
Embedded isolated singularities of flat surfaces in hyperbolic 3-space

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Abstract

We give a complete description of the flat surfaces in hyperbolic 3-space that are regularly embedded around an isolated singularity. Specifically, we show that there is a one-to-one explicit correspondence between this class and the class of regular analytic convex Jordan curves in the 2-sphere. Previously, the only known examples of such surfaces were rotational ones. To achieve this result, we first solve the geometric Cauchy problem for flat surfaces in hyperbolic 3-space.

1 Introduction

The study of flat surfaces in the hyperbolic 3-space \mathbb{H}^3 is a classical subject that traces back to the beginning of the 20th century, but which has been overlooked for a long time, mainly because the only complete examples of the theory are horospheres and hyperbolic cylinders. However, this situation has changed in the last few years, since the discovery in [GMM1] of a Weierstrass representation for the flat surfaces of \mathbb{H}^3 , like the classic one for minimal surfaces in \mathbb{R}^3 .

The just mentioned lack of complete examples has contributed to the recent study of singularities of flat surfaces in \mathbb{H}^3 [KRSUY, KUY, Roi]. This study has revealed a particular kind of singularity that does not take place in minimal surface theory, and which is very special even among flat surfaces: the isolated singularities around which

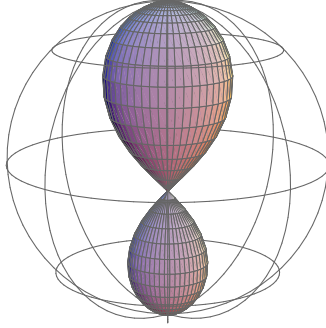


Figure 1: A rotational flat surface in the Poincaré model of \mathbb{H}^3 with an embedded isolated singularity

the flat surface is regularly embedded. Actually, up to now the only known flat surfaces with this type of singularities are rotational ones (see Figure 1), and it has been wondered whether there are other examples or not.

The main objective of this paper is to give a general explicit description of all the flat surfaces in \mathbb{H}^3 that are embedded around an isolated singularity. This task is worthy and by no means trivial, as highlighted by the following facts:

- The main theorem in [KRSUY] shows that isolated singularities are much of a rare phenomenon among the possible singularities that a flat surface in \mathbb{H}^3 might have.
- The Weierstrass representation of flat surfaces is quite intricate, which makes it difficult to obtain from the Weierstrass data information about the embeddedness of the surface, or about when a curve of singular points degenerates into an isolated singularity.

The fundamental tool to obtain this structure theorem is the previous solution to the *geometric Cauchy problem* for flat surfaces in \mathbb{H}^3 , which consists of finding all flat surfaces in \mathbb{H}^3 that pass through a given curve in \mathbb{H}^3 with a given unit normal along it. We shall actually solve this problem for a wider class of flat surfaces with *admissible singularities*, called *flat fronts* ([KUY, KRSUY]).

The paper is organized as follows. In Section 2 we review the theory of flat surfaces (and flat fronts) in \mathbb{H}^3 , together with their associated Weierstrass representations in [GMM1] and [KUY], respectively. We shall prove there that any complete flat front with the topology of a cylinder and (not necessarily embedded) regular ends must be a finite-fold covering of a rotational one.

In Section 3 we prove the existence and uniqueness of the solution to the geometric Cauchy problem for flat fronts, and show how to construct such solution explicitly in

terms of the initial data. We also characterize which initial data span (regular) flat surfaces in \mathbb{H}^3 , and which do not.

In Section 4 we derive some basic applications of the solution to the Cauchy problem for flat fronts. These applications concern symmetries and period problems of flat surfaces, and the explicit construction of flat fronts from a regular curve in \mathbb{H}^3 of singular points of the front.

Finally, in Section 5 we obtain the main theorem of this work. This result asserts that the class of flat surfaces that have $a \in \mathbb{H}^3$ as an embedded isolated singularity admits an explicit one-to-one correspondence with the class of analytic regular convex Jordan curves in the 2-sphere \mathbb{S}^2 . In this description, the known rotational examples of Figure 1 correspond to circles of \mathbb{S}^2 . To prove this theorem we use the solution to the geometric Cauchy problem for flat fronts, as well as a careful study of the embeddedness condition for isolated singularities of convex surfaces in Euclidean 3-space.

We would like to thank Prof. Wayne Rossman for several useful comments, and particularly for suggesting the reflection property of flat fronts of Corollary 16.

2 The geometry of flat fronts in \mathbb{H}^3

A basic property of any flat surface in \mathbb{H}^3 is that its second fundamental form $d\sigma^2$ is definite, and so it inherits a canonical Riemann surface structure via $d\sigma^2$. This fact was used in [GMM1] to provide a conformal representation for flat surfaces in \mathbb{H}^3 , that recovers any such simply-connected surface in terms of holomorphic data. Later, this representation was extended by Kokubu, Umehara and Yamada [KUY] to the case of flat fronts (that is, flat surfaces admitting a special kind of singularities) in \mathbb{H}^3 . To explain these representation formulae we will consider the Hermitian model for \mathbb{H}^3 .

First, let us identify the Lorentz-Minkowski 4-space \mathbb{L}^4 with Lorentzian metric $\langle, \rangle = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$ with the set of 2×2 Hermitian matrices by means of

$$(x_0, x_1, x_2, x_3) \in \mathbb{L}^4 \longleftrightarrow \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \in \text{Herm}(2).$$

In this way $\langle m, m \rangle = -\det(m)$ for all $m \in \text{Herm}(2)$. Therefore the group $\mathbf{SL}(2, \mathbb{C})$ acts isometrically on \mathbb{L}^4 by $\Phi \cdot m = \Phi m \Phi^*$, with $\Phi \in \mathbf{SL}(2, \mathbb{C})$, $\Phi^* = \bar{\Phi}^t$, and $m \in \text{Herm}(2)$. In this model the hyperbolic 3-space $\mathbb{H}^3 = \{x \in \mathbb{L}^4 : \langle x, x \rangle = -1, x_0 > 0\}$ is

$$\mathbb{H}^3 = \{\Phi \Phi^* : \Phi \in \mathbf{SL}(2, \mathbb{C})\}.$$

Given three vectors $a, b, c \in \mathbb{L}^4$, their *cross product* $a \times b \times c \in \mathbb{L}^4$ is determined by the equality

$$\langle a \times b \times c, w \rangle = \det(a, b, c, w) \quad \text{for all } w \in \mathbb{L}^4.$$

Here \langle, \rangle denotes the Lorentzian scalar product in \mathbb{L}^4 , and \det is the usual determinant of the four vectors.

If $a = (a_0, a_1, a_2, a_3)$, $b = (b_0, b_1, b_2, b_3)$, $c = (c_0, c_1, c_2, c_3)$, then using a natural extension of the usual formalism for the cross product in \mathbb{R}^3 we have that $a \times b \times c$ is given in explicit coordinates by

$$a \times b \times c = \begin{vmatrix} -\vec{i} & \vec{j} & \vec{k} & \vec{l} \\ a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \end{vmatrix}.$$

Among several properties, $a \times b \times c$ is orthogonal in \mathbb{L}^4 to the space spanned by $\{a, b, c\}$. Moreover, the usual exterior product \wedge in \mathbb{H}^3 is related to \times as follows: if $p \in \mathbb{H}^3$ and $u, v \in T_p \mathbb{H}^3 \equiv \{p\}^\perp \subset \mathbb{L}^4$, then $u \wedge v = p \times u \times v$.

A holomorphic map $F : \Sigma \rightarrow \mathbf{SL}(2, \mathbb{C})$ from a Riemann surface Σ into $\mathbf{SL}(2, \mathbb{C})$ is called a *Legendrian curve* if

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \text{so that } DdA - BdC = 0.$$

Equivalently, F is a Legendrian curve if there are holomorphic 1-forms θ, ω on Σ so that

$$F^{-1}dF = \begin{pmatrix} 0 & \theta \\ \omega & 0 \end{pmatrix}.$$

On the other hand a *flat front* is a map $\psi : \Sigma \rightarrow \mathbb{H}^3$ from a surface Σ into $\mathbb{H}^3 \subset \mathbb{L}^4$ whose curvature at regular points vanishes, and for which there exists a map $\eta : \Sigma \rightarrow \mathbb{S}_1^3 = \{x \in \mathbb{L}^4 : \langle x, x \rangle = 1\}$ (called the *unit normal* of ψ) such that $\langle \psi, \eta \rangle = \langle d\psi, \eta \rangle = 0$ and, in addition, $\langle d\psi, d\psi \rangle + \langle d\eta, d\eta \rangle$ is non-degenerate at every point.

With this, we have

Theorem 1 ([GMM1, KUY]) *Let $F : \Sigma \rightarrow \mathbf{SL}(2, \mathbb{C})$ be a holomorphic Legendrian immersion. Then $\psi = FF^* : \Sigma \rightarrow \mathbb{H}^3$ is a flat front in \mathbb{H}^3 . If, in addition, $|\theta| \neq |\omega|$ at every point, ψ is a flat surface.*

Conversely, every simply-connected flat front and every simply-connected flat surface in \mathbb{H}^3 can be described in this way.

If we allow the presence of singular points of the Legendrian curve F , then those points are called *branch points*, and ψ is called a *branched flat front*. Besides, the unit normal of the flat front ψ is given by

$$\eta = F \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F^* : \Sigma \rightarrow \mathbb{S}_1^3.$$

Remark 2 *The definition of flat front in \mathbb{H}^3 has a degenerate situation, which was implicitly observed in [KUY]: any geodesic of \mathbb{H}^3 is a flat front!*

This fact can be seen analytically via Theorem 1: let $f(z) : \Sigma \rightarrow \mathbb{C} \setminus \{0\}$ be a regular holomorphic map, and consider the Legendrian immersion $F : \Sigma \rightarrow \mathbf{SL}(2, \mathbb{C})$

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} f & f \\ -1/f & 1/f \end{pmatrix}.$$

Then the image of $\psi = FF^* : \Sigma \rightarrow \mathbb{H}^3$ is a piece of the geodesic of \mathbb{H}^3 given by $x_1 = x_2 = 0$. Thus ψ parametrizes a piece of a geodesic in \mathbb{H}^3 , but it is also a flat front by Theorem 1.

Let \mathbb{N}^3 denote the positive light cone in \mathbb{L}^4 , which can be seen in $\text{Herm}(2)$ as $\mathbb{N}^3 = \{w\bar{w}^t : w \in \mathbb{C}^2 \setminus \{(0,0)\}\} \subset \text{Herm}(2)$. Then we can identify the quotient $\mathbb{N}^3/\mathbb{R}_+$ with the Riemann sphere $\mathbb{C} \cup \{\infty\}$ by means of $[(x_0, x_1, x_2, x_3)] \equiv (x_1 + ix_2)/(x_0 - x_3)$.

If $\psi : \Sigma \rightarrow \mathbb{H}^3$ is a flat front with unit normal $\eta : \Sigma \rightarrow \mathbb{S}_1^3$, then the maps $G, G_* : \Sigma \rightarrow \mathbb{C} \cup \{\infty\}$ given by $G = [\psi + \eta]$ and $G_* = [\psi - \eta]$ are meromorphic functions [GMM1], called the *hyperbolic Gauss maps* of ψ . Then, it was proved in [KUY] that every flat front in \mathbb{H}^3 is expressed in terms of its hyperbolic Gauss maps as $\psi = FF^*$, where $F : \Sigma \rightarrow \mathbf{SL}(2, \mathbb{C})$ is

$$F = \begin{pmatrix} G/\xi & \xi G_*/(G - G_*) \\ 1/\xi & \xi/(G - G_*) \end{pmatrix}, \quad \xi = c \exp \left(\int_{z_0}^z \frac{dG}{G - G_*} \right), \quad c \in \mathbb{C} \setminus \{0\}. \quad (2.1)$$

A flat front ψ in \mathbb{H}^3 is *complete* if there is a symmetric tensor of compact support on Σ such that $T + ds^2$ is a complete Riemannian metric, ds^2 being the first fundamental form of ψ . It was shown in [KUY] that if $\psi : \Sigma \rightarrow \mathbb{H}^3$ is a complete flat front, then Σ is conformally equivalent to a finitely punctured compact Riemann surface, $\Sigma \equiv \bar{\Sigma} \setminus \{p_1, \dots, p_r\}$. The points p_j are called the *ends* of ψ , and any such end is called *regular* if G and G_* extend meromorphically across it.

If a complete flat front has two ends, both of them regular and embedded, then it has genus zero [KUY]. As a consequence, it was also proved in [KUY] that any such front is a rotational (embedded) flat front. We conclude the present section by generalizing this result in the following way.

Theorem 3 *Any complete flat front in \mathbb{H}^3 of genus zero and two regular ends is a finite-fold covering of a rotational flat front.*

Proof: Let $\psi : \Sigma \rightarrow \mathbb{H}^3$ be a complete flat front of genus zero and two regular ends. Then $\Sigma \equiv \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and the holomorphic 1-forms ω, θ associated to ψ , which are multivalued on \mathbb{C}^* , satisfy that $|\omega|, |\theta|$ are well defined on \mathbb{C}^* .

Consider the covering map $\exp(z) : \mathbb{C} \rightarrow \mathbb{C}^*$. Then, arguing as in [GMM1, Lemma2] we get the existence of two holomorphic 1-forms ω_1, θ_1 that are single-valued on \mathbb{C}^* , and of two real numbers $\lambda, \mu \in \mathbb{R}$, such that

$$\theta = \zeta^\lambda \theta_1, \quad \omega = \zeta^\mu \omega_1. \quad (2.2)$$

Moreover, by [GMM1, Lemma2] and the completeness of the front, both ω_1, θ_1 extend meromorphically to the ends $0, \infty$ of \mathbb{C}^* .

In addition, the Hopf differential $Q = \omega \theta$ is well defined on \mathbb{C}^* and holomorphic, and as the ends are regular, it has at most a pole of order two at them [GMM1]. So,

$$Q = \frac{a}{\zeta^2} d\zeta^2, \quad a \in \mathbb{C}. \quad (2.3)$$

If $a = 0$, then $Q = 0$ and the front is totally umbilic, i.e. it is a horosphere. But this is impossible, since the flat front has two ends. So $a \neq 0$.

In this way none of ω_1 and θ_1 vanishes on \mathbb{C}^* . Putting together this condition with the fact that ω_1, θ_1 extend meromorphically to the ends we conclude from (2.2) that

$$\theta = c \zeta^\alpha d\zeta, \quad \omega = d \zeta^\delta d\zeta,$$

where $c, d \in \mathbb{C}$ and $\alpha, \delta \in \mathbb{R}$. Finally, by (2.3),

$$\theta = c \zeta^\alpha d\zeta, \quad \omega = \frac{a}{c} \zeta^{-2-\alpha} d\zeta.$$

These are the data of a finite-fold covering of a rotational flat front (see [GMM1, KUY]), and we are done. □

3 The geometric Cauchy problem for flat fronts

The theory of flat surfaces (and flat fronts) in \mathbb{H}^3 is connected with the surfaces in \mathbb{H}^3 with mean curvature one, for which a conformal representation due to R.L. Bryant [Bry] is available (see also [GMM2]). In [GaMi] the authors studied an initial value problem, the *geometric Cauchy problem*, for the class of mean curvature one surfaces in \mathbb{H}^3 (see also [ACM]). Here, we introduce the *geometric Cauchy problem* for flat fronts in \mathbb{H}^3 . From now on, I will denote an open real interval.

Let $\beta : I \rightarrow \mathbb{H}^3$ and $V : I \rightarrow \mathbb{S}_1^3$ be two analytic maps such that $\langle \beta, V \rangle \equiv \langle \beta', V \rangle \equiv 0$. Find all flat fronts $\psi : \Sigma \rightarrow \mathbb{H}^3$ for which there is a regular analytic curve $\Gamma \subset \Sigma$ so that $\psi(\Gamma) = \beta$ and the unit normal of ψ along Γ is given by V .

Any pair of maps β, V in the above conditions will be called a pair of *initial data*. The same problem can be formulated for flat surfaces, but this time asking β to be a regular curve.

Theorem 4 *Let $\beta(s), V(s)$ be a pair of initial data such that $\beta'(s), V'(s)$ do not vanish simultaneously at any point. There exists a unique solution to the geometric Cauchy problem for flat fronts in \mathbb{H}^3 with initial data β, V . This solution can be explicitly constructed in a neighbourhood of β as $\psi = FF^* : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{H}^3$, where $F : \Omega \subseteq \mathbb{C} \rightarrow \mathbf{SL}(2, \mathbb{C})$ is given by (2.1) for*

$$G(z) = \frac{\mathcal{N}_1(z) + i\mathcal{N}_2(z)}{\mathcal{N}_0(z) - \mathcal{N}_3(z)}, \quad G_*(z) = \frac{\mathcal{L}_1(z) + i\mathcal{L}_2(z)}{\mathcal{L}_0(z) - \mathcal{L}_3(z)}, \quad c = \frac{\sqrt{2}}{\sqrt{\mathcal{N}_0(s_0) - \mathcal{N}_3(s_0)}}. \quad (3.1)$$

Here $s_0 \in I$ is fixed but arbitrary satisfying $\mathcal{N}_0(s_0) \neq \mathcal{N}_3(s_0)$ and $\mathcal{N}(z), \mathcal{L}(z)$ are meromorphic extensions of $\mathcal{N}(s) = \beta(s) + V(s)$ and $\mathcal{L}(s) = \beta(s) - V(s)$.

Proof: We start with uniqueness. Let $\psi : \Sigma \rightarrow \mathbb{H}^3$ be a flat front that solves the geometric Cauchy problem for the initial data β, V . Thus there exists a regular analytic curve $\Gamma \subset \Sigma$ with $\psi(\Gamma(s)) = \beta(s)$ and $\eta(\Gamma(s)) = V(s)$, where here $\eta : \Sigma \rightarrow \mathbb{S}_1^3$ is the unit normal to ψ . Fix $s_0 \in I$. Then we can parametrize a piece of the flat front as $\psi : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{H}^3$ so that a certain real interval I containing s_0 lies in Ω , and $\psi(s, 0) = \beta(s)$ and $\eta(s, 0) = V(s)$ for all $s \in I$.

Denote now $\mathcal{N}(s) = \beta(s) + V(s)$ and $\mathcal{L}(s) = \beta(s) - V(s)$, and suppose for clarity that $\mathcal{N}_0(s_0) \neq \mathcal{N}_3(s_0)$. Then the hyperbolic Gauss maps G, G_* of ψ are described along I by

$$G(s, 0) = \frac{\mathcal{N}_1(s) + i\mathcal{N}_2(s)}{\mathcal{N}_0(s) - \mathcal{N}_3(s)}, \quad G_*(s, 0) = \frac{\mathcal{L}_1(s) + i\mathcal{L}_2(s)}{\mathcal{L}_0(s) - \mathcal{L}_3(s)}, \quad (3.2)$$

and so, by meromorphic extension, G, G_* are as in (3.1). Therefore, the flat front ψ is recovered in terms of G, G_* and a certain $c \in \mathbb{C} \setminus \{0\}$ as $\psi = FF^*$, where F is the holomorphic Legendrian curve in (2.1). In addition, its unit normal η is described as

$$\eta = F \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F^*. \quad (3.3)$$

Finally, since

$$\psi + \eta = 2 F \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} F^* = \frac{2}{|\xi|^2} \begin{pmatrix} |G|^2 & G \\ \bar{G} & 1 \end{pmatrix}, \quad (3.4)$$

by evaluating this expression in s_0 we infer that

$$|c|^2 = \frac{2}{\mathcal{N}_0(s_0) - \mathcal{N}_3(s_0)}.$$

As the change $c \mapsto e^{i\theta}c$ for $\theta \in \mathbb{R}$ has no effect on the value of FF^* , we conclude that c can be chosen as in (3.1). Therefore, since this can be done for every $s_0 \in I$, the value of the flat front ψ is completely determined by β, V in a neighbourhood of Γ in Σ , and since flat fronts in \mathbb{H}^3 are analytic, we obtain the uniqueness part.

For existence, we start with initial data β, V , and define $\mathcal{N}(s) = \beta(s) + V(s)$ and $\mathcal{L}(s) = \beta(s) - V(s)$, as well as $G(s)$ and $G_*(s)$ by means of (3.2). Let now be $s_0 \in I$ fixed but arbitrary with $\mathcal{N}_0(s_0) \neq \mathcal{N}_3(s_0)$, and define

$$\xi(s) = c \exp \left(\int_{s_0}^s \frac{G'(r)}{G(r) - G_*(r)} dr \right), \quad c = \frac{\sqrt{2}}{\sqrt{\mathcal{N}_0(s_0) - \mathcal{N}_3(s_0)}}. \quad (3.5)$$

Then we can consider the map $F(s) : I \rightarrow \mathbf{SL}(2, \mathbb{C})$ given by

$$F(s) = \begin{pmatrix} G(s)/\xi(s) & \xi(s)G_*(s)/(G(s) - G_*(s)) \\ 1/\xi(s) & \xi(s)/(G(s) - G_*(s)) \end{pmatrix}.$$

Even though at a first sight this map might take infinite value at some point, we shall show later that this cannot happen. So there exists a holomorphic extension $F : \Omega \subseteq$

$\mathbb{C} \rightarrow \mathbf{SL}(2, \mathbb{C})$ of $F(s)$ over an open set Ω containing I , and we can thus define the map $\psi = FF^* : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{H}^3$. The shape of F indicates that it is a holomorphic Legendrian curve in $\mathbf{SL}(2, \mathbb{C})$. Therefore, ψ is a flat front (possibly with branch points), with unit normal $\eta : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{S}_1^3$ given by (3.3).

Let us check that $\psi(s, 0) = \beta(s)$ and $\eta(s, 0) = V(s)$. This is equivalent to showing that $(\psi + \eta)(s, 0) = \mathcal{N}(s)$ and $(\psi - \eta)(s, 0) = \mathcal{L}(s)$, and will indicate that $F(s)$ takes finite values at all points.

To see this, we note that the metric conditions on β, V imply that

$$\langle \mathcal{N}, \mathcal{N} \rangle = \langle \mathcal{L}, \mathcal{L} \rangle = 0, \quad \langle \mathcal{N}, \mathcal{L} \rangle = -2, \quad \langle \mathcal{N}', \mathcal{L} \rangle = \langle \mathcal{N}', \mathcal{N} \rangle = 0.$$

Now, by means of these relations, a long but direct computation shows that

$$\operatorname{Re} \left(\frac{G'}{G - G_*} \right) = \frac{-2(\mathcal{L}_0 - \mathcal{L}_3)(\mathcal{N}_0 - \mathcal{N}_3)^2(\mathcal{N}'_0 - \mathcal{N}'_3)}{4(\mathcal{N}_0 - \mathcal{N}_3)^3(\mathcal{L}_0 - \mathcal{L}_3)} = -\frac{\mathcal{N}'_0 - \mathcal{N}'_3}{2(\mathcal{N}_0 - \mathcal{N}_3)},$$

where we have suppressed the s parameter for clarity. This indicates that

$$\left| \exp \left(\int_{s_0}^s \frac{G'(r)}{G(r) - G_*(r)} dr \right) \right|^2 = \frac{\mathcal{N}_0(s_0) - \mathcal{N}_3(s_0)}{\mathcal{N}_0(s) - \mathcal{N}_3(s)}.$$

Therefore, by (3.5) we see that

$$|\xi|^2(s) = \frac{2}{\mathcal{N}_0(s) - \mathcal{N}_3(s)}.$$

Now, it is immediate that

$$(\psi + \eta)(s, 0) = 2 F \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} F^* = (\mathcal{N}_0 - \mathcal{N}_3) \begin{pmatrix} |G|^2 & G \\ \bar{G} & 1 \end{pmatrix} = \mathcal{N}(s).$$

In the same way

$$(\psi - \eta)(s, 0) = 2 F \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} F^* = \frac{4}{(\mathcal{N}_0(s) - \mathcal{N}_3(s))|G - G_*|^2} \begin{pmatrix} |G_*|^2 & G_* \\ \bar{G}_* & 1 \end{pmatrix} = \mathcal{L}(s),$$

where we have used the general relation

$$|G(s) - G_*(s)|^2 = \frac{4}{(\mathcal{N}_0(s) - \mathcal{N}_3(s))(\mathcal{L}_0(s) - \mathcal{L}_3(s))}.$$

Thus, the branched flat front ψ satisfies the desired initial conditions.

To end up the proof, we just need to check that ψ does not have branch points along I , since in that case by shrinking Ω we can assure that ψ is a flat front free of branch points. So, suppose that $s_0 \in I$ is a branch point of ψ . Then $F'(s_0) = 0$, and by $\beta(s) = F(s)F(s)^*$ and (3.3) we get that $\beta'(s_0) = V'(s_0) = 0$, which is impossible by the conditions imposed to the initial data. Let us note additionally that if we would have let $\beta'(s_0) = V'(s_0) = 0$ for some point, then $G'(s_0) = G'_*(s_0) = 0$ and the solution to the geometric Cauchy problem for the data β, V would have had a branch point at s_0 (see [KUY]).

□

Remark 5 *Let β, V be a pair of initial data, such that $\beta'(s), V'(s)$ do not vanish simultaneously at any point. Then, the flat front that β, V generate via Theorem 4 coincides with the one generated by $\beta, -V$. This happens because, by the definition of flat front in \mathbb{H}^3 , the unit normal of a flat front is defined up to sign.*

To close this section, we treat the geometric Cauchy problem for the regular case of flat surfaces in \mathbb{H}^3 .

Let $\psi : \Sigma \rightarrow \mathbb{H}^3$ be a flat front with unit normal η , and let z be a complex parameter on Σ . Then $z_0 \in \Sigma$ is a singular point of Σ if and only if $\langle \psi_z, \eta_{\bar{z}} \rangle = 0$ at z_0 (see [KUY, GMM1]). That is, if $z = s + it$, then z_0 is a singular point if and only if $\langle \psi_s, \eta_s \rangle = 0$. So we see that *a necessary and sufficient condition for the regularity of the solution to the geometric Cauchy problem for flat fronts with initial data β, V in a neighbourhood of I is that $\langle \beta'(s), V'(s) \rangle \neq 0$ for all $s \in I$.* In other words,

Corollary 6 *Let $\beta : I \rightarrow \mathbb{H}^3$ be a regular analytic curve and $V : I \rightarrow \mathbb{S}_1^3$ be an analytic map with $\langle \beta, V \rangle = \langle \beta', V \rangle = 0$. There exists a flat surface in \mathbb{H}^3 that contains β and whose unit normal along β is V if and only if $\langle \beta'(s), V'(s) \rangle \neq 0$ for all $s \in I$. If this condition is fulfilled, the flat surface is unique, and can be explicitly constructed as in Theorem 4.*

From these comments we may also conclude that *a regular curve $\beta(s)$ of a flat front in \mathbb{H}^3 is a curve of singularities of this flat front if and only if $\langle \beta'(s), V'(s) \rangle = 0$ for all s , where here $V(s)$ is the unit normal of the flat front along $\beta(s)$.*

4 Some applications

Let Φ be a rigid motion of \mathbb{H}^3 , and let β, V be initial data. We say that Φ is a *symmetry* of the data β, V if there is an analytic diffeomorphism $\Psi : I \rightarrow I$ such that $\beta \circ \Psi = \Phi \circ \beta$ and $V \circ \Psi = d\Phi \circ V$. Then Theorem 4 has the following two consequences:

Theorem 7 (Generalized symmetry principle) *Any symmetry in the initial data of the geometric Cauchy problem for flat fronts induces a global symmetry of the flat front they generate.*

Corollary 8 *If the initial data $\beta(s), V(s)$ are T -periodic for some $T > 0$, the flat front $\psi : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{H}^3$ they generate via Theorem 4 has in a neighbourhood of the real line the topology of a cylinder, namely, it can be defined on $\Omega/(T\mathbb{Z})$.*

Conversely, every flat front in \mathbb{H}^3 with the topology of a cylinder is constructed in this way.

The proofs of these results are analogous to the corresponding ones obtained in [GaMi], and so we omit them. Another direct consequence of the generalized symmetry principle is the following result, obtained simultaneously in [Mir] and [Roi] with two different approaches:

Corollary 9 (Reflection principle) *If a flat surface in \mathbb{H}^3 intersects orthogonally a hyperbolic plane in \mathbb{H}^3 , then it is symmetric with respect to that plane.*

Now, we turn our attention to singularities of flat fronts. As was proved in [KRSUY], the singularities of a flat front are generically distributed along regular curves in \mathbb{H}^3 . So it is natural to adapt the formulation of the geometric Cauchy problem for flat fronts to the specific situation in which we ask a regular curve in \mathbb{H}^3 to be a curve of singular points of the flat front. In this direction, we get the following direct consequence of Theorem 4, which generalizes a result in [Roi].

Corollary 10 *Let $\beta(s)$ be a regular analytic curve in \mathbb{H}^3 . There is a unique flat front in \mathbb{H}^3 that contains $\beta(s)$ as a curve of singularities, and it can be explicitly constructed in terms of $\beta(s)$.*

Proof: First, suppose that $\beta(s)$ is not a geodesic of \mathbb{H}^3 , and choose a piece of β so that the vectors $\beta(s), \beta'(s), \beta''(s)$ are linearly independent at every point. Let also $V(s)$ be the curve in \mathbb{S}_1^3 such that $\langle \beta, V \rangle = \langle \beta', V \rangle = 0$ and in addition $\langle \beta', V' \rangle = 0$. Then, recalling the definition of cross product in \mathbb{L}^4 , we have

$$V(s) = \pm \frac{\beta(s) \times \beta'(s) \times \beta''(s)}{\|\beta(s) \times \beta'(s) \times \beta''(s)\|}, \quad (4.1)$$

where $\|v\| := +\sqrt{|\langle v, v \rangle|}$ for any $v \in \mathbb{L}^4$.

It follows from the last paragraph of Section 3 that if a flat front ψ in \mathbb{H}^3 contains β as a curve of singularities, the unit normal of ψ along $\beta(s)$ must be then given by (4.1). But since the choice of sign in (4.1) only means a change of orientation in ψ (see Remark 5), we conclude by means of Theorem 4 that ψ is uniquely determined by $\beta(s)$. This shows uniqueness, and tells how to construct the resulting flat front (in case it exists) via Theorem 4. In addition, existence follows directly from Theorem 4 and the regularity of β .

Now, suppose that β is a geodesic of \mathbb{H}^3 that is a curve of singularities of a flat front ψ , and let V denote the unit normal of ψ along β . As $\langle \beta, V \rangle = \langle \beta', V \rangle = \langle \beta', V' \rangle = 0$, by composing with a rigid motion of \mathbb{H}^3 if necessary, we get the existence of an analytic function $f : I \rightarrow \mathbb{R}$ such that

$$\beta(s) = (\cosh s, 0, 0, \sinh s), \quad V(s) = (0, \cos f(s), \sin f(s), 0).$$

Then $G(s) = -G_*(s) = \exp(s + if(s))$ and $\xi(s)$ can be chosen to be $\xi(s) = \sqrt{2G(s)}$. A straightforward computation by means of Theorem 4 shows then that ψ coincides with the geodesic β (see Remark 2). This completes the proof. \square

5 Embedded isolated singularities

At last, we turn to the description of the flat surfaces in \mathbb{H}^3 that are embedded around an isolated singularity. For exposition clarity we have divided this section into

three parts. First, we study the immersed case, in which we determine the conformal type of the flat surfaces around the singularity, and characterize the initial data of the geometric Cauchy problem that give rise to isolated singularities. Then, in order to study when the isolated singularity is embedded, we investigate in the second part of the section a closely related problem for convex surfaces in \mathbb{R}^3 . In essence, we prove that any such surface is locally a graph around any embedded isolated singularity. We remark that this second part is completely independent from the rest of the paper. Finally, putting together the previous results and by means of the Klein ball model for \mathbb{H}^3 , we provide the desired structure theorem for embedded isolated singularities of flat surfaces in \mathbb{H}^3 .

The immersed case: conformal type and initial data

Definition 11 *Let $\psi : D \rightarrow \mathbb{H}^3$ be a continuous map on a topological disk D , and assume there is some $q \in D$ such that $\psi|_{D \setminus \{q\}}$ is a regular flat surface in \mathbb{H}^3 , but ψ is not C^1 at q . Then q (or $a = \psi(q)$) is called an isolated singularity of ψ .*

We will say that the isolated singularity is embedded provided there is a punctured neighbourhood $U^ \subset D \setminus \{q\}$ of q such that $\psi|_{U^*}$ is embedded.*

We start working in the above situation. Let $\psi : D \setminus \{q\} \rightarrow \mathbb{H}^3$ be an immersed flat surface in \mathbb{H}^3 around an isolated singularity, and let us endow $D \setminus \{q\}$ with the conformal structure induced by its second fundamental form. Then $D \setminus \{q\}$ is conformally a punctured disk or an annulus.

First we show that the punctured disk conformal type is impossible. To see this, assume that $D \setminus \{q\}$ is biholomorphic to the punctured unit disk $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$, and let $F : \widetilde{\mathbb{D}}^* \rightarrow \mathbf{SL}(2, \mathbb{C})$ be the map in (2.1), where $\widetilde{\mathbb{D}}^*$ is the universal covering of \mathbb{D}^* . Then

$$\psi_0 - \psi_3 = \frac{1}{|\xi|^2} + \frac{|\xi|^2}{|G - G_*|^2} \geq \frac{1}{|\xi|^2}.$$

Thus, by continuity of ψ , the map $|\xi|^2$ (which is well defined on \mathbb{D}^*) is bounded from below by a positive constant $d > 0$ in a neighbourhood of the origin. With this, and as $\psi_0 - \psi_3 \geq |\xi|^2/|G - G_*|^2$, the same happens to $|G - G_*|^2$. As a result, the meromorphic function $G - G_* : \mathbb{D}^* \rightarrow \mathbb{C} \cup \{\infty\}$ has at most a pole at 0. Taking now into account that

$$\psi_0 + \psi_3 = \frac{|G|^2}{|\xi|^2} + |\xi|^2 \frac{|G_*|^2}{|G - G_*|^2} \geq |\xi|^2 \frac{|G_*|^2}{|G - G_*|^2},$$

we see in the same way that $G_*/(G - G_*)$ does not have an essential singularity at 0. Therefore, G and G_* have at most a pole at 0.

On the other hand, as ψ is not C^1 at 0, we get that its unit normal η is not continuous at 0. Hence, by (3.4) we obtain that $|\xi|^2$ cannot be continuous at 0. But now, if ω denotes the Weierstrass 1-form of the flat surface ψ , it holds (see [KUY]) that $\xi^2 = -dG/\omega$. As by [GMM1, Lemma 2], $\omega = z^\nu \omega_1$, where $\nu \in [0, 1)$ and ω_1 is a single valued holomorphic 1-form on \mathbb{D}^* , the non-continuity of $|\xi|^2$ at 0 ensures that ω_1 has an essential singularity

at 0. But then $|\xi(z)|^2 = |z|^{-\nu}|h(z)|$, where h has an essential singularity at 0. This is impossible, since $|\xi|^2 \geq d > 0$ near 0 for some $d > 0$.

So, $D \setminus \{q\}$ must have the conformal type of an annulus, $\mathcal{A} = \{z : 0 < r < |z| < R\}$. In other words, we have a conformally immersed flat surface $\psi : \mathcal{A} \rightarrow \mathbb{H}^3$ which extends continuously to $C_r = \{z : |z| = r\}$, and with $\psi(C_r) = a \in \mathbb{H}^3$.

In order to achieve a classification result, we will make the following good-behaviour assumption on isolated singularities of flat surfaces: *the map $\psi : \mathcal{A} \rightarrow \mathbb{H}^3$ extends analytically to C_r* . This is the case of any singular curve of a flat front whose image is a single point, like the rotational examples in Figure 1.

Let us also note that \mathcal{A} is conformally equivalent to the quotient $\tilde{\Omega}/2\pi\mathbb{Z}$, where $\tilde{\Omega} = \{z : 0 < \text{Im}(z) < R\}$, and that the extension to the boundary of this equivalence maps C_r into $\text{Im}(z) = 0$. With all of this, we have the following structure theorem for *immersed* isolated singularities of flat surfaces in \mathbb{H}^3 .

Theorem 12 *Let $\psi : D \setminus \{q\} \rightarrow \mathbb{H}^3$ be an isolated singularity of a flat surface in \mathbb{H}^3 with $\psi(q) = a$, and consider a conformal parametrization as $\psi : \Omega/2\pi\mathbb{Z} \rightarrow \mathbb{H}^3$, where $\Omega = \{z \in \mathbb{C} : 0 \leq \text{Im}(z) \leq R\}$, so that $\psi(\mathbb{R}) = a$. Then, the unit normal of ψ along \mathbb{R} , denoted V , is a regular analytic 2π -periodic convex curve in $\mathbb{S}^2 \equiv \mathbb{S}_1^3 \cap \{a\}^\perp$, and the flat surface ψ is recovered via Theorem 4 by means of the initial data $\beta(s) = a$ and $V(s)$.*

Conversely, if $\beta(s) = a \in \mathbb{H}^3$ and $V(s)$ is a regular analytic 2π -periodic convex curve in $\mathbb{S}^2 \equiv \mathbb{S}_1^3 \cap \{a\}^\perp$, then the solution to the Cauchy problem for flat fronts with initial data β, V has $a \in \mathbb{H}^3$ as an isolated singularity.

Proof: As $\psi : \Omega/2\pi\mathbb{Z} \rightarrow \mathbb{H}^3$ extends analytically to $\mathbb{R} \equiv \text{Im}(z) = 0$, so does its unit normal $\eta : \Omega/2\pi\mathbb{Z} \rightarrow \mathbb{S}_1^3$. Therefore, if $G = [\psi + \eta]$ and $G_* = [\psi - \eta]$ denote the hyperbolic Gauss maps of ψ we obtain that $G(\mathbb{R})$ and $G_*(\mathbb{R})$ are analytic curves in $\mathbb{C} \cup \{\infty\}$. Thus, G, G_* extend meromorphically across \mathbb{R} to $\text{Im}(z) < 0$, what indicates that ψ extends across \mathbb{R} as a flat front. This fact together with Theorem 4 assure that ψ is recovered explicitly by means of the data $\beta(s) = \psi(s, 0) = a$ and $V(s) = \eta(s, 0)$. In addition $V(s)$ is analytic, 2π -periodic, and takes its values in $\mathbb{S}_1^3 \cap \{a\}^\perp \equiv \mathbb{S}^2$.

We need to show that $V(s)$ is regular and convex. For this, let θ, ω denote the Weierstrass 1-forms of ψ . Then, we know that the singular points of ψ in $\Omega/2\pi\mathbb{Z}$ are given by the condition $|\theta/\omega| = 1$. In other words, the set of singular points of ψ is the nodal set of the harmonic function $\phi = \log |\theta/\omega|$. As the real line \mathbb{R} is made up of singular points, it follows that, close enough to \mathbb{R} , ψ is an immersion in the interior of $\Omega/2\pi\mathbb{Z}$ if and only if \mathbb{R} is not crossed by any other nodal curve of ϕ , if and only if $\nabla\phi(s, 0) \neq 0$ for all $s \in \mathbb{R}$. Here ∇ stands for the usual Euclidean gradient.

In other words, sufficiently close to \mathbb{R} , ψ is an immersion in the interior of $\Omega/2\pi\mathbb{Z}$ if and only if

$$\theta'(s, 0)\omega(s, 0) - \omega'(s, 0)\theta(s, 0) \neq 0 \quad \forall s \in \mathbb{R}. \quad (5.1)$$

Suppose now without loss of generality that $a = (1, 0, 0, 0) \in \mathbb{H}^3$, and let $s_0 \in \mathbb{R}$. Obviously, we can also assume that $V_3(s_0) > 0$. Therefore, we can view $V(s) = (0, V_1(s), V_2(s), V_3(s))$ in a neighbourhood of s_0 as a curve $(x(s), y(s))$ in \mathbb{R}^2 by means of

$$x(s) = V_1(s)/V_3(s), \quad y(s) = V_2(s)/V_3(s)$$

(and thus $V_3(s) = (1 + x(s)^2 + y(s)^2)^{-1/2}$). In addition, it is well known that with the above identification the curve $V(s)$ is convex in \mathbb{S}^2 (i.e. it has non-vanishing geodesic curvature) if and only if the curve $(x(s), y(s))$ is convex in \mathbb{R}^2 .

Observe also that the Gauss maps along \mathbb{R} are given by

$$G(s) = \frac{V_1(s) + iV_2(s)}{1 - V_3(s)}, \quad G_*(s) = -\frac{V_1(s) + iV_2(s)}{1 + V_3(s)},$$

and that the 1-forms θ, ω are recovered in \mathbb{R} as (see [KUY])

$$\omega(s, 0) = -\frac{G'(s)}{\xi(s)^2}, \quad \theta(s, 0) = \frac{G'_*(s)\xi(s)^2}{(G(s) - G_*(s))^2},$$

where $\xi(s)$ verifies in \mathbb{R} the differential equation

$$\xi'(s) = \frac{G'(s)}{G(s) - G_*(s)} \xi(s).$$

Taking all of this into account, a direct calculation shows that

$$\theta'(s, 0)\omega(s, 0) - \omega'(s, 0)\theta(s, 0) = \frac{i(x''(s)y'(s) - x'(s)y''(s))}{2(1 + x(s)^2 + y(s)^2)}.$$

Consequently, (5.1) holds at s_0 if and only if the curve $(x(s), y(s))$, and thus $V(s)$, is regular and convex at s_0 . Finally, as ψ is an immersion in $\Omega \setminus \mathbb{R}$, the curve $V(s)$ must be regular and convex.

Now consider the converse. Since $V(s)$ is 2π -periodic, Corollary 8 shows that the solution to the Cauchy problem for the initial data $\beta(s) = a$ and $V(s)$ is well defined in $\Omega/2\pi\mathbb{Z}$, where $\Omega \subset \mathbb{C}$ is a strip of the form $\Omega = \{z \in \mathbb{C} : 0 \leq \text{Im}(z) \leq R\}$. In addition, $\psi(s, 0) = a$ and $\eta(s, 0) = V(s)$, where $\eta : \Omega/2\pi\mathbb{Z} \rightarrow \mathbb{S}_1^3$ is the unit normal to ψ in \mathbb{H}^3 . Finally, the above computations show that, as $V(s)$ is regular and convex in $\mathbb{S}^2 \equiv \mathbb{S}_1^3 \cap \{a\}^\perp$, equation (5.1) holds for ψ , and therefore ψ is regular away from \mathbb{R} . As a result, ψ is a flat surface in \mathbb{H}^3 with $a \in \mathbb{H}^3$ as an isolated singularity. \square

Embedded singularities of convex annuli in \mathbb{R}^3

Next, we prove the following result on convex surfaces in \mathbb{R}^3 , which will be the key for understanding the embeddedness of isolated singularities of flat surfaces in \mathbb{H}^3 .

Theorem 13 *Let $X : A \rightarrow \mathbb{R}^3$ be a smooth map from an annulus $A = \{p \in \mathbb{R}^2 : 0 < r \leq \|p\| \leq R\}$, such that $X(A \setminus C_r)$ is a regular embedded convex surface in \mathbb{R}^3 , and $X(C_r) = 0 \in \mathbb{R}^3$. Then, there exists a neighbourhood $\mathcal{U} \subset A$ of C_r and a plane Π in \mathbb{R}^3 passing through 0 such that $X(\mathcal{U})$ is a continuous graph over Π .*

Proof: Given X with the above conditions, we can regard it as a smooth injective map from a closed punctured disk $\mathcal{D} \setminus \{p\}$ into \mathbb{R}^3 , so that X extends continuously to p and $X(p) = 0$. Choose now a plane H_0 passing through 0, but whose two unit normals do not belong to the set $N(C_r) \subset \mathbb{S}^2$. As X is convex, it is clear that the intersection $H_0 \cap X(A \setminus C_r)$ is made up of points (at which the intersection is non-transversal), closed convex curves, convex curves with boundary in the exterior boundary $X(C_R)$, and convex curves for which at least one of its ends is 0. By starting with a different H_0 if necessary, it is obvious that we can assume that there actually exist curves of this type with 0 as an endpoint.

So, viewed on \mathcal{D} , the curves $\gamma \subset \mathcal{D}$ such that $X(\gamma) \subset H_0$ are of one of the types drawn in Figure 2.

Figure 2: Connected components of $H_0 \cap X(A \setminus C_r)$ viewed in \mathcal{D} .

We claim that there is only a finite number of such curves. To see this, we first note that as X is convex, N is a local diffeomorphism on $A \setminus C_r$, and so the above curves are isolated on $A \setminus C_r$. This property has the following consequence by an elementary compactness argument: if the number of curves is infinite, and we choose a point on each curve, we obtain a sequence in $\mathcal{D} \setminus \{p\}$ for which any converging subsequence has limit p . Consequently, choosing this sequence in an adequate way we see that the number of the above curves meeting the exterior boundary of \mathcal{D} has to be finite.

Besides, at any isolated point in $H_0 \cap X(A \setminus C_r)$ the unit normal N points in the normal direction of H_0 . Moreover, let γ be a closed curve, or a curve with both ends at 0, belonging to the above class. Then the interior region of γ that is exterior to all closed curves in the class that are contained in the interior region of γ must lie in a closed halfspace determined by H_0 . So, as the boundary of such a region lies in H_0 , there is at least one point in it at which the unit normal N is also normal to H_0 . So, if the number of curves (and isolated points) $\gamma \subset \mathcal{D} \setminus \{p\}$ with $X(\gamma) \subset H_0$ was infinite, we would have a sequence of points in \mathcal{D} whose unit normals are normal to H_0 . By the condition imposed to $N(C_r)$, the existing convergent subsequence has limit in $\mathcal{D} \setminus \{p\}$, but this contradicts that N is a local diffeomorphism. This proves the claim.

In this way, by shrinking \mathcal{D} if necessary, we can assume that the intersection $X(\mathcal{D}) \cap H_0$ is made up of a finite number of convex curves $X(\gamma_i)$, where each $\gamma_i \subset \mathcal{D}$ has p as an endpoint, and the other endpoint in the boundary of \mathcal{D} (see Figure 3).

Therefore we can number the γ_i 's according to the order of their endpoints in $\partial\mathcal{D}$,

Figure 3: The curves γ_i in \mathcal{D} such that $X(\gamma_i) \in H_0$.

once fixing an orientation on $\partial\mathcal{D}$.

It is easy to check that the number of curves γ_i is actually even. This follows from the fact that $\mathcal{D} \setminus \{\gamma_1, \dots, \gamma_k\}$ has k connected components and that, as the intersection of X with H_0 is transversal along all such curves, any of the above connected components lies in a halfspace determined by H_0 , and the adjacent ones lie in the *other* halfspace. In addition, we can suppose that if i is odd, the region in $X(\mathcal{D})$ situated between $X(\gamma_i)$ and $X(\gamma_{i+1})$ lies in the positive halfspace $H_0^+ \subset \mathbb{R}^3$ determined by H_0 (having previously fixed an orientation on H_0). Here we take the convention that $\gamma_{k+1} = \gamma_1$.

Our next aim is to prove that every convex curve $X(\gamma_i) \subset H_0$ has a well defined limit unit tangent vector at 0.

For this, we consider for every $\varepsilon > 0$ the parallel plane to H_0 at a distance ε that lies in H_0^+ , denoted H_ε . Then for any fixed odd number $i \in \{1, \dots, 2n-1\}$ we have a family of connected curves Γ_ε^i lying in $H_\varepsilon \cap X(\mathcal{D})$ and with $\lim_{\varepsilon \rightarrow 0} \Gamma_\varepsilon^i = X(\gamma_i) \cup \{0\} \cup X(\gamma_{i+1})$. Obviously, the curves Γ_ε^i are regular and convex for sufficiently small ε . Now, if for every $\varepsilon > 0$ we let $T_1^\varepsilon, T_2^\varepsilon$ denote the unit tangent vectors to Γ_ε^i at its endpoints (which lie in $\Gamma_\varepsilon^i \cap X(\partial\mathcal{D})$), and $\alpha_\varepsilon \in [0, 2\pi)$ is the angle between T_1^ε and T_2^ε , then by continuity and for any arbitrary $L > 0$ there is some $\varepsilon_0 > 0$ such that

$$|\alpha_0 - \alpha_\varepsilon| < L, \quad 0 < \varepsilon < \varepsilon_0. \quad (5.2)$$

Here, α_0 is the angle between the tangent unit vectors to $X(\gamma_i)$ and $X(\gamma_{i+1})$ at their respective endpoints in the boundary $X(\partial\mathcal{D})$.

Furthermore, the *accumulated turning* of the unit tangent vector of the convex curve Γ_ε^i is of the form $2\pi l + \alpha_\varepsilon$ for some $l \in \mathbb{N}$. By (5.2) we see then that for small $\varepsilon > 0$ the number l is constant, i.e. it does not depend on ε .

As $X(\gamma_i)$ is convex, its unit tangent vector always rotates in \mathbb{S}^1 in the same direction. Also, if this unit tangent vector winded infinitely many times, there would exist a compact piece of $X(\gamma_i)$ away from 0 with accumulated turning greater than $2\pi(l+1)$. But this would imply the existence of some $\varepsilon > 0$ such that a compact piece of Γ_ε^i has accumulated turning greater than $2\pi(l+1)$, which is impossible since the whole Γ_ε^i is convex and has accumulated turning $2\pi l + \alpha_\varepsilon$.

Therefore, $X(\gamma_i)$ has a limit unit tangent vector at 0, which we will denote by T_i . Obviously, the same argument works if i is even.

Consequently, the situation is as follows: if we have three consecutive curves $\gamma_i, \gamma_{i+1}, \gamma_{i+2}$, then $X(\gamma_i) \cup \{0\} \cup X(\gamma_{i+1})$ and $X(\gamma_{i+1}) \cup \{0\} \cup X(\gamma_{i+2})$ are convex at 0, which implies that the angle between T_i and T_{i+1} is in $(0, \pi]$ clockwise (up to a change of orientation), and the angle between T_{i+1} and T_{i+2} is also in $(0, \pi]$ but anticlockwise. In other words, the angle between T_{i+1} and T_{i+2} is in $[\pi, 2\pi)$ clockwise (see Fig. 4).

Figure 4: The curves $X(\gamma_i), X(\gamma_{i+1}), X(\gamma_{i+2})$ and their limit unit tangents T_i, T_{i+1}, T_{i+2} .

More generally, we can assume that the angle between T_i and T_{i+1} is in $(0, \pi]$ clockwise if i is odd, and in $[\pi, 2\pi)$ clockwise if i is even. This is a fundamental property that will let us prove that the number of curves γ_i is actually two. In order to do so, we shall first justify the following fact:

FACT: Let $C \subset \mathbb{R}^2$ be a Jordan curve meeting the unit circle \mathbb{S}^1 transversally at $2n$ points, $n \geq 1$. Let $\{p_1, \dots, p_{2n} = p_0\}$ be such points, ordered according to an orientation of C , and let $\beta_i \in [0, 2\pi)$ denote the angle in \mathbb{S}^1 between p_i and p_{i+1} . If $(\beta_i - \pi)(\beta_{i+1} - \pi) < 0$ for all i , then $n = 1$.

To see this, suppose that there are at least four points. Then, up to a rotation we may assume that $p_2 = (1, 0)$, and as $(\alpha_1 - \pi)(\alpha_2 - \pi) < 0$, both p_1, p_3 lie (up to a symmetry) in the upper halfplane $x > 0$. Also note that as the intersection is transversal, every time C meets \mathbb{S}^1 , it must go from the interior region determined by \mathbb{S}^1 to the exterior one, or vice-versa. Then, p_3 must lie in the smaller or the bigger arc joining p_2 and p_1 . Nevertheless, by reversing the orientation in C and labelling p_3 as the initial point, we see that we may assume without loss of generality that p_3 lies in the smaller arc between p_1 and p_2 (see Figure 5).

Obviously, after these rigid motions and orientation changes, we see that $\alpha_1 \in (\pi, 2\pi)$ and $\alpha_2 \in (0, \pi)$. Therefore, $\alpha_3 \in (\pi, 2\pi)$. In particular, p_4 must lie in the bigger arc joining p_1 to p_3 . But as C cannot intersect, we deduce that p_4 actually lies in the smaller arc between p_2 and p_3 . If we repeat this process we obtain in general that p_{2n} lies in the smaller arc between p_{2n-2} and p_{2n-1} . Therefore, in the final step C cannot link p_{2n} to p_1 without self-intersecting.

Figure 5: The situation with three points, up to certain identifications. A similar picture is obtained if at p_1 the curve C goes *from the inside to the outside*.

This proves the stated fact.

Now, let us show that the curves γ_i are just two. As all $X(\gamma_i)$ have well defined unit tangent vectors T_i at 0, we have:

- (a) $X(\gamma_i)$ is a graph sufficiently close to 0 over an appropriately chosen line. In particular, sufficiently near from $p \in \mathcal{D}$ the distance from $X(q)$, $q \in \gamma_i$, to the origin is strictly decreasing.
- (b) Given an angle $\theta_0 > 0$ there exist open neighbourhoods $I_i \subset \gamma_i \cup \{p\}$ of p such that for every i and every point $q^* \in I_i \setminus p$ the angle between $X(q^*)$ and $-T_i$ is smaller than θ_0 .

As $X(\gamma_i \setminus I_i) \cup X(\partial\mathcal{D})$ is compact and does not meet the origin, we can define $d_0 > 0$ to be the distance of this set to the origin. Now, by the Sard-Brower transversality theorem, there is some $d_1 \in (0, d_0)$ such that $\mathbb{S}^2(d_1)$ meets $X(\mathcal{D})$ transversally. Therefore, $X(\mathcal{D}) \cap \mathbb{S}^2(d_1)$ is made up of regular Jordan curves, and by (a), $\mathbb{S}^2(d_1)$ meets every $X(\gamma_i)$ exactly once. So, there is a Jordan curve $C \subset \mathcal{D} \setminus \{p\}$ surrounding p with the property that $X(C)$ lies in $\mathbb{S}^2(d_1) \cap X(\mathcal{D})$ and meets each $X(\gamma_i)$ exactly once. As a result, $X(C) \subset \mathbb{S}^2(d_1)$ meets the great circle $H_0 \cap \mathbb{S}^2(d_1)$ exactly $2n$ times, one for each γ_i , at points $X(q_i)$ where $q_i \in \gamma_i$. Also observe that the order given to the γ_i 's imposes an orientation on $X(C)$.

Now assume that the angle between $T_i \in \mathbb{S}^1$ and $T_{i+1} \in \mathbb{S}^1$ is clockwise smaller than π for some (necessarily odd) i . Then it follows directly from (b) that, possibly starting with a smaller θ_0 , the angle between $X(q_i)$ and $X(q_{i+1})$ is smaller than π clockwise. The same argument shows that if the angle between T_i and T_{i+1} is greater than π , then so is the angle between $X(q_i)$ and $X(q_{i+1})$ for small enough θ_0 . Finally, if the angle between T_i and T_{i+1} is π , the curve $\Gamma = X(\gamma_i) \cup \{0\} \cup X(\gamma_{i+1})$ is a globally regular convex curve, and thus 0 has in it a convex neighbourhood lying in a vector halfplane. Hence, the angle between $X(q_i)$ and $X(q_{i+1})$ is smaller (resp. greater) than π clockwise if i is odd (resp. even).

Next, we use stereographic projection from $\mathbb{S}^2(d_1)$ into \mathbb{R}^2 , so that $X(C)$ turns into a plane Jordan curve $C^* \subset \mathbb{R}^2$ and $H_0 \cap \mathbb{S}^2(d_1)$ is mapped into a circle $\mathbb{S}^1(r)$ in \mathbb{R}^2 . To do so, we may first need to slightly rectify the curve $X(C)$ so that it does not pass through the point in $\mathbb{S}^2(d_1)$ from which we are projecting, without losing embeddedness. It is then evident that the just deduced condition on the angles between $X(q_i)$ and $X(q_{i+1})$ shows that the hypothesis of the above proved *fact* are fulfilled by C^* and $\mathbb{S}^1(r)$. Therefore, there are just two curves γ_i .

With all of this, we have proved that $H_0 \cap X(\mathcal{D})$ is $X(\gamma_1) \cup \{0\} \cup X(\gamma_2)$. Therefore it is a global graph in a neighbourhood of 0 over the vector line in H_0 spanned by $T_1 - T_2$.

Now, if we consider the x_3 -axis in \mathbb{R}^3 to be the vector line in H_0 orthogonal to $T_1 - T_2$, then the interior normal of the curves $X(\gamma_1)$, $X(\gamma_2)$ point upwards, and we can assume that they wind anticlockwise. Consequently, the interior unit normal of $X(\mathcal{D} \setminus \{p\})$ also points upwards around p (see Figure 6).

Figure 6: The curves $X(\gamma_1)$, $X(\gamma_2)$ and their interior normals.

Next, we consider again the Γ_ε curves in the H_ε -planes, that are regular, convex, embedded and with $\lim_{\varepsilon \rightarrow 0} \Gamma_\varepsilon = X(\gamma_1) \cup \{0\} \cup X(\gamma_2)$. Let $\mathbb{B}(0, d_1) \subset \mathbb{R}^3$ denote the open ball of radius d_1 centered at the origin, where $d_1 > 0$ is as we described above. Then, for small enough ε , the part Γ_ε^* of Γ_ε outside $\mathbb{B}(0, d_1)$ is made up of two compact connected components, each one meeting $X(\partial\mathcal{D})$ at a single point, P_1^ε and P_2^ε , respectively. So, by continuity, there is some $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then the interior normal of Γ_ε^* always points upwards and rotates counterclockwise. Therefore, every Γ_ε^* is a global graph in H_ε in the x_3 -axis direction.

Furthermore, it is clear from these conditions and (5.2) that we can join P_1^ε and P_2^ε by means of a regular embedded convex curve in H_ε with the following properties:

- It does not meet $\mathbb{B}(0, d_1)$.
- It extends Γ_ε^* to a smooth embedded convex curve in H_ε .

In other words, we obtain a closed convex curve $\tilde{\Gamma}_\varepsilon \subset H_\varepsilon$ that has Γ_ε as a proper subset. Moreover, $\tilde{\Gamma}_\varepsilon$ is embedded outside $\mathbb{B}(0, d_1)$ by its construction, but also in $\mathbb{B}(0, d_1)$

because Γ_ε is embedded. Thus $\tilde{\Gamma}_\varepsilon$ is a convex Jordan curve in H_ε for every small enough $\varepsilon > 0$. In particular, the interior normal of Γ_ε does not perform a complete loop, and by the conditions derived at the endpoints $P_1^\varepsilon, P_2^\varepsilon$ we deduce that it must always point upwards.

Therefore, the unit normal to the surface also points upwards along all these curves, and hence it points upwards in a neighbourhood of the origin. So finally we are led to the conclusion that $X(\mathcal{D})$ is a global graph in a neighbourhood of the origin, as we wished to prove. □

The statement of the next Lemma might not be new. However, as we could not find it anywhere in the literature, we have decided to include a proof here.

Lemma 14 *Any regular convex Jordan curve in \mathbb{S}^2 lies in an open hemisphere. Consequently, its interior region is a geodesically convex set in \mathbb{S}^2 .*

Proof: Let $\Gamma \subset \mathbb{S}^2$ be a convex Jordan curve, and take $p \in \Gamma$. Then its tangent great circle in \mathbb{S}^2 at p leaves Γ at one side in a neighbourhood of p . Let us assume that this great circle is $\mathbb{S}^1 = \mathbb{S}^2 \cap \{x_3 = 0\}$, and that Γ is locally contained in \mathbb{S}_+^2 near p . Obviously, if $\mathbb{S}^1 \cap \Gamma = \{p\}$, then Γ lies in an open hemisphere of \mathbb{S}^2 which is sufficiently close to \mathbb{S}_+^2 .

Suppose now that $\mathbb{S}^1 \cap \Gamma \neq \{p\}$, and let us endow Γ with an orientation. Then, starting at p , there exists a first point $q_1 \in \mathbb{S}^1 \cap \Gamma \setminus \{p\}$ and a last point $q_2 \in \mathbb{S}^1 \cap \Gamma \setminus \{p\}$ (which obviously can coincide). We shall denote by Γ_1 (resp. Γ_2) the open piece of Γ with endpoints p and q_1 (resp. q_2 and p). Obviously, both Γ_1, Γ_2 are regular convex curves properly embedded in \mathbb{S}_+^2 which do not intersect. Besides, by convexity of Γ at p , there are points $p_1 \in \Gamma_1$ and $p_2 \in \Gamma_2$ such that the smaller piece of great circle in \mathbb{S}^2 that joins p_1, p_2 does not meet \mathbb{S}^1 . Also note that this arc lies in the convex part of Γ at both p_1 and p_2 .

With all of this, if we let $\pi : \mathbb{S}_+^2 \rightarrow \mathbb{R}^2$ be the totally geodesic diffeomorphism given by $\pi(x_1, x_2, x_3) = (x_1/x_3, x_2/x_3)$, then $\pi(\Gamma_1)$ and $\pi(\Gamma_2)$ are regular convex curves properly embedded in \mathbb{R}^2 and that do not intersect. Thus $\pi(\Gamma_i)$ divides \mathbb{R}^2 into two connected components, one of them (say Ω_i) convex. Therefore the line segment joining $\pi(p_1)$ with $\pi(p_2)$ must lie in $\Omega_1 \cap \Omega_2$, which is convex. But as $\pi(\Gamma_1) \cap \pi(\Gamma_2) = \emptyset$, $\Omega_1 \cap \Omega_2$ must have two boundary components. This is possible for a convex set only if these boundary components are parallel straight lines, which is not the present situation, as $\pi(\Gamma_i)$ must have positive curvature. Thus $\mathbb{S}^1 \cap \Gamma = \{p\}$ and Γ lies in an open hemisphere of \mathbb{S}^2 .

Once proved this, the last assertion follows directly from the corresponding result for convex Jordan curves in \mathbb{R}^2 . □

The classification

At last, we are in the position to prove our main result.

Theorem 15 *Let $a \in \mathbb{H}^3$, and consider the sets*

$$\mathcal{C} = \{\text{regular analytic convex Jordan curves in } \mathbb{S}^2\},$$

$$\mathcal{E}_a = \{\text{flat surfaces in } \mathbb{H}^3 \text{ having } a \text{ as embedded isolated singularity}\}.$$

Then the map assigning to each $V(s) : \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{S}^2$ in \mathcal{C} the solution to the Cauchy problem for the initial data $\beta(s) = a$ and $V(s) : \mathbb{R} \rightarrow \mathbb{S}_1^3 \cap \{a\}^\perp \equiv \mathbb{S}^2$ is a one-to-one explicit correspondence between \mathcal{C} and \mathcal{E}_a .

Moreover, curves in \mathcal{C} differing only by a rigid motion in \mathbb{S}^2 correspond to flat surfaces in \mathcal{E}_a differing only by a rigid motion in \mathbb{H}^3 fixing a .

Before coming to the proof, let us describe two fundamental tools that we shall use.

- The *Klein model* for \mathbb{H}^3 : it consists on the diffeomorphism from \mathbb{H}^3 into the open unit ball $\mathbb{B}^3(0, 1) \subset \mathbb{R}^3$ given by $\Phi : \mathbb{H}^3 \rightarrow \mathbb{B}^3(0, 1) \subset \mathbb{R}^3$,

$$\Phi(x_0, x_1, x_2, x_3) = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0} \right).$$

This map is totally geodesic, and thus preserves convexity. In particular, flat surfaces in \mathbb{H}^3 are mapped into convex surfaces in \mathbb{R}^3 . Moreover, $\Phi(1, 0, 0, 0) = 0 \in \mathbb{R}^3$, and the unit tangent bundle to \mathbb{H}^3 at $(1, 0, 0, 0)$, given by $\mathbb{S}_1^3 \cap \{x_0 = 0\}$, is taken via Φ to the the unit 2-sphere \mathbb{S}^2 , seen as the unit tangent bundle of \mathbb{R}^3 at 0.

- The *Legendre transform*: let $X = (X_1, X_2, X_3)$ denote a convex surface in \mathbb{R}^3 which is a local graph in the x_3 -axis direction around any point, and let $N = (N_1, N_2, N_3)$ be its interior unit normal, which lies in the upper halfsphere \mathbb{S}_+^2 . Then the map (see [LSZ, pg. 89] for instance)

$$\mathcal{L}_X = \left(\frac{-N_1}{N_3}, \frac{-N_2}{N_3}, -X_1 \frac{N_1}{N_3} - X_2 \frac{N_2}{N_3} - X_3 \right) \quad (5.3)$$

is a convex surface in \mathbb{R}^3 that is a local graph in the x_3 -axis direction around any point. In addition, its interior unit normal is given by

$$\mathcal{N}_{\mathcal{L}} = \frac{1}{\sqrt{1 + X_1^2 + X_2^2}} (-X_1, -X_2, 1). \quad (5.4)$$

Proof of Theorem 15: Let $V(s)$ be a curve in \mathcal{C} . Then, by Theorem 4 and Theorem 12, the solution to the Cauchy problem for flat fronts with initial data $\beta(s) = a$, $V(s)$, is a (unique) flat surface ψ in \mathbb{H}^3 that has a as an isolated singularity. Our first task is to show that the stated assignment is a well defined mapping, i.e. it takes its values in \mathcal{E}_a . In other words, we need to show that the embeddedness of $V(s)$ guarantees that ψ is embedded around a .

Given $V(s) \in \mathcal{C}$, let $\psi : \Omega/2\pi\mathbb{Z} \rightarrow \mathbb{H}^3$ be the flat surface spanned via Theorem 12 by the data $\beta(s) = a$ and $V(s)$. Without loss of generality, we shall suppose that $a = (1, 0, 0, 0)$. So, if $X = \Phi \circ \psi$ is the surface ψ in the Klein model, we get that X is convex in $\Omega \cap \{\text{Im}(z) > 0\}$, and $X(\mathbb{R}) = 0 \in \mathbb{R}^3$. As $V(\mathbb{R})$ is a convex Jordan curve in $\mathbb{S}_1^3 \cap \{a\}^\perp$ which is the restriction to \mathbb{R} of the unit normal of ψ in \mathbb{H}^3 , and as $\psi(\mathbb{R}) = (1, 0, 0, 0)$, we see that the unit normal N of X in \mathbb{R}^3 verifies that $N(\mathbb{R})$ is a regular convex Jordan curve in \mathbb{S}^2 . Thus, by Lemma 14 $N(\mathbb{R})$ lies in an open hemisphere, say, the upper hemisphere $\mathbb{S}_+^2 = \mathbb{S}^2 \cap \{x_3 > 0\}$. Therefore, by shrinking Ω away from \mathbb{R} if necessary, we may assume that N always points upwards in \mathbb{S}^2 . Thus, X is a local graph in the x_3 -axis direction around any point in $\Omega \cap \{\text{Im}(z) > 0\}$.

Now let $\mathcal{L}_X : (\Omega/2\pi\mathbb{Z}) \cap \{\text{Im}(z) > 0\} \rightarrow \mathbb{R}^3$ denote the Legendre transform (5.3) of X . Then $\mathcal{L}_X(\mathbb{R}/2\pi\mathbb{Z})$ is a regular convex Jordan curve in the x_1, x_2 -plane, and by (5.4) the unit normal of \mathcal{L}_X at $\mathbb{R}/2\pi\mathbb{Z}$ takes the limit value $(0, 0, 1)$. Therefore, \mathcal{L}_X lies in the upper halfspace of \mathbb{R}^3 , and there is some $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ the intersection $\Upsilon_\varepsilon = \mathcal{L}_X(\Omega/2\pi\mathbb{Z}) \cap \{x_3 = \varepsilon\}$ is a regular convex Jordan curve. Now let $S_{\varepsilon_1, \varepsilon_2}$ denote the portion of \mathcal{L}_X that lies in the slab given by the planes $\{x_3 = \varepsilon_1\}$ and $\{x_3 = \varepsilon_2\}$, where $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon_0$. Then, as $S_{\varepsilon_1, \varepsilon_2}$ is convex, it is known that the convexity and embeddedness of the curves Υ_ε assure that the unit normal $\mathcal{N}_{\mathcal{L}}$ of \mathcal{L}_X in this slab is not only a local diffeomorphism, but a global diffeomorphism onto its image in \mathbb{S}^2 . Letting $\varepsilon_1 \rightarrow 0$, $\mathcal{N}_{\mathcal{L}}$ is a global diffeomorphism on a neighbourhood of $\mathbb{R}/2\pi\mathbb{Z}$ in $(\Omega/2\pi\mathbb{Z}) \cap \{\text{Im}z > 0\}$ into its spherical image in \mathbb{S}^2 . Therefore, by (5.4) we conclude that X must be a global graph about the origin in the x_3 -axis direction. Consequently, ψ is embedded around the singularity, as we wished to prove.

With this, we have a map $\mathcal{C} \rightarrow \mathcal{E}_a$, which by its construction is injective. To see that it is onto, we just need to show that if ψ is embedded around an isolated singularity, then the curve $V(s)$ that recovers ψ via Theorem 12 is embedded in \mathbb{S}^2 .

So, suppose that ψ is embedded around the isolated singularity $a = (1, 0, 0, 0)$. Then, using the Klein model, the convex surface $X = \Phi \circ \psi$ is parametrized on a compact annulus $A = \{z \in \mathbb{C} : 0 < r \leq |z| \leq R\}$, and satisfies $X(C_r) = 0 \in \mathbb{R}^3$. Consequently, X is in the conditions of Theorem 13, and therefore it is a global graph in a neighbourhood $\mathcal{U} \subset A$ of C_r . We shall assume then that $X(\mathcal{U})$ is graph in the x_3 -axis direction, what implies that the unit normal to X , $N(\mathcal{U})$, always point upwards in $\mathcal{U} \setminus C_r$. By continuity, $N(C_r) \subset \mathbb{S}^2 \cap \{x_3 \geq 0\}$. As $N(C_r)$ is analytic and convex, it only meets the great circle $\mathbb{S}^2 \cap \{x_3 = 0\}$ at a finite number of points $N(q_1), \dots, N(q_m)$.

Let now $\mathcal{L}_X : \mathcal{U} \setminus C_r \rightarrow \mathbb{R}^3$ denote the Legendre transform of X . Then by (5.3) and (5.4) the unit normal $\mathcal{N}_{\mathcal{L}}$ of \mathcal{L}_X extends continuously to C_r with $\mathcal{N}_{\mathcal{L}}(C_r) = (0, 0, 1)$, and the third coordinate $(\mathcal{L}_X)_3$ of \mathcal{L}_X also extends continuously to C_r , so that $(\mathcal{L}_X)_3(C_r) = 0$.

With all of this, and as \mathcal{L}_X is convex, there is an open annulus $\mathcal{R} = \{z \in \mathbb{C} : 0 < r < |z| < r_0\}$ such that $\mathcal{L}_X(\mathcal{R})$ is properly immersed in the upper halfspace $\mathbb{R}_+^3 = \{x_3 > 0\}$. In particular, if $\varepsilon_0 = \text{dist}(0, \mathcal{L}_X(C_{r_0})) > 0$, then for all $\varepsilon \in (0, \varepsilon_0)$ the curves $\Upsilon_\varepsilon = \mathcal{L}_X(\mathcal{R}) \cap \{x_3 = \varepsilon\}$ are regular, convex and closed.

Moreover, these curves Υ_ε must be embedded, as we show next. If Υ_ε were not embedded, its interior normal would describe at least two loops. So, the projection into

the x_1, x_2 -plane of the interior unit normal $\mathcal{N}_{\mathcal{L}}$ along Υ_ε would also wind at least twice. But as the image of $\mathcal{N}_{\mathcal{L}}(\mathcal{R})$ lies in the upper hemisphere of \mathbb{S}^2 , this would mean that $\mathcal{N}_{\mathcal{L}}$ self-intersects along Υ_ε . This, however is not possible, since by (5.4) it would contradict the fact that X is a graph in the x_3 -axis direction.

Therefore, all the curves Υ_ε are regular convex Jordan curves lying in parallel planes. If now we view these curves in the upper hemisphere of \mathbb{S}^2 by means of the totally geodesic diffeomorphism $\mathbb{S}_+^2 \rightarrow \mathbb{R}^2$ given by $(x_1, x_2, x_3) \rightarrow (x_1/x_3, x_2/x_3)$, we obtain a family of regular convex Jordan curves V^ε that converge to $V = N(C_r)$ as $\varepsilon \rightarrow 0$. This assures that $V = N(C_r)$ is embedded, i.e. it is a Jordan curve.

Finally, the last assertion of the Theorem is a straightforward consequence of the generalized symmetry principle in Theorem 7. This concludes the proof. \square

Theorem 15 suggests that a flat front is symmetric with respect to point reflection in \mathbb{H}^3 through any of its embedded isolated singularities. Indeed, the following statement follows directly from the generalized symmetry principle and Remark 5:

Corollary 16 *Let $\psi : \Sigma \rightarrow \mathbb{H}^3$ be a flat front, and assume that there is an analytic curve $\Gamma \subset \Sigma$ such that $\psi(\Gamma(s)) = a \in \mathbb{H}^3$. Then $\psi(\Sigma)$ is symmetric with respect to point reflection in \mathbb{H}^3 through a .*

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