by \( |dF/dG| < 1 \).

Therefore, by Theorem 1 the holomorphic functions \( F, G \) define a parabolic affine sphere

\[
\psi : \Omega \setminus \{z_0\} \to \mathbb{R}^3,
\]
where $\Omega \setminus \{z_0\}$ is the conformal universal covering of $\Omega \setminus \{z_0\}$. The affine sphere $\psi$ is actually well defined on $\Omega \setminus \{z_0\}$. To see this we have to check that

$$\text{Re} \int_{\Gamma} F dG = 0 \quad (2.6)$$

along any loop $\Gamma$ in $\Omega \setminus \{z_0\}$. First, observe that as $p(z)$ and $q(z)$ map each $C_k$ into a piece of a straight line, the functions $F, G$ can be holomorphically extended across each $C_k$ by Schwarzian reflection. In this way, we may extend $\Omega \setminus \{z_0\}$ slightly so that the curves $C_k$ are interior to the extended domain, and $F, G$ are holomorphic there. But in this larger domain, the curves $C_1, \ldots, C_n$ provide a basis of its first homology group. So, we just need to check condition (2.6) for $\Gamma = C_1, \ldots, C_n$. To do so, we first observe that by (2.5) it holds for all $k \in \{1, \ldots, n\}$

$$\int_{C_k} F dG = \int_{C_k} (p \, dq - q \, dp).$$

In addition,

$$0 = \text{Re} \int_{C_k} d(pq) = -\text{Re} \int_{C_k} (p \, dq - q \, dp) + 2 \int_{C_k} (\text{Im} \, q \, \text{Im} \, \psi' + \text{Re} \, p \, \text{Re} \, \psi') \, dz.$$

As both $\text{Re} \, p(z)$ and $\text{Im} \, q(z)$ are constant along $C_k$, we obtain that $\text{Re} \int_{C_k} F dG = 0$ for all the $C_k$’s, as we wished.

Finally, we show that $\psi = (\psi_1, \psi_2, \psi_3) : \Omega \setminus \{z_0\} \to \mathbb{R}^3$ is actually an embedding. For this, observe first of all that $F$ is holomorphic at $z_0$, and $G$ has a simple pole at $z_0$. Thus, $(\psi_1, \psi_2)$ is a homeomorphism from a small enough punctured disk about $z_0$ onto the exterior of a Jordan curve in $\mathbb{R}^2$.

Next, observe that $\mathcal{F} := (\psi_1, \psi_2) : \Omega \setminus \{z_0\} \to \mathbb{R}^2$ is well defined and continuous on the once punctured topological sphere $E$ which results when each curve $C_k$ in $\overline{\Omega} \setminus \{z_0\}$ is identified with a single point. This happens because by (2.1) we have that $\mathcal{F}(C_j) = p_j \in \mathbb{R}^2$ for every $j \in \{1, \ldots, n\}$ and for some points $p_1, \ldots, p_n \in \mathbb{R}^2$.

By the behaviour of $\mathcal{F}$ about $z_0$, there is a topological punctured disk $D \subset E$ with puncture at $z_0$ such that $\mathcal{F}$ maps $D$ bijectively onto the exterior of a closed disk $B$ in $\mathbb{R}^2$. Thus, $\mathcal{F}(E \setminus D)$ must fill the whole $\overline{B}$. As a consequence, $(\psi_1, \psi_2) : \overline{\Omega} \setminus \{z_0\} \to \mathbb{R}^2$ is onto.

Now, assume the existence of two points $z_1, z_2 \in \Omega \setminus \{z_0\}$ with $\mathcal{F}(z_1) = \mathcal{F}(z_2) = a \in \mathbb{R}^2$, and let $C$ denote a divergent curve in $\mathbb{R}^2 \setminus \{p_1, \ldots, p_n\}$ with endpoint $a$ (that may belong to $\{p_1, \ldots, p_n\}$). As $\mathcal{F}$ is a local homeomorphism, there exist two paths $\gamma_1, \gamma_2$ in $\Omega$ with endpoints $z_1, z_0$ and $z_2, z_0$ respectively, such that $\mathcal{F}(\gamma_1) = C$. But $\mathcal{F}$ is a homeomorphism around $z_0$, and this implies that $\gamma_1, \gamma_2$ must coincide from a first point to the endpoint $z_0$. This contradicts that $\mathcal{F}$ is a local homeomorphism, unless $z_1 = z_2$. Hence, $\psi : \Omega \setminus \{z_0\} \to \mathbb{R}^3$ is one-to-one.

Finally, we need to check that $\mathcal{F}(\Omega \setminus \{z_0\}) \subseteq \mathbb{R}^2 \setminus \{p_1, \ldots, p_n\}$. If $\mathcal{F}(\tilde{z}) = p_j$ for some $\tilde{z} \in \Omega$, we may take disjoints neighbourhoods $A_1, A_2$ of $\tilde{z}$ and $C_j$ in $\overline{\Omega} \setminus \{z_0\}$,
respectively. There exist then points $\zeta_i \in A_i$ with $F(\zeta_1) = F(\zeta_2) \in \mathbb{R}^2 \setminus \{p_1, \ldots, p_n\}$. Thus both points lie in $\Omega$, which is impossible.

Summing up, we have an embedded parabolic affine sphere $\mathcal{S} = \psi(\Omega \setminus \{z_0\}) \subset \mathbb{R}^3$ such that $(\psi_1, \psi_2) : \Omega \setminus \{z_0\} \to \mathbb{R}^2 \setminus \{p_1, \ldots, p_n\}$ is a global homeomorphism. This produces a regular solution to (1.1) globally defined in $\mathbb{R}^2 \setminus \{p_1, \ldots, p_n\}$ and with isolated singularities exactly at the points $p_1, \ldots, p_n$. All these isolated singularities are non-removable by Lemma 5.

3 The classification theorem

This Section is devoted to show that any solution to (1.1) in a finitely punctured plane must be equiaffinely equivalent to one of the examples of Theorem 2. In order to do so, it is convenient to introduce the following terminology. By definition, a global solution to (1.1) in $\mathbb{R}^2 \setminus \{p_1, \ldots, p_n\}$ will be a $C^2$ function $\phi(x, y) : \mathbb{R}^2 \setminus \{p_1, \ldots, p_n\} \to \mathbb{R}$ verifying (1.1) and that is not $C^1$ at the $p_j$’s. We remark (see [Jor2]) that if a solution to (1.1) extends as a $C^1$ function across an isolated singularity, then it actually extends analytically across it.

With this, we shall prove

Theorem 3 Any global solution $\phi(x, y) : \mathbb{R}^2 \setminus \{p_1, \ldots, p_n\} \to \mathbb{R}$ to (1.1) is one of the examples in Theorem 2 up to equiaffine transformations.

In particular, there exists an explicit bijective correspondence between the conformal equivalence classes of once punctured bounded domains in $\mathbb{C}$ of connectivity $n-1$, and the space of global solutions to (1.1) with $n$ punctures, modulo equiaffine transformations.

Let $\mathcal{M}_n$ be the quotient space of global solutions to (1.1) with $n > 1$ punctures, modulo equiaffine transformations. As an interesting consequence of Theorem 3 we can endow $\mathcal{M}_n$ with a finite dimensional analytic manifold structure:

Corollary 4 The space $\mathcal{M}_n$ can be naturally identified with an open subset of $\mathbb{R}^{3n-4}$.

Proof. Let $\Omega \subset \mathbb{C}$ be a bounded domain in $\mathbb{C}$ of connectivity $n-1$, and take $z_0 \in \Omega$. There exists then a Möbius transformation of the Riemann sphere $\mathbb{C} \cup \{\infty\}$ taking $\Omega \setminus \{z_0\}$ to a new complex domain $\Lambda \subset \mathbb{C}$ consisting of $\mathbb{C}$ with the interior of $n$ Jordan curves removed. Thus, by Koebe’s uniformization theorem, $\Lambda$ is conformally equivalent to $\mathbb{C}$ with $n$ disjoint disks removed. Moreover, these disks $D_1, \ldots, D_n$ are uniquely determined if we assume the following two restrictions: $D_1 = \mathbb{D}$ is the unit disk, and $D_2$ has its center in the positive real line.

Let now $\mathcal{B}_n$ denote the space of conformal equivalence classes of once punctured bounded domains in $\mathbb{C}$ of connectivity $n-1$. The above comments show that $\mathcal{B}_n$ can be canonically identified with the open subset in $(1, +\infty) \times \mathbb{C}^{n-2} \times (\mathbb{R}^+)^{n-1} \subset \mathbb{R}^{3n-4}$ consisting of those points $a = (c_1, \ldots, c_{n-1}, r_1, \ldots, r_{n-1})$ for which the open disks of
centers $c_j$ and associated radii $r_j$ (with $c_1 \in (1, +\infty)$) together with the unit disk $\mathbb{D}$ are pairwise disjoint. Once here, the proof follows immediately from Theorem 3.

Before coming to the proof of Theorem 3, we shall give a criterion to determine when an isolated singularity of a solution to (1.1) can be removed. Our perspective here is completely different from the usual results on removing isolated singularities of Monge-Ampère equations that can be found in the literature. However this description is to some extent implicit in the result by Jörgens [Jor2] on removing isolated singularities of (1.1).

**Lemma 5** An isolated singularity of a solution $\phi(x, y)$ to (1.1) is removable if and only if the underlying conformal structure of $\phi(x, y)$ around the singularity is that of a punctured disk.

**Proof:** Let $\phi(x, y)$ be a solution to (1.1) in a once punctured topological disk $U \setminus \{(x_0, y_0)\} \subset \mathbb{R}^2$. Let $\zeta: \Omega \subset \mathbb{C} \to U$ be a global holomorphic parameter with respect to the underlying conformal structure of $\phi(x, y)$. Then, shrinking $U$ if necessary, $\Omega$ is biholomorphic to a punctured disk or an annulus. Suppose that $(x_0, y_0)$ is a removable isolated singularity of $\phi$, i.e. $\phi$ extends as a $C^1$ function across $(x_0, y_0)$. As the holomorphic data verify (see [FMM1] for instance) $G + F = 2x + 2i\phi_y, \ G - F = 2\phi_x + 2iy$, both $F, G$ have a well defined value at the singularity. This implies that we must have the punctured disk conformal type (otherwise, $F, G$ would be constant).

Now, suppose that the underlying conformal structure is that of a punctured disk, say $\Omega = \{z : 0 < |z| < 1\}$, and let $F, G$ be the holomorphic data of the conformal representation. As was proved in [FMM2], it must hold on $\mathbb{D}^*$ that $dx^2 + dy^2 \leq |dG|^2$. Hence $G$ has at most a pole at 0, and $dG(0) \neq 0$. In addition, as $|dF/dG| < 1$ on $\mathbb{D}^*$, we conclude that $F$ has at most a pole at 0. But now, if $F$ or $G$ had a pole at 0, the same would happen to $G + F$ or $G - F$. This is not possible, by (2.1). So $F, G$ extend holomorphically to 0. Finally $|dF/dG|(0) \neq 1$, because $|dF/dG| < 1$ on $\mathbb{D}^*$. Therefore, the conformal representation assures that the parabolic affine sphere extends regularly across 0, i.e. $(x_0, y_0)$ is a removable singularity of $\phi(x, y)$.

**Remark 6** Lemma 5 indicates that if $\phi(x, y)$ is a solution to (1.1) in a punctured neighbourhood of a non-removable isolated singularity $(x_0, y_0)$, it has the underlying conformal structure of an annulus $A_r = \{z : 1 < |z| < r\}$. Moreover, it is deduced from the conformal representation that $(G + F)|_{S^1} = x_0 + iy_0$, where $S^1 = \{z : |z| = 1\}$. So, both $G$ and $F$ can be analytically continued across $S^1$ by Schwarzian reflection. In particular, $\phi(x, y)$ extends continuously to $(x_0, y_0)$, although it is not $C^1$ at this point.

**Proof of Theorem 3:** Let $\phi(x, y): \mathbb{R}^2 \setminus \{p_1, \ldots, p_n\} \to \mathbb{R}$ be a global solution to (1.1), and let $\Sigma$ denote its underlying Riemann surface structure. By Lemma 5, the underlying
conformal type of \( \phi(x, y) \) about any of the isolated singularities is that of an annulus. Moreover, it was proved in \( \text{[FMM2]} \) that the conformal type of \( \phi(x, y) \) in the exterior of a sufficiently large disk \( x^2 + y^2 > R^2 \) is that of a punctured disk. Putting together these conditions and applying Koebe’s uniformization theorem we deduce that \( \Sigma \) is conformally equivalent to \( \mathbb{C} \) with \( n \) disjoint disks removed. We shall denote by \( \zeta \) the complex parameter of this domain. At last, by reflecting this domain with respect to the circle \( C \) that bounds one of these disks, \( \Sigma \) can be assumed to be a once punctured bounded domain \( \Omega \setminus \{z_0\} \subset \mathbb{C} \) whose boundary is made up by \( n \) disjoint circles, the exterior one being precisely \( C \).

Once here, by the representation formula we obtain that the parabolic affine sphere \( \mathcal{S}_\phi = (x, y, \phi(x, y)) \) can be conformally parametrized as \( \psi: \Omega \setminus \{z_0\} \rightarrow \mathbb{R}^3 \), where \( \psi \) is given by (2.1) in terms of two holomorphic functions \( F, G: \Omega \setminus \{z_0\} \rightarrow \mathbb{C} \). Moreover, the map \( G + \overline{F}: \Omega \setminus \{z_0\} \rightarrow \mathbb{R}^2 \setminus \{p_1, \ldots, p_n\} \) is a global diffeomorphism, \( G + \overline{F} \) is constant along each boundary circle in \( \overline{\Omega} \), and \( G \) has a pole of order one at \( z_0 \) \( \text{[FMM2]} \).

We recall at this point another result in \( \text{[FMM2]} \), which describes the asymptotic behaviour of parabolic affine spheres at infinity. This result tells in our situation that the solution \( \phi(x, y) \) to (1.1) we started with has the decomposition

\[
\phi(x, y) = E_\phi(x, y) + \alpha \log |\zeta|^2 + O(1).
\]  

Here \( \zeta \) is the conformal parameter of the above mentioned complex domain consisting of \( \mathbb{C} \) with \( n \) disks removed, \( \alpha \in \mathbb{R} \), \( O(1) \) stands for a term bounded in absolute value by a constant, and \( E_\phi(x, y) \) is a quadratic polynomial. When \( k > 0 \) is a large positive number, the ellipse \( E_\phi(x, y) = k \) describes the shape of \( \phi(x, y) \) at infinity, and is called the ellipse at infinity of \( \phi(x, y) \).

Now let \( \Phi \in \mathcal{E} \) be an equiaffine transformation as in (1.2), and let \( \phi'(x', y') \) be the solution to (1.1) associated to the embedded parabolic affine sphere \( \mathcal{S}' = \Phi \circ \mathcal{S}_\phi \). Then \( \phi'(x', y'): \mathbb{R}^2 \setminus \{q_1, \ldots, q_n\} \rightarrow \mathbb{R} \) is a global solution to (1.1), where the isolated singularities \( q_1, \ldots, q_n \) are in general different from \( p_1, \ldots, p_n \). Thus, \( \phi'(x', y') \) has all the properties established for \( \phi(x, y) \), and it becomes clear from the decomposition (3.1) that its ellipse at infinity differs from the one of \( \phi \) only by an affine transformation of \( \mathbb{R}^2 \). With this, it is easy to choose an adequate equiaffine transformation \( \Phi \) so that the ellipse at infinity of \( \phi'(x', y') \) is actually a circle.

Consider next the holomorphic data of the embedded parabolic affine sphere \( \mathcal{S}' \), denoted by \( G^*, F^* \), and defined on \( \Omega \setminus \{z_0\} \). Then \( G^* \) has a simple pole at \( z_0 \) and, as the ellipse at infinity of \( \mathcal{S}' \) is a circle, \( F^* \) extends holomorphically about \( z_0 \) (see [FMM2]).

Furthermore, by performing an adequate dilatation and rotation of \( \Omega \) if necessary, we may assume that the residue of \( G^* \) at \( z_0 \) is 2.

With all of this we get that

\[
\psi_1 = \frac{1}{2} \text{Re} (G^* + F^*): \Omega \setminus \{z_0\} \rightarrow \mathbb{R}
\]

is harmonic, constant on each boundary component of \( \Omega \), and \( (G^* + F^*)/2 \) has a simple pole of residue one at \( z_0 \). Thus, by uniqueness of the holomorphic map \( p(z) \) defined
in Section 2 we infer that $G^*(z) + F^*(z) = 2p(z) + c_1$, where $c_1 \in \mathbb{C}$. Analogously, $G^*(z) - F^*(z) = 2q(z) + c_2$ for $c_2 \in \mathbb{C}$. Finally we conclude that

$$G^*(z) = p(z) + q(z) + d_1, \quad F^*(z) = p(z) - q(z) + d_2,$$

for constants $d_1, d_2 \in \mathbb{C}$. This lets us conclude that the global solution $\phi'(x', y')$, and thus the original global solution $\phi(x, y)$, differs from one of the examples in Theorem 2 only by an equiaffine transformation. This proves the first claim.

The second claim is essentially direct. We assign to a specific once punctured complex domain in $\mathbb{C}$ the example constructed in Theorem 2. As changing the domain within its conformal equivalence class does not change the surface, we have a well defined mapping from conformal equivalence classes of once punctured domains with $n$ boundary curves into global solutions to (1.1) in the $n$-times punctured plane, modulo equiaffine transformations. This mapping is surjective by the first part of the theorem, and injective because solutions to (1.1) differing just by an equiaffine transformation have the same underlying conformal structure. This ends up the proof.

\[ \square \]

**Remark 7** The two classical Jörgens theorems indicate that there is exactly one (up to equiaffine transformations) global solution to (1.1) in the whole plane $\mathbb{R}^2$, and in the once punctured plane. This does not hold anymore when we consider the $n > 1$ times punctured plane, as shown by Corollary 4. In any case, given $n > 1$ and a conformal equivalence class of $\mathbb{C}$ with $n$ disjoint disks removed, there exists a unique (up to equiaffine transformations) global solution to (1.1) in an $n$-times punctured plane whose underlying conformal structure is the previously given one. So, unicity in the $n = 1$ case comes from the fact that all domains of the form $\mathbb{C} \setminus D$, where $D$ is a disk in $\mathbb{C}$, are conformally equivalent.

Theorem 3 can also be stated in the geometric context of parabolic affine spheres. Indeed, if a parabolic affine sphere with a finite number of singularities is the boundary of a convex body then it must be regular at infinity (see [FMM2]), and reasoning as in Theorem 2 we see that it must be an entire graph. So, we have:

**Corollary 8** All embedded parabolic affine spheres in $\mathbb{R}^3$ that have a finite number of isolated singularities and are the boundary of a convex body of $\mathbb{R}^3$ must be equiaffinely equivalent to one of the examples in Theorem 2.

Actually, the condition of being the boundary of a convex body can be substituted by the property of being closed in $\mathbb{R}^3$. Without going into detail, we just indicate that this fact can be proved as follows: given a closed parabolic affine sphere $S$, its Legendre transform surface must lie on the boundary of a convex body, and so $S$ must be globally convex. Thus, Corollary 8 can be applied.
4 Explicit construction for one and two punctures

To begin this last Section, we show that Theorem 3 for $n = 1$ recovers the classical description by Jörgens of the global solutions to (1.1) in the once punctured plane. To do so, we just need to analyze the example with $n = 1$ in Theorem 2. Obviously, we may assume there that $\Omega = \mathbb{D}$, the unit circle, and $z_0 = 0$.

Let $z_0 \in \mathbb{D}$. The Green function in $\mathbb{D}$ with respect to $z_0$ is then

$$G(z, z_0) = \log \left| \frac{1 - zz_0}{z - z_0} \right|.$$  

Deriving this expression with respect to $z_0$ as in (2.2), (2.4) we get the harmonic functions

$$u(z, z_0) = \Re \left( \frac{1}{z - z_0} - \frac{z}{1 - zz_0} \right), \quad v(z, z_0) = \Im \left( \frac{1}{z - z_0} + \frac{z}{1 - zz_0} \right).$$

Therefore, letting $z_0 = 0$ we see that the canonical functions $p(z), q(z)$ in $\mathbb{D}^*$ are

$$p(z) = \frac{1}{z} - z, \quad q(z) = \frac{1}{z} + z.$$  

Finally, by (2.5) the holomorphic data of the solution to (1.1) are $G(z) = 2/z$ and $F(z) = -2z$. These are known to be the holomorphic data (up to the conformal change $w = 2/z$) of the rotational example in the introduction (see [FMM1]). Also observe that $(G + F)|_{S^1} = 0$. Hence, Jörgens’ theorem is recovered.

![Figure 1: The unique solution to the Monge-Ampère equation in the once punctured plane, modulo equiaffine transformations.](image)

It is also possible to write down explicitly in terms of theta functions all global solutions to (1.1) in a twice punctured plane $\mathbb{R}^2 \setminus \{p_1, p_2\}$, as we show next.

Let $\theta_0(w)$ be the Jacobi theta function (see [CoHi] for instance)

$$\theta_0(w) = C \prod_{k=1}^{\infty} \left( 1 - r^{2k-1} e^{2\pi i w} \right) \left( 1 - r^{2k-1} e^{-2\pi i w} \right),$$

where $C$ is a normalization constant and $0 < r < 1$. The theta function $\theta_0(w)$ satisfies the following properties:

- **Periodicity:** $\theta_0(w + 2\pi i) = e^{i \pi w} \theta_0(w)$.
- **Quasi-modularity:** $\theta_0(w) = e^{i \pi} \theta_0(w + 1)$.
- **Multiplicative property:** $\theta_0(w) = e^{i \pi \frac{w^2}{2}} \theta_0(2w)$.

These properties make $\theta_0(w)$ a powerful tool in the study of holomorphic functions and their applications in complex analysis.
where \( C = \prod_{k=1}^{\infty} (1 - r^{2k}) \) and \( 0 < r < 1 \). Consider also \( \mathcal{A}_r \) to be the annulus given by \( \mathcal{A}_r = \{ z : r < |z| < 1 \} \), and let \( \vartheta_1 (z) \) denote the annular Jacobi theta function given by \( \vartheta_1 (e^{2\pi iw}) = \theta_0 (rw) \). In other words,

\[
\vartheta_1 (z) = C \left( 1 - \frac{1}{z} \right) \prod_{k=1}^{\infty} \left( 1 - r^{2k}z \right) \left( 1 - r^{2k}/z \right).
\]

(4.1)

It comes then clear from this expression that

\[
\vartheta_1 (z) = \vartheta_1 (\bar{z}) = -r^2 z \vartheta_1 (r^2 z), \quad \vartheta_1 (z/r^2) = -z \vartheta_1 (z),
\]

(4.2)

and thus that

\[
\vartheta_1' (z) = -r^2 \vartheta_1 (r^2 z) - r^4 z \vartheta_1' (r^2 z), \quad \vartheta_1' (z/r^2) = -r^2 \vartheta_1 (z) - r^2 z \vartheta_1' (z).
\]

(4.3)

A formula for the Green function of an annulus of the type \( \{ r^{1/2} < |z| < r^{-1/2} \} \) for \( r \in (0, 1) \) can be found in [CoHi, pg. 386-387]. By the conformal invariance property of the Green function, we obtain from this formula that the Green function \( G(z, z_0) \) of \( \mathcal{A}_r \) with respect to a point \( z_0 \in \mathcal{A}_r \) can be obtained in terms of \( \vartheta_1 \) as

\[
G(z, z_0) = \log |z| \left( 1 + \frac{\log |z_0|}{\log r} \right) - \log \left| \frac{\vartheta_1(z_0/z)}{\vartheta_1(z/z_0)} \right|.
\]

In other words,

\[
G(z, z_0) = \text{Re} \ h(z) = \text{Re} \left\{ \log \left( \frac{z^{1+\log |z_0|}/\log r}{\vartheta_1(z_0/z)/\vartheta_1(z/z_0)} \right) \right\}.
\]
At this point we can find the harmonic function in (2.2) by differentiation. We obtain

\[ u(z) = \text{Re} \left\{ \log \frac{z}{\log r \left( \frac{1}{z_0} + \frac{1}{z_0} \right)} \right\} \text{Re} \left\{ \frac{\vartheta_1'(z_0/z)}{z\vartheta_1(z_0/z)} \frac{\vartheta_1'(z_0z)}{\vartheta_1(z_0z)} \right\}. \]

Noting that

\[ \frac{\vartheta_1'(z_0/z)}{z\vartheta_1(z_0/z)} - \frac{\vartheta_1'(z_0z)}{\vartheta_1(z_0z)} \]

is a well defined meromorphic function on \( \mathbb{A}_r \), the way we defined the holomorphic function \( p(z) \) in (2.3) suggests that

\[ p(z) = \frac{\vartheta_1'(z_0z)z}{\vartheta_1(z_0z)} - \frac{\vartheta_1'(z_0/z)}{z\vartheta_1(z_0/z)}. \] (4.4)

The validity of this assertion is easily checked: \( p(z) \) is holomorphic on \( \mathbb{A}_r \setminus \{z_0\} \), it has a simple pole of residue 1 at \( z_0 \), and using (4.2) and (4.3) we see that

\[ p(1/\bar{z}) + p(z) = 0, \quad p(r^2/\bar{z}) + p(z) = -\frac{1}{z_0} - \frac{1}{\bar{z}_0}. \] (4.5)

So, the real part of \( p \) is constant on the boundary curves \( |z| = r, 1 \).

The corresponding process starting this time with the harmonic function defined via (2.4) lets us recover the other canonical holomorphic function \( q(z) : \mathbb{A}_r \setminus \{z_0\} \to \mathbb{C} \) as

\[ q(z) = -\frac{\vartheta_1'(z_0/z)}{z\vartheta_1(z_0/z)} - \frac{\vartheta_1'(z_0z)}{\vartheta_1(z_0z)}. \] (4.6)

Putting together (4.4) and (4.6), and recalling the formula (2.5), we obtain the holomorphic data on \( \mathbb{A}_r \setminus \{z_0\} \)

\[ G(z) = -\frac{2\vartheta_1'(z_0/z)}{z\vartheta_1(z_0/z)}, \quad F(z) = \frac{2\vartheta_1'(z_0z)}{\vartheta_1(z_0z)}. \] (4.7)

Finally, by Theorem 3, all solutions to (1.1) in a twice punctured plane are recovered, up to equiaffine transformations, by the holomorphic data in (4.7). The 2-parametric family of such solutions described by Corollary 4 is given by the variation of \( r \in (0, 1) \) and \( |z_0| \in (r, 1) \). In the solution to (1.1) we have just constructed, the singularities are located at the points \((0, 0)\) and \((\text{Re}(-2/\bar{z}_0), \text{Im}(-2/\bar{z}_0))\) of \( \mathbb{R}^2 \). This follows from (2.1) and the relation (4.5).

References


