

Complete Surfaces in the Hyperbolic Space with a Constant Principal Curvature

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Abstract. In this paper we study complete orientable surfaces with a constant principal curvature R in the 3-dimensional hyperbolic space \mathbf{H}^3 . Thus, we prove that if $R^2 > 1$, such a surface is totally umbilical or umbilically free and it can be described in terms of a complete regular curve in \mathbf{H}^3 . When $R^2 \leq 1$, we show that this result is not true by means of several examples, which contradicts a previous theorem by Zhisheng [6].

Mathematics Subject Classification (2000): Primary 53042, Secondary 53C45

Key words: hyperbolic space, complete surface, constant principal curvature.

1 Introduction and Statement of Main Results

Let \mathbf{L}^4 be the 4-dimensional *Lorentz-Minkowski space*, that is, the real vector space \mathbf{R}^4 endowed with the Lorentzian metric tensor \langle, \rangle given by

$$\langle, \rangle = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2,$$

where (x_1, x_2, x_3, x_4) are the canonical coordinates of \mathbf{R}^4 . The 3-dimensional unitary *hyperbolic space* is given as the following hyperquadric of \mathbf{L}^4 ,

$$\mathbf{H}^3 = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 : x_1^2 + x_2^2 + x_3^2 - x_4^2 = -1, x_4 > 0\}.$$

As it is well known, \mathbf{H}^3 inherits from \mathbf{L}^4 a Riemannian metric which makes it the standard model of a space form with constant sectional curvature -1.

Throughout this work, we let M be a 2-dimensional connected complete orientable Riemannian manifold of class C^∞ . We will denote by $\psi : M \rightarrow \mathbf{H}^3 \subset \mathbf{L}^4$ an isometric immersion of M into \mathbf{H}^3 , and by N its (globally defined) Gauss map. We will also denote by λ_1, λ_2 the principal curvatures of M associated to N .

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The second author is partially supported by DGICYT Grant No PB97-0785.
 \mathcal{PT}^2

The global study of the curvatures of surfaces in space forms is intended to determine what surfaces have the simplest behaviour of their curvatures. Our aim is the study of complete orientable surfaces with a constant principal curvature in \mathbf{H}^3 .

For the case of complete orientable surfaces with a zero principal curvature in the Euclidean space \mathbf{E}^3 , Hartman and Nirenberg [1] proved in 1959 that they are cylinders on a complete planar regular curve (see also [3] and [5]). Later, in 1970, Shiohama and Takagi [4] showed that a complete orientable surface with a constant positive principal curvature in \mathbf{E}^3 is either totally umbilical or else umbilically free.

Following the ideas of Shiohama and Takagi, Zhisheng [6] stated in 1989 that every complete orientable surface in \mathbf{H}^3 with a constant principal curvature R is totally umbilical or umbilically free. Unfortunately, this result is false, as the examples of Section 2 become plane. Actually, the above result is only true if $R^2 > 1$. Even more, in this case the surface can be described as follows:

Theorem 1 *Let $\psi : M \rightarrow \mathbf{H}^3$ be a complete orientable surface with a constant principal curvature R such that $R^2 > 1$. Then $\psi(M)$ is a totally umbilical round sphere or umbilically free. In the second case the surface can be described as*

$$\psi(x, y) = \frac{1}{\sqrt{R^2 - 1}} (-R\alpha(y) + \cos(x)v_1(y) + \sin(x)v_2(y)) \quad (1)$$

where α is a C^∞ complete regular curve in \mathbf{H}^3 and $\{v_1(y), v_2(y)\}$ is an orthonormal frame of the normal plane along α .

Conversely, given a regular curve α in \mathbf{H}^3 , (1) defines an umbilically free orientable immersion in \mathbf{H}^3 with a constant principal curvature R such that $R^2 > 1$.

Moreover, if the surface is compact we prove

Theorem 2 *Let $\psi : M \rightarrow \mathbf{H}^3$ be a compact orientable surface with a constant principal curvature R . Then $R^2 > 1$ and $\psi(M)$ is a totally umbilical round sphere or an (umbilically free) torus which can be described as*

$$\psi(x, y) = \frac{1}{\sqrt{R^2 - 1}} (-R\alpha(y) + \cos(x)v_1(y) + \sin(x)v_2(y))$$

where α is a C^∞ closed regular curve in \mathbf{H}^3 and $\{v_1(y), v_2(y)\}$ is an orthonormal frame of the normal plane along α .

2 Examples

Recall that given a Riemannian metric I and a symmetric $(2,0)$ -tensor II on a simply-connected 2-dimensional manifold M , if I and II satisfy the Gauss and Mainardi-Codazzi equations of the hyperbolic space \mathbf{H}^3 , then there exists an only immersion (up to an isometry) $\psi : M \rightarrow \mathbf{H}^3$ such that I and II are its first and second fundamental forms, respectively.

In the following example we construct a family of complete orientable surfaces in \mathbf{H}^3 with umbilical and non umbilical points, and a constant principal curvature $R = 1$.

Example 3 Let us consider $M = \mathbf{R}^2$ and ψ_h the only immersion (up to an isometry) with first and second fundamental forms given by

$$I_h = dx^2 + \left(1 + \frac{x^2 h(y)}{2}\right)^2 dy^2$$

and

$$II_h = dx^2 + \left(1 + \frac{x^2 h(y)}{2}\right) \left(1 + \left(\frac{x^2}{2} - 1\right) h(y)\right) dy^2$$

respectively, where $h : \mathbf{R} \rightarrow \mathbf{R}$ is a non negative C^∞ function which vanishes at some point. Since $I_h \geq dx^2 + dy^2$, the immersion ψ_h is complete, with principal curvatures

$$1 \quad \text{and} \quad 1 - \frac{2h(y)}{2 + x^2 h(y)}.$$

Thus, the set of umbilical points is $\Omega = \{(x, y) : h(y) = 0\}$.

It is worth pointing out that the interior of Ω may be non empty. ■

Next we are going to construct a family of complete orientable surfaces in \mathbf{H}^3 with umbilical and non umbilical points, and a constant principal curvature $0 \leq R < 1$.

Example 4 Let us consider again $M = \mathbf{R}^2$ and $\psi_{R,h}$ the only immersion (up to an isometry) with first and second fundamental forms

$$I_{R,h} = dx^2 + \left(\frac{Rh(y)}{1-R^2} + e^{\sqrt{1-R^2}x}\right)^2 dy^2$$

and

$$II_{R,h} = R dx^2 + \frac{\left(h(y) + R(1-R^2)e^{\sqrt{1-R^2}x}\right) \left(Rh(y) + (1-R^2)e^{\sqrt{1-R^2}x}\right)}{(1-R^2)^2} dy^2$$

respectively, where $h : \mathbf{R} \rightarrow \mathbf{R}$ is a non negative C^∞ function which vanishes at some point, and $0 \leq R < 1$. Since $I_{R,h} \geq dx^2 + e^{\sqrt{1-R^2}x} dy^2$, the immersion ψ_h is complete, with principal curvatures

$$R \quad \text{and} \quad \frac{h(y) + R(1-R^2)e^{\sqrt{1-R^2}x}}{Rh(y) + (1-R^2)e^{\sqrt{1-R^2}x}}.$$

Hence, the set of umbilical points is $\Omega = \{(x, y) : h(y) = 0\}$.

As above, note that this surface can meet a totally umbilical surface in an open set. ■

3 Proofs of Main Results

Let $\psi : M^2 \rightarrow \mathbf{H}^3 \subset \mathbf{L}^4$ be a complete surface in \mathbf{H}^3 with a constant principal curvature $\lambda_1 = R \geq 0$ (up to a change of orientation). If there exists a non umbilical point $p \in M$, then we can consider local isothermal parameters (u, v) in a neighborhood U of p without umbilical points, such that

$$\begin{aligned} \langle d\psi, d\psi \rangle &= E du^2 + G dv^2 \\ \langle d\psi, -dN \rangle &= RE du^2 + \lambda_2 G dv^2 \end{aligned}$$

where the principal curvature $\lambda_2 \neq R$. Then, the structure equations are given by

$$\begin{aligned}\psi_{uu} &= \frac{E_u}{2E} \psi_u - \frac{E_v}{2G} \psi_v + REN + E\psi \\ \psi_{uv} &= \frac{E_v}{2E} \psi_u + \frac{G_u}{2G} \psi_v \\ \psi_{vv} &= -\frac{G_u}{2E} \psi_u + \frac{G_v}{2G} \psi_v + \lambda_2 GN + G\psi \\ N_u &= -R \psi_u \\ N_v &= -\lambda_2 \psi_v\end{aligned}$$

and the Mainardi-Codazzi equations for the immersion ψ are

$$\begin{aligned}(R - \lambda_2)E_v &= 0 \\ (R - \lambda_2)\frac{G_u}{2G} + (R - \lambda_2)_u &= 0.\end{aligned}$$

Since $\lambda_2 \neq R$, the coefficient E does not depend on v , that is, $E = E(u)$. If we consider the new parameters

$$x = \int \sqrt{E(u)} \, du, \quad y = v$$

the structure equations become

$$\begin{aligned}\psi_{xx} &= RN + \psi \\ \psi_{xy} &= \frac{G_x}{2G} \psi_y \\ \psi_{yy} &= -\frac{G_x}{2} \psi_x + \frac{G_y}{2G} \psi_y + \lambda_2 GN + G\psi \\ N_x &= -R \psi_x \\ N_y &= -\lambda_2 \psi_y\end{aligned} \tag{2}$$

and the Mainardi-Codazzi equation is

$$(R - \lambda_2)\frac{G_x}{2G} + (R - \lambda_2)_x = 0, \tag{3}$$

whence the Gauss equation results

$$\left(\frac{G_x}{2G}\right)_x + \left(\frac{G_x}{2G}\right)^2 = 1 - R\lambda_2 = 1 - R(\lambda_2 - R) - R^2. \tag{4}$$

Thus, if we take

$$\varphi = \frac{1}{R - \lambda_2}$$

we obtain from (3) and (4) that

$$\varphi_x = \frac{G_x}{2G} \varphi$$

$$\varphi_{xx} = \left(\left(\frac{G_x}{2G} \right)_x + \left(\frac{G_x}{2G} \right)^2 \right) \varphi = R + (1 - R^2)\varphi. \quad (5)$$

Let γ_q be the maximal integral curve passing through a point $q = \psi(x_0, y_0) \in U$ for the principal curvature R . Then, from (2) it follows that $\gamma_q(t) = \psi(x_0 + t, y_0)$ satisfies

$$\begin{aligned} (\gamma_q)_{tt} &= R(N \circ \gamma_q) + \gamma_q \\ (N \circ \gamma_q)_t &= -R(\gamma_q)_t \end{aligned}$$

so that γ_q is a geodesic curve, which is a solution of the differential equation

$$(\gamma_q)_{tt} + (R^2 - 1)\gamma_q = R w_o$$

for a vector $w_o \in \mathbf{L}^4$. Therefore, γ_q is given by

$$\gamma_q = \cos(\sqrt{R^2 - 1} t) w_1 + \sin(\sqrt{R^2 - 1} t) w_2 + \frac{R}{R^2 - 1} w_o \quad \text{when } R > 1 \quad (6)$$

$$\gamma_q = w_1 + t w_2 + \frac{1}{2} t^2 w_o \quad \text{when } R = 1 \quad (7)$$

$$\gamma_q = \cosh(\sqrt{1 - R^2} t) w_1 + \sinh(\sqrt{1 - R^2} t) w_2 + \frac{R}{R^2 - 1} w_o \quad \text{when } 0 \leq R < 1 \quad (8)$$

for suitable vectors $w_1, w_2 \in \mathbf{L}^4$.

From (5), the principal curvature λ_2 can be calculated on γ_q as

$$R - \lambda_2 = \left(a \cos(\sqrt{R^2 - 1} t) + b \sin(\sqrt{R^2 - 1} t) + \frac{R}{R^2 - 1} \right)^{-1} \quad \text{when } R > 1 \quad (9)$$

$$R - \lambda_2 = \left(a + b t + \frac{1}{2} t^2 \right)^{-1} \quad \text{when } R = 1 \quad (10)$$

$$R - \lambda_2 = \left(a \cosh(\sqrt{1 - R^2} t) + b \sinh(\sqrt{1 - R^2} t) + \frac{R}{R^2 - 1} \right)^{-1} \quad \text{when } 0 \leq R < 1 \quad (11)$$

for real constants a, b .

Hence, if $\gamma_q(t_1)$ is the first umbilical point on γ_q , we obtain from (9), (10), (11) and the continuity of λ_2 that

$$0 = R - \lambda_2(\gamma_q(t_1)) = \lim_{t \rightarrow t_1} R - \lambda_2(\gamma_q(t)) \neq 0,$$

which is a contradiction. Therefore, there is not any umbilical point on γ_q . Moreover, since M is complete it follows that the geodesic γ_q is defined for all $t \in \mathbf{R}$.

Let \tilde{U} be the connected component of non umbilical points containing p . Note that \tilde{U} is an open set, and from the above reasoning, can be parametrized by $(x, y) \in (-\infty, \infty) \times (\beta_1, \beta_2)$ for certain β_1, β_2 , where $-\infty \leq \beta_1 < \beta_2 \leq \infty$, so that, since $R > 1$, the immersion can be expressed from (6) as

$$\psi(x, y) = \cos(\sqrt{R^2 - 1} x) w_1(y) + \sin(\sqrt{R^2 - 1} x) w_2(y) + \frac{R}{R^2 - 1} w_o(y) \quad (12)$$

Let us suppose now that there exists an umbilical point $\tilde{q} \in \partial\psi(\tilde{U})$. Then there exists a sequence of points $q_n = \psi(x_n, y_n)$ tending to \tilde{q} , being $(x_n, y_n) \in [0, 2\pi/\sqrt{R^2 - 1}] \times (\beta_1, \beta_2)$. Therefore the sequence of compact geodesics γ_n of length $2\pi/\sqrt{R^2 - 1}$ passing through q_n associated to the principal curvature R , converges to a compact geodesic $\gamma_{\tilde{q}}$ passing through \tilde{q} which is also a line of curvature for the eigenvalue R .

Now, from the above argument, it is sufficient to prove that there exists a non umbilical point on $\gamma_{\tilde{q}}$. In fact, from (9), we are able to choose a point $p_n \in \gamma_n$ such that $\lambda_2(p_n) = 1/R \neq R$. Finally, from an argument of compactness, there exists a subsequence $\{p_k\}$ of $\{p_n\}$ converging to a non umbilical point $\tilde{p} \in \gamma_{\tilde{q}}$.

Consequently M is umbilically free or totally umbilical.

Observe that (12) can be rewritten as

$$\psi(x, y) = \frac{1}{\sqrt{R^2 - 1}} \left(-R\alpha(y) + \cos(\sqrt{R^2 - 1}x) v_1(y) + \sin(\sqrt{R^2 - 1}x) v_2(y) \right)$$

where

$$\alpha = \frac{-1}{\sqrt{R^2 - 1}} w_o, \quad v_1 = \sqrt{R^2 - 1} w_1, \quad v_2 = \sqrt{R^2 - 1} w_2.$$

From the expressions of ψ and ψ_{xx} , the Gauss map N can be calculated using the first equation in (2). Thus, since $\langle \psi, \psi \rangle = -1$, $\langle \psi_x, \psi_x \rangle = 1$, $\langle N, N \rangle = 1$ and they are mutually orthogonal, it follows that $\{\alpha, v_1, v_2\}$ are orthogonal, and $\langle \alpha, \alpha \rangle = -1$, $\langle v_1, v_1 \rangle = 1$ and $\langle v_2, v_2 \rangle = 1$.

On the other hand, since ψ_y is orthogonal to ψ, ψ_x and N , we get that

$$\alpha' = \mu_o P, \quad v_1' = \mu_1 P, \quad v_2' = \mu_2 P,$$

where P is the vectorial product of α, v_1 and v_2 in \mathbf{L}^4 , and μ_o, μ_1, μ_2 are C^∞ functions. Hence, since

$$\langle \psi_y, \psi_y \rangle = \frac{1}{R^2 - 1} \left(-R\mu_o + \cos(\sqrt{R^2 - 1}x) \mu_1 + \sin(\sqrt{R^2 - 1}x) \mu_2 \right)^2$$

and it is positive, it follows that $\mu_o \neq 0$ and therefore α is a regular curve with tangent vector P . In particular, $\{v_1(y), v_2(y)\}$ is an orthonormal frame of the normal plane along α . Finally, the completeness of α follows from the completeness of M .

The converse is a straightforward computation, which finishes the proof of Theorem 1.

On the other hand, if M is compact the cases (7) and (8) are not possible, that is, necessarily $R > 1$. Moreover, if M is not totally umbilical then from Theorem 1 we have that M is umbilically free. Now, the Poincaré Theorem ([2, Theorem II, p. 103]) allows us to state that M is a torus since the field of line elements associated to the principal curvature R has not got any singularity.

Besides, M is foliated by the geodesic circles in \mathbf{H}^3 associated to the principal curvature R with center α . Therefore, α is a closed curve and Theorem 2 follows from Theorem 1.

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