

Some Estimates for the Curvatures of Complete Spacelike Hypersurfaces in Generalized Robertson-Walker spacetimes

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Abstract. In this paper we obtain some estimates for the higher order mean curvatures, the scalar curvature and the Ricci curvature of a complete spacelike hypersurface in a Generalized Robertson-Walker (GRW) spacetime, under certain assumptions on the warped function of the ambient space. Our results will be an application of a generalized maximum principle due to Omori.

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1 Introduction

Our results will be an application of the following generalized maximum principle for Riemannian manifolds given by Omori [2] (see also Yau's paper [3]).

A generalized maximum principle: *Let M be a complete Riemannian manifold whose sectional curvatures are bounded away from $-\infty$ and let $u : M \rightarrow \mathbf{R}$ be a smooth function bounded from above. Then, for each $\varepsilon > 0$ there exists a point $p_\varepsilon \in M$ such that*

- i) $|\nabla u(p_\varepsilon)| < \varepsilon$,*
- ii) $\nabla^2 u_{p_\varepsilon}(v, v) < \varepsilon$, for all tangent vector $v \in T_p M$, $|v| = 1$,*
- iii) $\sup u - \varepsilon < u(p_\varepsilon) \leq \sup u$,*

where ∇u and $\nabla^2 u$ denote, respectively, the gradient and the Hessian of u .

2 Preliminaries

Let (F, g) be an n -dimensional ($n \geq 2$) Riemannian manifold and let $I \subset \mathbf{R}$ be an open interval in \mathbf{R} endowed with the metric $-dt^2$. The warped product $\overline{M} = I \times_f F$ endowed with the Lorentzian metric

$$\langle \cdot, \cdot \rangle = \pi_I^*(-dt^2) + f^2(\pi_I)\pi_F^*(g),$$

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where $f > 0$ is a smooth function on I , and π_I and π_F denote the projections onto I and F respectively, is said to be a *Generalized Robertson-Walker* (GRW) spacetime with *base* $(I, -dt^2)$, *fiber* (F, g) and *warping function* f (see [1]).

A smooth immersion $\psi : M^n \rightarrow \overline{M}$ of an n -dimensional connected manifold M is said to be a *spacelike hypersurface* if the induced metric via ψ is a Riemannian metric on M , which, as usual, is also denoted by $\langle \cdot, \cdot \rangle$.

Observe that the timelike vector field $\partial_t = \partial/\partial t \in \mathfrak{X}(\overline{M})$ determines a time-orientation on \overline{M} . Thus, if $\psi : M^2 \rightarrow \mathbf{S}_1^3 \subset \mathbf{L}^4$ is a spacelike hypersurface, then we can choose a unique (globally defined) timelike unit normal field N on M , which is in the same time-orientation that ∂_t , that is, $\langle \partial_t, N \rangle \leq -1$, and hence we may assume that M is oriented by N . We will refer to N as the *Gauss map* of M .

In order to set up the notation to be used later, we will denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections of \overline{M} and M , respectively. Then the Gauss and Weingarten formulas for M in \overline{M} are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y - \langle AX, Y \rangle N \quad (1)$$

and

$$A(X) = -\overline{\nabla}_X N \quad (2)$$

for all tangent vector fields $X, Y \in \mathfrak{X}(M)$, where A stands for the shape operator of M in \overline{M} with respect to N .

Associated to the shape operator of M there are n algebraic invariants, which are the elementary symmetric functions σ_r of its principal curvatures k_1, \dots, k_n , given by

$$\sigma_j(k_1, \dots, k_n) = \sum_{i_1 < \dots < i_j} k_{i_1} \cdots k_{i_j}, \quad 1 \leq j \leq n.$$

The j -th mean curvature H_j of the spacelike hypersurface is then defined by

$$\binom{n}{j} H_j = (-1)^j \sigma_j(k_1, \dots, k_n) = \sigma_j(-k_1, \dots, -k_n). \quad (3)$$

When $j = 1$, $H_1 = -(1/n)\text{tr}(A) = H$ is the mean curvature of M . The choice of the sign $(-1)^j$ in our definition of H_j is motivated by the fact that in that case the mean curvature vector is given by $\vec{H} = HN$. Therefore, $H(p) > 0$ at a point $p \in M$ if and only if $\vec{H}(p)$ is in the time-orientation determined by $N(p)$. On the other hand, when $j = n$, $H_n = (-1)^n \det(A)$ defines the Gauss-Kronecker curvature of the spacelike hypersurface.

The spacelike slices $F(t_o) = \{t_o\} \times F$, $t_o \in I$, will play an important role in our work. As it can be easily seen, $F(t_o)$ has shape operator $A = -f'(t_o)/f(t_o)$, so that $F(t_o)$ is a totally umbilical hypersurface of \overline{M} with constant j -th mean curvature

$$h_j(r) = \left(\frac{f'(t_o)}{f(t_o)} \right)^j.$$

3 Main results

Theorem 1 Let $\overline{M} = I \times_f F$ a be GRW spacetime, $\psi: M^2 \rightarrow \mathbf{S}_1^3 \subset \mathbf{L}^4$ a complete spacelike hypersurface whose sectional curvatures are bounded away from $-\infty$, and let us orient M in the same time-orientation as ∂_t . Let us denote by u the smooth function on M given by $u = \pi_I \circ \psi$.

a) If $f'(u) \gg 0$ on M and there exists $\alpha = \inf(u)$, then

$$\sup H_j \geq \left(\frac{f'(\alpha)}{f(\alpha)} \right)^j.$$

b) If $f'(u) \ll 0$ on M and there exists $\beta = \sup(u)$, then

$$\sup H_j \geq \left(\frac{f'(\beta)}{f(\beta)} \right)^j$$

if j is even, and

$$\inf H_j \leq \left(\frac{f'(\beta)}{f(\beta)} \right)^j$$

if j is odd.

Proof: From the Gauss formula (1) it is not difficult to see that the gradient of u is $-\partial_t^\top$, where

$$\partial_t^\top = \partial_t + \langle \partial_t, N \rangle N \in \mathfrak{X}(M)$$

denotes the tangential component of ∂_t . On the other hand, using (1) and (2), we also obtain that the Hessian of u is given by

$$\nabla^2 u(X, Y) = -\frac{f'(u)}{f(u)} (\langle X, Y \rangle + \langle X, \partial_t^\top \rangle \langle Y, \partial_t^\top \rangle) + \langle N, \partial_t \rangle \langle AX, Y \rangle \quad (4)$$

for all tangent vector field $X, Y \in \mathfrak{X}(M)$.

a) Since u is a smooth function on M bounded from below by $\alpha = \inf(u)$, we know from the generalized maximum principle that for each $\varepsilon_m = 1/m$ there exists a point $p_m \in M$ such that

$$|\nabla u(p_m)| < \varepsilon_m, \quad (5)$$

$$\nabla^2 u_{p_m}(v, v) > -\varepsilon_m, \quad (6)$$

for all tangent vector $v \in T_{p_m} M$, $|v| = 1$, and

$$\alpha + \varepsilon_m > u(p_m) \geq \alpha. \quad (7)$$

Let $\{e_i^m\}_{i=1, \dots, n}$ an orthonormal basis of principal directions at the point p_m satisfying $A_{p_m}(e_i^m) = k_i(p_m)e_i^m$. We obtain from (6) and equation (4) that

$$\nabla^2 u(e_i^m, e_i^m) = -\frac{f'(u(p_m))}{f(u(p_m))} \left(1 + \langle e_i^m, \partial_t^\top(p_m) \rangle^2 \right) + \langle N(p_m), \partial_t(p_m) \rangle k_i(p_m)$$

and, since $\langle N(p_m), \partial_t(p_m) \rangle < 0$, it follows that

$$k_i(p_m) < \frac{1}{\langle N(p_m), \partial_t(p_m) \rangle} \left(\frac{-1}{m} + \frac{f'(u(p_m))}{f(u(p_m))} \left(1 + \langle e_i^m, \partial_t^\top(p_m) \rangle^2 \right) \right). \quad (8)$$

Since $f'(u) \ll 0$ on M , $k_i(p_m)$ is negative for m sufficiently big. Let us assume from now on that m is big enough such that $k_i(p_m) < 0$. Then we get

$$\binom{n}{j} H_j(p_m) \geq \binom{n}{j} \left(\frac{-1}{\langle N(p_m), \partial_t(p_m) \rangle} \left(\frac{-1}{m} + \frac{f'(u(p_m))}{f(u(p_m))} \left(1 + \langle e_i^m, \partial_t^\top(p_m) \rangle^2 \right) \right) \right)^j$$

and letting $m \rightarrow \infty$ it follows that

$$\sup H_j \geq \left(\frac{f'(\alpha)}{f(\alpha)} \right)^j.$$

b) From a similar argument, for each $\varepsilon_m = 1/m$ there exists a point $q_m \in M$ such that

$$|\nabla u(q_m)| < \varepsilon_m, \quad (9)$$

$$\nabla^2 u_{q_m}(v, v) < \varepsilon_m, \quad (10)$$

for all tangent vector $v \in T_{(q_m)}M$, $|v| = 1$, and

$$\beta - \varepsilon_m < u(q_m) \leq \beta. \quad (11)$$

An analogous reasoning allows us to obtain

$$k_i(q_m) > \frac{1}{\langle N(q_m), \partial_t(q_m) \rangle} \left(\frac{1}{m} + \frac{f'(u(q_m))}{f(u(q_m))} \left(1 + \langle e_i^m, \partial_t^\top(q_m) \rangle^2 \right) \right), \quad (12)$$

being $k_i(q_m)$ positive for m sufficiently big. Hence we have:

- If j is even, letting $m \rightarrow \infty$ it follows that

$$\sup H_j \geq \left(\frac{f'(\beta)}{f(\beta)} \right)^j.$$

- If j is odd, letting $m \rightarrow \infty$ it results

$$\inf H_j \leq \left(\frac{f'(\beta)}{f(\beta)} \right)^j.$$

■

Remark 2 Deberíamos remarcar que las estimaciones de las H_j son los valores de las curvaturas medias de orden superior de una slice.

The classical Maximum principle allows us to obtain the following estimates for the higher order mean curvatures of a spacelike hypersurface when the function u arise a local maximum or minimum on M and f' has a suitable sign at such points:

Theorem 3 Let $\overline{M} = I \times_f F$ a be GRW spacetime, $\psi : M^2 \rightarrow \mathbf{S}_1^3 \subset \mathbf{L}^4$ a spacelike hypersurface and let us orient M in the same time-orientation as ∂_t . Let us denote by u the smooth function on M given by $u = \pi_I \circ \psi$.

a) If u attains a local minimum at a point $p_1 \in M$ and $f'(u(p_1)) \geq 0$, then

$$\sup H_j \geq \left(\frac{f'(u(p_1))}{f(u(p_1))} \right)^j.$$

b) If u attains a local maximum at a point $p_2 \in M$ and $f'(u(p_2)) \leq 0$, then

$$\sup H_j \geq \left(\frac{f'(u(p_2))}{f(u(p_2))} \right)^j.$$

if j is even, and

$$\inf H_j \leq \left(\frac{f'(u(p_2))}{f(u(p_2))} \right)^j$$

if j is odd.

Proof: Since u attains a local minimum at p_1 we have that $\nabla u(p_1) = -\partial_t^\top = 0$ and $\nabla^2(p_1)$ is negative semidefinite. A similar argument as in Theorem 1 allows us to obtain the result.

The proof of *ii*) is analogous. ■

If the ambient space \overline{M} has constant sectional curvature \bar{c} , then the Ricci curvature of M is given by

$$\text{Ric}(X, Y) = \bar{c}(n-1)\langle X, Y \rangle - \text{tr}(A)\langle A(X), Y \rangle + \langle A(X), A(Y) \rangle, \quad (13)$$

for $X, Y \in \mathfrak{X}(M)$. Under this additional hypothesis, the second higher order mean curvature H_2 is, up to a constant, the scalar curvature S of M ; indeed

$$S = n(n-1)(\bar{c} - H_2). \quad (14)$$

The spacelike slices $F(t_o)$ of such spacetimes have constant Ricci curvature given by

$$(n-1) \left(\bar{c} - \left(\frac{f'(t_o)}{f(t_o)} \right)^2 \right) \langle \cdot, \cdot \rangle$$

and constant scalar curvature

$$n(n-1) \left(\bar{c} - \left(\frac{f'(t_o)}{f(t_o)} \right)^2 \right).$$

We are able to give the following estimations for the Ricci and scalar curvatures of complete spacelike hypersurfaces in GRW spacetimes of constant sectional curvature:

Theorem 4 Let $\overline{M} = I \times_f F$ a be GRW spacetime of constant sectional curvature \bar{c} , $\psi : M^2 \rightarrow \mathbf{S}_1^3 \subset \mathbf{L}^4$ a complete spacelike hypersurface whose sectional curvatures are bounded away from $-\infty$, and let us orient M in the same time-orientation as ∂_t . Let us denote by u the smooth function on M given by $u = \pi_I \circ \psi$.

a) If $f'(u) \gg 0$ on M and there exists $\alpha = \inf(u)$, then

$$\inf_{\substack{p \in M \\ v \in T_p M \\ |v|=1}} \text{Ric}_p(v, v) \leq (n-1) \left(\bar{c} - \left(\frac{f'(\alpha)}{f(\alpha)} \right)^2 \right)$$

and

$$\inf S \leq n(n-1) \left(\bar{c} - \left(\frac{f'(\alpha)}{f(\alpha)} \right)^2 \right).$$

b) If $f'(u) \ll 0$ on M and there exists $\beta = \sup(u)$, then

$$\inf_{\substack{p \in M \\ v \in T_p M \\ |v|=1}} \text{Ric}_p(v, v) \leq (n-1) \left(\bar{c} - \left(\frac{f'(\beta)}{f(\beta)} \right)^2 \right)$$

and

$$\inf S \leq n(n-1) \left(\bar{c} - \left(\frac{f'(\beta)}{f(\beta)} \right)^2 \right).$$

Proof: Under the assumptions of a), we obtain from (13) and (8) that

$$\begin{aligned} \text{Ric}(e_j^m, e_j^m) &= \bar{c}(n-1) - \sum_{i=1}^n k_i(p_m)k_j(p_m) + k_j^2(p_m) \\ &= \bar{c}(n-1) - \sum_{i \neq j} k_i(p_m)k_j(p_m) \\ &\leq \bar{c}(n-1) - (n-1) \left(\frac{1}{\langle N(p_m), \partial_t(p_m) \rangle} \left(\frac{-1}{m} + \frac{f'(u(p_m))}{f(u(p_m))} \left(1 + \langle e_i^m, \partial_t^\top(p_m) \rangle^2 \right) \right) \right)^2, \end{aligned}$$

and letting $m \rightarrow \infty$ we get that

$$\inf_{\substack{p \in M \\ v \in T_p M \\ |v|=1}} \text{Ric}_p(v, v) \leq (n-1) \left(\bar{c} - \left(\frac{f'(\alpha)}{f(\alpha)} \right)^2 \right).$$

Tacking into account that $S = \text{tr}(\text{Ric})$, it follows the result on the scalar curvature.

The case b) is analogous. ■

Similarly, we have:

Theorem 5 Let $\bar{M} = I \times_f F$ be GRW spacetime of constant sectional curvature \bar{c} , $\psi : M^2 \rightarrow \mathbf{S}_1^3 \subset \mathbf{L}^4$ a spacelike hypersurface, and let us orient M in the same time-orientation as ∂_t . Let us denote by u the smooth function on M given by $u = \pi_I \circ \psi$.

a) If u attains a local minimum at a point $p_1 \in M$ and $f'(u(p_1)) \geq 0$, then

$$\inf_{\substack{p \in M \\ v \in T_p M \\ |v|=1}} \text{Ric}_p(v, v) \leq (n-1) \left(\bar{c} - \left(\frac{f'(u(p_1))}{f(u(p_1))} \right)^2 \right)$$

and

$$\inf S \leq n(n-1) \left(\bar{c} - \left(\frac{f'(u(p_1))}{f(u(p_1))} \right)^2 \right).$$

b) If u attains a local maximum at a point $p_2 \in M$ and $f'(u(p_2)) \leq 0$, then

$$\inf_{\substack{p \in M \\ v \in T_p M \\ |v|=1}} \text{Ric}_p(v, v) \leq (n-1) \left(\bar{c} - \left(\frac{f'(u(p_2))}{f(u(p_2))} \right)^2 \right)$$

and

$$\inf S \leq n(n-1) \left(\bar{c} - \left(\frac{f'(u(p_2))}{f(u(p_2))} \right)^2 \right).$$

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