

A Weierstrass Representation for Linear Weingarten Spacelike Surfaces of Maximal Type in the Lorentz-Minkowski Space

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Abstract. In this work we extend the Weierstrass representation for maximal spacelike surfaces in the 3-dimensional Lorentz-Minkowski space to spacelike surfaces whose mean curvature is proportional to its Gaussian curvature (linear Weingarten surfaces of maximal type). We use this representation in order to study the Gaussian curvature and the Gauss map of such surfaces when the immersion is complete, proving that the surface is a plane or the supremum of its Gaussian curvature is a negative constant and its Gauss map is a diffeomorphism onto the hyperbolic plane. Finally, we classify the rotation linear Weingarten surfaces of maximal type.

1 Introduction

The global study of surfaces with constant mean curvature or Gaussian curvature in space forms has been of a special interest in Submanifolds Geometry. In this sense, Liebmann (1899) and Hilbert (1901) proved independently that the totally umbilical round spheres are the unique ovaloids with constant mean curvature H and the unique compact surfaces with constant Gaussian curvature K in the Euclidean space.

If we consider the class of Weingarten surfaces, that is, surfaces whose principal curvatures λ_1, λ_2 satisfy the relation $W(\lambda_1, \lambda_2) = 0$ for a function W , then the above results were improved by H. Hopf [6]. Indeed, he proved that an orientable analytic closed Weingarten surface of genus zero for which one principal curvature is a monotone decreasing function of the other in a neighbourhood of

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an umbilical point must be a round sphere. Later, Hartman and Wintner [5] and Chern [2] proved that Hopf's result remains true if the surface is only of class C^2 .

Recently, some results of a great interest on Weingarten surfaces have been proved by Rosenberg, Sa Earp and Toubiana. They consider surfaces in the Euclidean or hyperbolic space whose mean curvature H and extrinsic curvature K_e satisfy an elliptic equation $H = f(H^2 - K_e)$, being f a function defined on a connected interval containing zero, and show that the behaviour of such surfaces depends strongly on the value of $f(0)$.

On the one hand, when $f(0)$ is positive Rosenberg and Sa Earp [13] prove that the surfaces behave like a nonzero constant mean curvature surface. Moreover, they extend some results of Meeks [10] and Korevaar, Kusner and Solomon [8] about annular ends when the surface satisfies height estimates. Even more, in this case Sa Earp and Toubiana characterize and classify the complete rotation surfaces, and extend some results about constant mean curvature and constant Gaussian curvature using the Alexandrov reflection technique (see [15], [16]).

On the other hand, when $f(0) = 0$ the surface behaves like a minimal surface. Indeed, Sa Earp and Toubiana [14] show that a "half space theorem" and a "Bernstein theorem" also hold for these surfaces. Moreover, they study and classify the revolution examples. In particular this family includes the surfaces satisfying $2aH + bK = 0$, $a \neq 0$, which verify that the mean of their curvature radii is constant.

In this paper we consider spacelike Weingarten surfaces in the Lorentz-Minkowski space \mathbf{L}^3 satisfying $-2aH + bK = 0$, being a , b two real constants with $a \neq 0$. These surfaces, which generalize the maximal ones in \mathbf{L}^3 , have a special interest as solutions of a variational problem [13].

Our main goal is to obtain a conformal representation for this kind of surfaces (Theorem 9) which extends the known one for maximal surfaces ([9], [7]). As it is well-known (see [11]), one of the most important keys in order to obtain the Weierstrass representation of a minimal surface in the Euclidean space is that its Gauss map is conformal for the induced metric. In our case, the difficulty is to take a special metric on the surface adapted to the problem, such that the Gauss map is also conformal (Lemma 1) and the Laplacian of the immersion with respect to this metric has a good behaviour (Lemma 5).

In comparison with the maximal case, we prove that for any $b \neq 0$ there exist non flat complete spacelike surfaces. Thus, we spend Section 3 on proving some properties about the geometric behaviour of complete spacelike surfaces. Actually, we prove that a non flat complete spacelike surface satisfying $-2aH + bK = 0$ has negative curvature everywhere, in such a way that out of every compact set on the surface the supremum of its Gaussian curvature is $-4a^2/b^2$ (Theorem 12 and Remark 13).

On the other hand, the study of the Gauss map is of special interest. So, we show that the Gauss map of a non flat complete spacelike surface satisfying $-2aH + bK = 0$ is a global diffeomorphism onto the unit disk (Theorem 15).

Finally, in Section 4, as an application of the above results we construct explicitly the rotation examples (Examples 17, 18 and 19), study their completeness and classify them (Theorem 20).

2 Weierstrass Representation

Let \mathbf{L}^3 be the 3-dimensional *Lorentz-Minkowski space*, that is, the real vector space \mathbf{R}^3 endowed with the Lorentzian metric tensor \langle, \rangle given by

$$\langle, \rangle = dx_1^2 + dx_2^2 - dx_3^2,$$

where (x_1, x_2, x_3) are the canonical coordinates of \mathbf{R}^3 . An immersion $\psi : M^2 \rightarrow \mathbf{L}^3$ of a 2-dimensional connected manifold M is said to be a *spacelike surface* if the induced metric via ψ is a Riemannian metric on M , which, as usual, is also denoted by \langle, \rangle . It is well-known that such a surface is orientable, namely, we can choose a unit timelike normal vector field N globally defined on M that we will call the *Gauss map* of the immersion.

We will denote by $H = -\text{trace}(A)/2$ and $K = -\det(A)$ the mean and Gaussian curvatures of M respectively, where $A : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ stands for the shape operator of M in \mathbf{L}^3 associated to N , given by $A = -dN$.

Following the ideas of Rosenberg, Sa Earp and Toubiana in [13] and [14], we will say that $\psi : M^2 \rightarrow \mathbf{L}^3$ is a *linear Weingarten spacelike surface of maximal type*, in short, an LWM-spacelike surface, if there exist $a, b \in \mathbf{R}$, $a \neq 0$, satisfying

$$-2aH + bK = 0. \quad (1)$$

Note that if we consider the homothety of \mathbf{L}^3 $h(x) = \mu x$, for $\mu > 0$, then it is well-known that

$$H = \mu \tilde{H}, \quad K = \mu^2 \tilde{K}$$

where \tilde{H}, \tilde{K} are the mean and Gaussian curvatures of the new immersion $\tilde{\psi} = h \circ \psi$. Therefore, $\tilde{\psi}$ is also a LWM-spacelike surface satisfying

$$-2a\tilde{H} + \mu b\tilde{K} = 0.$$

Now, let us denote by σ the symmetric tensor on M for the immersion ψ

$$\sigma(X, Y) = a\langle X, Y \rangle - b\langle AX, Y \rangle, \quad X, Y \in \mathfrak{X}(M),$$

which is a definite metric on M . In fact, if we take a local orthonormal frame $\{E_1, E_2\}$ for the induced metric \langle, \rangle such that $AE_i = \lambda_i E_i$, $i = 1, 2$, being λ_1, λ_2 the principal curvatures associated to A , then

$$\sigma(E_i, E_j) = (a - b\lambda_i)\delta_{ij}, \quad i, j = 1, 2,$$

where δ_{ij} is the Kronecker delta, so that

$$\det(\sigma) = (a - b\lambda_1)(a - b\lambda_2) = a^2 - b(-2aH + bK) = a^2 > 0. \quad (2)$$

Moreover, we can assume that σ is positive definite; otherwise, just replace N by $-N$ and a by $-a$ to (1) be satisfied. From now on, we will choose N such that σ is a Riemannian metric.

Note that, by parallel translation to the origin in \mathbf{L}^3 , we can regard the unit normal vector field N as a map $N : M^2 \rightarrow \mathbf{H}^2$, where \mathbf{H}^2 denotes the two sheeted hyperboloid

$$\mathbf{H}^2 = \{x \in \mathbf{L}^3 : \langle x, x \rangle = -1\}.$$

Then we have the following:

Lemma 1 *The Gauss map $N : (M, \sigma) \rightarrow \mathbf{H}^2$ of an LWM-spacelike surface in \mathbf{L}^3 is conformal.*

Proof: Let $\{E_1, E_2\}$ be a local orthonormal frame for the induced metric such that $AE_i = \lambda_i E_i$, $i = 1, 2$. Then N is conformal if and only if

$$\frac{\lambda_1^2}{a - b\lambda_1} = \frac{\lambda_2^2}{a - b\lambda_2},$$

or equivalently

$$(\lambda_1 - \lambda_2)(-2aH + bK) = 0,$$

which is true. ■

Remark 2 From the Cauchy-Schwarz inequality, $H^2 + K \geq 0$, and (1), it follows that

$$0 \leq \frac{b^2 K^2}{4a^2} + K = K \left(\frac{b^2}{4a^2} K + 1 \right).$$

Therefore, either $K \geq 0$ everywhere, or $b \neq 0$ and $K \leq -4a^2/b^2$ on M .

Thus, it is clear that N reverses the orientation if and only if $K \geq 0$ everywhere. ■

Observe that, up to a symmetry of \mathbf{L}^3 , we can suppose that the image of N lies on

$$\mathbf{H}_+^2 = \{x \in \mathbf{H}^2 : x_3 > 0\}.$$

Let us introduce complex coordinates in \mathbf{H}_+^2 using the usual stereographic projection $\pi : \mathbf{H}_+^2 \rightarrow \mathbb{D}$ from the hyperbolic plane \mathbf{H}_+^2 onto the unit disk \mathbb{D} given by

$$\pi(x_1, x_2, x_3) = \frac{x_1 - ix_2}{1 + x_3},$$

with inverse map

$$\pi^{-1}(z) = \left(\frac{z + \bar{z}}{1 - |z|^2}, i \frac{z - \bar{z}}{1 - |z|^2}, \frac{1 + |z|^2}{1 - |z|^2} \right). \quad (3)$$

From the above comments one has

Lemma 3 *Let $\psi : M^2 \rightarrow \mathbf{L}^3$ be an LWM-spacelike surface. If we consider M as a Riemann surface with the conformal structure induced by σ , then $g = \pi \circ N$ is a conformal map from M in \mathbb{D} which preserves the orientation if and only if the Gaussian curvature of the immersion is non-negative everywhere.*

Moreover, if M is simply-connected then M is conformally equivalent to the unit disk or $\psi(M)$ is a plane.

Proof: Bearing in mind that π is a reversing orientation conformal map, it is clear from Remark 2 that g preserves the orientation if and only if the Gaussian curvature of the immersion is non-negative everywhere.

Moreover, if M is simply-connected then from Kokubu uniformization Theorem M is conformally equivalent to the unit disk or the complex plane since M cannot be compact.

But, if M is conformal to the complex plane then the bounded conformal map g must be constant. Therefore, N is also constant and $\psi(M)$ lies on a plane. Besides, since the shape operator vanishes identically then its conformal structure is given by the induced metric and $\psi(M)$ must be the whole plane. ■

From now on, we will also refer to $g = \pi \circ N$ as the Gauss map of the surface.

Remark 4 It is worth pointing out that the conjugate map of π , $\bar{\pi}$, is also conformal and conserves the orientation. Thus, in the case of negative Gaussian curvature we can change π for $\bar{\pi}$ in order g to preserve the orientation. However, we have thought advisable to keep the same map in both cases to simplify the above development. ■

In order to obtain the Weierstrass representation, it is fundamental the fact that a linear combination of ψ and N is harmonic for σ .

Lemma 5 *Let $\psi : M^2 \rightarrow \mathbf{L}^3$ be an LWM-spacelike surface satisfying $-2aH + bK = 0$, $a \neq 0$. Then*

$$\Delta^\sigma \psi = -\frac{bK}{a^2} N,$$

where Δ^σ denotes the divergence of the gradient with respect to the Riemannian metric σ , that is, the Laplacian of σ .

Proof: Let (u, v) be isothermal parameters for the induced metric, that is,

$$\begin{aligned} \langle d\psi, d\psi \rangle &= E (du^2 + dv^2) \\ \langle d\psi, dN \rangle &= e du^2 + 2f dudv + g dv^2. \end{aligned}$$

Then the structure equations can be written as

$$\begin{aligned} \psi_{uu} &= \frac{E_u}{2E} \psi_u - \frac{E_v}{2E} \psi_v + eN \\ \psi_{uv} &= \frac{E_v}{2E} \psi_u + \frac{E_u}{2E} \psi_v + fN \\ \psi_{vv} &= -\frac{E_u}{2E} \psi_u + \frac{E_v}{2E} \psi_v + gN \\ N_u &= \frac{e}{E} \psi_u + \frac{f}{E} \psi_v \\ N_v &= \frac{f}{E} \psi_u + \frac{g}{E} \psi_v \end{aligned} \tag{4}$$

the mean and Gaussian curvatures are

$$H = \frac{e+g}{2E}, \quad K = \frac{f^2 - eg}{E^2}, \tag{5}$$

the Gauss equation is given by

$$K = -\frac{1}{2E} \left(\left(\frac{E_u}{E} \right)_u + \left(\frac{E_v}{E} \right)_v \right)$$

and the Codazzi-Mainardi equations are

$$\begin{aligned} e_v - f_u &= HE_v \\ g_u - f_v &= HE_u \end{aligned}$$

Then, if $h : M \rightarrow \mathbf{R}$ is a smooth function, a straightforward computation gives

$$\Delta^\sigma h = \frac{1}{a^2 E} \left[\frac{\partial}{\partial u} \left(\left(a + b \frac{g}{E} \right) h_u - b \frac{f}{E} h_v \right) + \frac{\partial}{\partial v} \left(\left(a + b \frac{e}{E} \right) h_v - b \frac{f}{E} h_u \right) \right]. \quad (6)$$

Hence, using the Codazzi-Mainardi equations we obtain

$$\langle \Delta^\sigma \psi, \psi_u \rangle = \frac{b}{a^2 E^2} \left(E(g_u - f_v) - gE_u + \frac{1}{2} E_u(g - e) \right) = 0$$

and analogously

$$\langle \Delta^\sigma \psi, \psi_v \rangle = 0.$$

As regards to the normal component of $\Delta^\sigma \psi$ we have

$$\langle \Delta^\sigma \psi, N \rangle = \frac{1}{a^2} (-2aH + 2bK) = \frac{bK}{a^2},$$

whence

$$\Delta^\sigma \psi = -\frac{bK}{a^2} N.$$

as we wanted to prove. ■

Lemma 6 *In the conditions of Lemma 5 we have*

$$\Delta^\sigma N = \frac{2}{a} KN$$

Proof: Analogously to Lemma 5 it can be seen, using the Codazzi-Mainardi equations that $\Delta^\sigma N$ is normal to the immersion. Anyway, it follows immediately because N is conformal.

Moreover, if we consider isothermal parameters (u, v) for the induced metric as in the above Lemma, the normal part of the immersion can be calculated as

$$\begin{aligned} \langle \Delta^\sigma N, N \rangle &= \frac{-1}{a^2 E} \left(\left\langle \left(a + b \frac{g}{E} \right) N_u - b \frac{f}{E} N_v, N_u \right\rangle + \left\langle \left(a + b \frac{e}{E} \right) N_v - b \frac{f}{E} N_u, N_v \right\rangle \right) \\ &= -\frac{1}{a^2} (a(4H^2 + 2K) - 2bHK) = -\frac{2}{a} K \end{aligned} \quad (7)$$

where (1), (4), (5) and (6) are used. ■

Remark 7 Since $\langle \Delta^\sigma N, N \rangle = -\langle \nabla^\sigma N, \nabla^\sigma N \rangle$, then we obtain from (7) that $K \geq 0$ if and only if $a > 0$. ■

As an immediate consequence of the two above Lemmas one gets

Corollary 8 Let $\psi : M^2 \longrightarrow \mathbf{L}^3$ be an LWM-spacelike surface satisfying $-2aH + bK = 0$, $a \neq 0$. Then

$$\Delta^\sigma (2a\psi + bN) = 0.$$

Now we are ready to obtain the conformal representation:

Theorem 9 Let $\psi : M^2 \longrightarrow \mathbf{L}^3$ be an LWM-spacelike surface such that $-2aH + bK = 0$, $a \neq 0$, and let us consider on M the conformal structure induced by σ . Then there exists a 1-form ω such that the immersion can be recovered as

$$\psi = -\frac{b}{2a} \left(\frac{g + \bar{g}}{1 - |g|^2}, i \frac{g - \bar{g}}{1 - |g|^2}, \frac{1 + |g|^2}{1 - |g|^2} \right) + \frac{1}{2a} \operatorname{Re} \int (1 + g^2, -i(1 - g^2), 2g) \omega, \quad (8)$$

where $g : M \longrightarrow \mathbb{D}$ is its Gauss map.

Moreover:

i) If $K \geq 0$ then g and ω are holomorphic and satisfy

$$|\omega|^2 > 4b^2 \frac{|dg|^2}{(1 - |g|^2)^4}. \quad (9)$$

ii) If $K < 0$ then g and ω are anti-holomorphic and satisfy

$$|\omega|^2 < 4b^2 \frac{|dg|^2}{(1 - |g|^2)^4}. \quad (10)$$

Conversely, given a simply-connected Riemann surface M , a map $g : M \longrightarrow \mathbb{D}$ and an 1-form ω on M

i) If g and ω are holomorphic and satisfy (9), then (8) defines an LWM-spacelike immersion.

ii) If g and ω are anti-holomorphic and satisfy (10), then (8) defines an LWM-spacelike immersion.

In both cases ψ verifies $-2aH + bK = 0$, $a \neq 0$, with Gaussian curvature

$$K = \frac{16a^2 |dg|^2}{(1 - |g|^2)^4 |\omega|^2 - 4b^2 |dg|^2}, \quad (11)$$

being g the Gauss map of the immersion, and

$$\begin{aligned} \langle d\psi, d\psi \rangle &= -\frac{b}{2a^2} dg \omega + \frac{(1 - |g|^2)^2}{4a^2} |\omega|^2 + \frac{b^2}{a^2(1 - |g|^2)^2} |dg|^2 - \frac{b}{2a^2} d\bar{g} \bar{\omega} \\ \langle d\psi, dN \rangle &= \frac{1}{2a} dg \omega - 2 \frac{b}{a(1 - |g|^2)^2} |dg|^2 + \frac{1}{2a} d\bar{g} \bar{\omega}. \end{aligned} \quad (12)$$

Proof: Let $\psi : M^2 \longrightarrow \mathbf{L}^3$ be an LWM-spacelike surface satisfying $-2aH + bK = 0$, $a \neq 0$, with non negative Gaussian curvature. Then, from Lemma 3, the Gauss map $g = \pi \circ N$ is holomorphic, that is, $g_{\bar{z}} = 0$ for a local conformal parameter z on M .

If we define

$$\Phi_i = \phi_i dz = (2a\psi_i + bN_i)_z dz, \quad i = 1, 2, 3, \quad (13)$$

then Φ_i , $i = 1, 2, 3$, are holomorphic 1-forms from Corollary 8.

On the other hand, since

$$\langle (\phi_1, \phi_2, \phi_3), N \rangle = 0$$

and from (3)

$$N = \left(\frac{g + \bar{g}}{1 - |g|^2}, i \frac{g - \bar{g}}{1 - |g|^2}, \frac{1 + |g|^2}{1 - |g|^2} \right), \quad (14)$$

we obtain

$$g(\phi_1 + i\phi_2) - \phi_3 + \bar{g}(\phi_1 - i\phi_2 - g\phi_3) = 0. \quad (15)$$

If we suppose that ψ is not planar, that is, g is not a constant, then deriving respect to \bar{z} it follows from (15) that

$$\phi_1 - i\phi_2 - g\phi_3 = 0$$

and

$$g(\phi_1 + i\phi_2) - \phi_3 = 0.$$

Hence

$$g = \frac{\phi_1 - i\phi_2}{\phi_3} = \frac{\phi_3}{\phi_1 + i\phi_2}$$

and

$$\phi_1^2 + \phi_2^2 - \phi_3^2 = 0.$$

Observe that these equalities hold although $\bar{g}_z = 0$ at some points.

Finally, if we define $\omega = \Phi_1 + i\Phi_2$ then

$$\Phi_1 - i\Phi_2 = \frac{\phi_3^2}{\phi_1 + i\phi_2} dz = \left(\frac{\phi_3}{\phi_1 + i\phi_2} \right)^2 (\phi_1 + i\phi_2) dz = g^2 \omega,$$

so that

$$\begin{aligned} \Phi_1 &= \frac{1}{2} (1 + g^2) \omega \\ \Phi_2 &= -\frac{i}{2} (1 - g^2) \omega \\ \Phi_3 &= g\omega. \end{aligned}$$

Consequently, from (13),

$$\psi_z = \frac{1}{2a} (-bN_z + (\phi_1, \phi_2, \phi_3)) \quad (16)$$

and, using (14), one has that the immersion can be recover as (8).

Moreover, since

$$N_z = \left(\frac{(1 + \bar{g}^2)g_z}{(1 - |g|^2)^2}, i \frac{(1 - \bar{g}^2)g_z}{(1 - |g|^2)^2}, \frac{2\bar{g}^2 g_z}{(1 - |g|^2)^2} \right) \quad (17)$$

one has from (16) and (17) the expressions of the induced metric and second fundamental form (12). In particular, K is given by (11).

Conversely, let M be a simply-connected Riemann surface, $g : M \rightarrow \mathbb{D}$ a holomorphic map, ω a holomorphic 1-form on M satisfying (9) and $\psi : M \rightarrow \mathbf{L}^3$ a map given by (8).

If we consider a conformal parameter z on M then

$$\psi_z = -\frac{b}{2a} \left(\frac{(1 + \bar{g}^2) g_z}{(1 - |g|^2)^2}, i \frac{(1 - \bar{g}^2) g_z}{(1 - |g|^2)^2}, \frac{2\bar{g}^2 g_z}{(1 - |g|^2)^2} \right) + \frac{1}{2a} \left(\frac{1 + g^2}{2} \frac{\omega}{dz}, -i \frac{1 - g^2}{2} \frac{\omega}{dz}, g \frac{\omega}{dz} \right)$$

and the induced metric $\langle d\psi, d\psi \rangle$ is given by (12). In particular, since (9) is satisfied, ψ is an immersion.

Moreover, if we take the vector field N given by (14) then $\langle \psi_z, N \rangle = 0$, that is, N is normal to ψ , and a straightforward computation give us that $\langle d\psi, dN \rangle$ is given by (12). Therefore,

$$H = \frac{8ab|dg|^2}{(1 - |g|^2)^4 |\omega|^2 - 4b^2 |dg|^2}, \quad K = \frac{16a^2 |dg|^2}{(1 - |g|^2)^4 |\omega|^2 - 4b^2 |dg|^2} \quad (18)$$

and $-2aH + bK = 0$.

The Weierstrass representation (8) is also valid if ψ is planar, taking g as a constant.

The case $K < 0$ is totally analogous. Observe that now g is an anti-holomorphic map and Φ_i must be now defined as

$$\Phi_i = \phi_i d\bar{z} = (2a\psi_i + bN_i)_{\bar{z}} d\bar{z}, \quad i = 1, 2, 3,$$

which are anti-holomorphic 1-forms. ■

Definition 10 Let $\psi : M^2 \rightarrow \mathbf{L}^3$ be an LWM-spacelike surface. The conformal map g and the 1-form ω given in the above Theorem, will be called the *Weierstrass data* of the immersion.

Remark 11 From (11) it follows that a point p of an LWM-spacelike surface is umbilic if and only if ω or dg vanish at p . ■

3 Complete LWM-Spacelike Surfaces

In 1970, Calabi [3] proved that the only complete maximal spacelike surfaces in \mathbf{L}^3 are the planes (see also [12]). This result is not true for complete LWM-spacelike surfaces, as it will be seen in Example 18. Anyway, we can assert the following:

Theorem 12 Let $\psi : M^2 \rightarrow \mathbf{L}^3$ be a complete LWM-spacelike surface such that $-2aH + bK = 0$, $a \neq 0$:

- i) If there exists a point where the Gaussian curvature is non negative, then $\psi(M)$ is a plane.
- ii) If there exists a point where the Gaussian curvature is negative (and therefore $K \leq -4a^2/b^2$), then $b \neq 0$ and there does not exist any constant c such that $K \leq c < -4a^2/b^2$ on M .

Proof: Since σ is a Riemannian metric, we have that $a - b\lambda_i > 0$ for $i = 1, 2$, where λ_1, λ_2 are the principal curvatures of ψ , and consequently $a + bH > 0$.

Let us see that, independently of the sign of K , we can find a complete metric τ conformal to σ :

- i)* As we have seen in Remark 7, if $K \geq 0$ then $a > 0$. Besides, the functions $2a + bH$, $2a - b\lambda_1$ and $2a - b\lambda_2$ are greater than the positive constant a on M .
- ii)* Let us suppose that there exists a constant c such that $K \leq c < -4a^2/b^2$ on M . Then, it can be easily shown that there exists a positive constant c_o satisfying that the functions $(2a + bH)^2$, $(2a - b\lambda_1)^2$ and $(2a - b\lambda_2)^2$ are greater than c_o^2 .

In both cases, if we consider a local orthonormal frame $\{e_1, e_2\}$ such that $Ae_i = \lambda_i e_i$, then

$$2(2a + bH) \sigma(e_1, e_1) = (2a + (a - b\lambda_1) + (a - b\lambda_2)) (a - b\lambda_1)$$

and using (2) we obtain

$$2(2a + bH) \sigma(e_1, e_1) = (2a - b\lambda_1)^2.$$

Analogously,

$$2(2a + bH) \sigma(e_2, e_2) = (2a - b\lambda_2)^2.$$

Thus, we deduce that the metric

$$\tau = 2(2a + bH) \sigma$$

verifies

$$\begin{aligned} \tau &\geq a^2 \langle \cdot, \cdot \rangle && \text{if } K \geq 0 \\ \tau &\geq c_o^2 \langle \cdot, \cdot \rangle && \text{if } K \leq c < -4a^2/b^2 \end{aligned}$$

and therefore τ is complete.

On the other hand, from (12)

$$\sigma = a \langle d\psi, d\psi \rangle + b \langle d\psi, dN \rangle = \frac{(1 - |g|^2)^2}{4a} |\omega|^2 - \frac{b^2}{a(1 - |g|^2)^2} |dg|^2,$$

where g, ω are the Weierstrass data of the immersion, and so, from (18),

$$\tau = \frac{1}{2} (1 - |g|^2)^2 |\omega|^2 \leq \frac{1}{2} |\omega|^2.$$

Thus, the flat metric $|\omega|^2/2$ is complete and conformal to σ . Then, from Cartan-Hadamard Theorem, it is deduced that M is conformally equivalent to the complex plane since M is simply-connected because of the completeness of the induced metric. Hence, from Lemma 3, $\psi(M)$ is a plane. In particular, there do not exist complete LWM-spacelike surfaces in \mathbf{L}^3 with negative Gaussian curvature such that $\sup K \neq -4a^2/b^2$. \blacksquare

Remark 13 It should be observed that given $\psi : M^2 \rightarrow \mathbf{L}^3$ a non flat complete LWM-spacelike surface such that $-2aH + bK = 0$ and P a compact set on M , then there does not exist any constant c satisfying

$$K \leq c < -\frac{4a^2}{b^2}$$

on $M - P$.

Otherwise, since, from Lemma 3, M can be conformally identified with the unit disk \mathbb{D} , then there would exist a disk centered at the origin of radius $0 < r < 1$, \mathbb{D}_r , such that $P \subseteq \mathbb{D}_r$. Then, arguing as in the above Theorem, $|\omega|^2/2$ is a complete metric on the annulus $\mathbb{D} - \mathbb{D}_r$. But, using [11, Lemma 9.3], $\mathbb{D} - \mathbb{D}_r$ should be conformally equivalent to a punctured disk, which is a contradiction. \blacksquare

Corollary 14 *The only complete ruled LWM-spacelike surfaces in the Lorentz-Minkowski space are the planes.*

Proof: It is a consequence of Theorem 12, because every ruled spacelike surface in \mathbf{L}^3 has non negative Gaussian curvature. ■

Theorem 9 also allows us to understand the behaviour of the Gauss map of a complete LWM-spacelike surface in \mathbf{L}^3 .

Theorem 15 *The Gauss map N of a non flat complete LWM-spacelike surface M in \mathbf{L}^3 is a conformal diffeomorphism preserving orientations onto \mathbf{H}_+^2 .*

Proof: Since M is non flat, then $K < 0$ on M and from (10) and (12) we get

$$\langle d\psi, d\psi \rangle \leq \frac{b^2}{a^2} \frac{4|dg|^2}{(1-|g|^2)^2} = \frac{b^2}{a^2} g^*(ds_P^2),$$

where the complete metric

$$ds_P^2 = \frac{4|dz|^2}{(1-|z|^2)^2}$$

is the Poincaré metric on \mathbb{D} . Thus, since $\langle d\psi, d\psi \rangle$ is complete, then $g^*(ds_P^2)$ is complete or, equivalently, g is a global diffeomorphism. ■

Remark 16 The converse of the above Theorem is not true. For instance, let us consider for any $a < 0$ the LWM-spacelike surface $\psi_a : \mathbb{D} \rightarrow \mathbf{L}^3$ satisfying $-2aH + bK = 0$ with $b = 1/2$ given by the Weierstrass data

$$g = \eta, \quad \omega = \frac{\eta}{(1-\eta^2)^2} d\eta, \quad \eta \in \mathbb{D}.$$

Then, since (10) is satisfied for the anti-holomorphic parameter η , ψ_a is well-defined and its Gauss map is a diffeomorphism onto \mathbf{H}_+^2 .

However, the divergent curve $\alpha : [0, 1) \rightarrow \mathbf{L}^3$ defined by $\alpha(t) = \psi_a(t)$ has finite length

$$l(\alpha) = \int_0^1 \frac{dt}{2|a|(1+t)} = \frac{\ln 2}{2|a|} < \infty$$

as it is followed from (12), and therefore the surface is not complete. ■

4 Rotation Surfaces

Let us start by constructing some examples of rotation LWM-spacelike surfaces, by setting suitable Weierstrass data:

Example 17 Set

$$g = z \quad \text{and} \quad \omega = \frac{A}{z^2} dz,$$

for a holomorphic parameter z , or analogously

$$g = \eta \quad \text{and} \quad \omega = \frac{A}{\eta^2} d\eta,$$

for an anti-holomorphic parameter η , where $A \in \mathbf{R}$. Since these Weierstrass data coincide with the Weierstrass data of a catenoid with vertical axis in \mathbf{R}^3 , we will refer to the corresponding immersion as the *LWM-catenoid of vertical axis*.

Now, integrating in (8) and writing in polar coordinates $z = r \exp(i\theta)$ or $\eta = r \exp(i\theta)$, $0 < r < 1$, $0 \leq \theta < 2\pi$, we obtain, up to a translation of \mathbf{L}^3 ,

$$\psi = \frac{1}{2a} \left(\left(\frac{-2br}{1-r^2} + \frac{A(r^2-1)}{r} \right) \cos \theta, - \left(\frac{-2br}{1-r^2} + \frac{A(r^2-1)}{r} \right) \sin \theta, \frac{-b(1+r^2)}{1-r^2} + 2A \ln r \right).$$

Therefore ψ is a rotation surface with temporal axis $(0, 0, 1)$.

- For the holomorphic case, the condition (9) implies that

$$\frac{A^2}{r^4} > \frac{4b^2}{(1-r^2)^4}$$

so that $A \neq 0$ and

$$0 < r < \frac{1}{2} \left(-c + \sqrt{4+c^2} \right) < 1, \quad c = \sqrt{2 \left| \frac{b}{A} \right|}.$$

- For the anti-holomorphic case, the condition (10) implies, when $A \neq 0$, that

$$0 < \frac{1}{2} \left(-c + \sqrt{4+c^2} \right) < r < 1, \quad c = \sqrt{2 \left| \frac{b}{A} \right|}.$$

Therefore, since g is not a diffeomorphism onto \mathbf{H}_+^2 , there does not exist any complete surface in this family (Theorem 15). If $A = 0$ the immersion is complete and it is, up to a homothety, \mathbf{H}_+^2 .

— FIGURE 1 —

■

Example 18 Set

$$g = \frac{z+1}{z-1} \quad \text{and} \quad \omega = \frac{A(1-z)^2}{2z^2} dz,$$

for a holomorphic parameter z , or analogously

$$g = \frac{\eta+1}{\eta-1} \quad \text{and} \quad \omega = \frac{A(1-\eta)^2}{2\eta^2} d\eta,$$

for an anti-holomorphic parameter η , where $A \in \mathbf{R}$. Since these Weierstrass data coincide with the Weierstrass data of a catenoid with a horizontal axis in \mathbf{R}^3 , we will refer to the corresponding immersion as the *LWM-catenoid of horizontal axis*.

Note that g is a conformal diffeomorphism from $\Omega = \{z \in \mathbf{C} : \operatorname{Re}(z) < 0\}$ onto \mathbb{D} . Integrating in (8) and writing in polar coordinates $z = r \exp(i\theta)$ or $\eta = r \exp(i\theta)$, $r > 0$, $\pi/2 < \theta < 3\pi/2$, we obtain, up to a translation of \mathbf{L}^3 ,

$$\psi = \frac{1}{2a} \left(\frac{b + 2A \cos^2 \theta}{\cos \theta} \frac{r^2 - 1}{2r}, b \tan \theta + 2A\theta, \frac{b + 2A \cos^2 \theta}{\cos \theta} \frac{r^2 + 1}{2r} \right).$$

Since

$$\left(\frac{r^2 - 1}{2r} \right)^2 - \left(\frac{r^2 + 1}{2r} \right)^2 = -1$$

we can put

$$\frac{r^2 - 1}{2r} = \sinh s, \quad \frac{r^2 + 1}{2r} = \cosh s$$

and therefore ψ is a rotation surface with spacelike axis $(0, 1, 0)$. Now, we may distinguish the two following cases:

- For the holomorphic case, the condition (9) implies that

$$\frac{A^2}{64} > b^2 \left(\frac{|z|}{|z-1|^2 - |z+1|^2} \right)^4$$

so that $A \neq 0$ and

$$-\cos \theta > c \geq 0, \quad c = \sqrt{\left| \frac{b}{2A} \right|}.$$

- For the anti-holomorphic case, the condition (10) implies, when $A \neq 0$, that

$$-\cos \theta < c.$$

Hence, the domain of g is Ω if and only if $c > 1$. In particular:

- If $c \leq 1$, the immersion is not complete, since g is not a global diffeomorphism (Theorem 15).
- If $c > 1$, taking $\zeta = (\eta + 1)/(\eta - 1)$, which is a conformal transformation of Ω in \mathbb{D} , we get

$$g = \zeta, \quad \omega = -\frac{4A}{(1 - \zeta^2)^2} d\zeta.$$

Let us see that the metric $\langle d\psi, d\psi \rangle$ is complete. If we put $\lambda = (1 - |\zeta|^2)^2$ and $\mu = |1 - \zeta^2|^2$, $0 < \lambda \leq \mu$, it follows from

$$\left(\frac{4A^2 \lambda}{\mu^2} + \frac{b^2}{\lambda} \right) - \left(\frac{4A^2}{\mu} + \frac{b^2}{\mu} \right) = \frac{\mu - \lambda}{\lambda \mu^2} (b^2 \mu - 4A^2 \lambda) \geq \frac{\mu - \lambda}{\lambda \mu} (b^2 - 4A^2)$$

and (12), that

$$\begin{aligned} a^2 \langle d\psi, d\psi \rangle &\geq \frac{2bA}{(1 - \zeta^2)^2} d\zeta^2 + \left(\frac{4A^2}{\mu} + \frac{b^2}{\mu} \right) |d\zeta|^2 \\ &\quad + \frac{2bA}{(1 - \bar{\zeta}^2)^2} d\bar{\zeta}^2 + \frac{\mu - \lambda}{\lambda \mu} (b^2 - 4A^2) |d\zeta|^2 \\ &= |\varphi|^2 + \frac{\mu - \lambda}{\lambda \mu} (b^2 - 4A^2) |d\zeta|^2, \end{aligned} \tag{19}$$

where

$$\varphi = \frac{2A}{1-\zeta^2} d\zeta + \frac{b}{1-\bar{\zeta}^2} d\bar{\zeta}.$$

We can estimate $|\varphi|$ in the way

$$|\varphi| \geq \left| \frac{|b|}{|1-\bar{\zeta}^2|} |d\bar{\zeta}| - \frac{2|A|}{|1-\zeta^2|} |d\zeta| \right| = \frac{|b|-2|A|}{|1-\zeta^2|} |d\zeta|$$

so that (19) becomes

$$a^2 \langle d\psi, d\psi \rangle \geq \frac{(|b|-2|A|)^2}{|1-\zeta^2|^2} |d\zeta|^2 + \frac{|1-\zeta^2|^2 - (1-|\zeta|^2)^2}{|1-\zeta^2|^2 (1-|\zeta|^2)^2} (b^2 - 4A^2) |d\zeta|^2.$$

Thus, if we call

$$m = \min \{ (|b|-2|A|)^2, b^2 - 4A^2 \}$$

which is a positive number because $c > 1$, then

$$\langle d\psi, d\psi \rangle \geq \frac{m}{a^2 (1-|\zeta|^2)^2} |d\zeta|^2.$$

Observe that the metric on the right-hand side is, up to a positive constant, the Poincaré metric in \mathbb{D} , so that the immersion ψ is complete.

Finally, when $A = 0$ the immersion is, up to a homothety, \mathbf{H}_+^2 .

— FIGURE 2 —

■

Example 19 Set

$$g = \frac{z+1}{z-1} \quad \text{and} \quad \omega = A(1-z)^2 dz,$$

for a holomorphic parameter z , or analogously

$$g = \frac{\eta+1}{\eta-1} \quad \text{and} \quad \omega = A(1-\eta)^2 d\eta,$$

for an anti-holomorphic parameter η , where $A \in \mathbf{R}$. These Weierstrass data coincide with the Weierstrass data of an Enneper's surface in \mathbf{R}^3 , so we will refer to the corresponding immersion as the *LWM-Enneper surface*.

Integrating in (8) and writing $z = x + iy$ or $\eta = x + iy$, we obtain, up to a translation of \mathbf{L}^3 ,

$$\psi = -\frac{b}{2a} \left(\frac{x^2 + y^2 - 1}{-2x}, \frac{-y}{x}, \frac{1 + x^2 + y^2}{-2x} \right) + \frac{A}{a} \left(x + \frac{x^3 - 3xy^2}{3}, -2xy, -x + \frac{x^3 - 3xy^2}{3} \right).$$

If we put

$$h_1(x) = \frac{3bx + 4Ax^3}{12a}, \quad h_2(x) = \frac{4Ax^2 - b}{4ax},$$

then

$$\psi = (h_1(x) + (1-y^2) h_2(x), -2y h_2(x), h_1(x) - (1+y^2) h_2(x)),$$

that is, ψ is a rotation surface with light-like axis $(1, 0, 1)$. Now, we have:

- For the holomorphic case, the condition (9) implies that

$$A^2 > \frac{b^2}{4x^4}$$

so that $A \neq 0$ and the domain of ψ is

$$x < -\sqrt{\left|\frac{b}{2A}\right|}.$$

- For the anti-holomorphic case, the condition (10) implies, when $A \neq 0$, that the domain of ψ is

$$-\sqrt{\left|\frac{b}{2A}\right|} < x < 0,$$

so that, from Theorem 15, there does not exist any complete surface in this family. On the other hand, if $A = 0$ the immersion is again, up to a homothety, \mathbf{H}_+^2 .

— FIGURE 3 —

■

Let us see that these examples and the plane, are all the rotation LWM-spacelike surfaces in \mathbf{L}^3 .

Theorem 20 *Every rotation LWM-spacelike surface in \mathbf{L}^3 is isometric to a piece of one of the following:*

- i) *Plane.*
- ii) *LWM-catenoid with vertical axis.*
- iii) *LWM-catenoid with horizontal axis.*
- iv) *LWM-Enneper surface.*

Proof: Let $\psi : M^2 \rightarrow \mathbf{L}^3$ be a rotation LWM-spacelike surface in \mathbf{L}^3 with timelike axis. We can suppose, up to a Lorentz transformation of \mathbf{L}^3 , that the axis of rotation is $(0, 0, 1)$, so that the immersion ψ can be written as

$$\psi(r, \theta) = (h_1(r) \cos \theta, h_1(r) \sin \theta, h_2(r)),$$

where $h_1, h_2 : I \rightarrow \mathbf{R}$ are differential functions defined on an open interval I of \mathbf{R} , being $h_1(r) > 0$ and $h_1'(r)^2 - h_2'(r)^2 > 0$. Then the first and second fundamental forms of the immersion are

$$\langle d\psi, d\psi \rangle = (h_1'(r)^2 - h_2'(r)^2) dr^2 + h_1(r)^2 d\theta^2 \tag{20}$$

and

$$\langle d\psi, dN \rangle = \frac{h_2'(r)h_1''(r) - h_1'(r)h_2''(r)}{\sqrt{h_1'(r)^2 - h_2'(r)^2}} dr^2 + \frac{-h_1(r)h_2'(r)}{\sqrt{h_1'(r)^2 - h_2'(r)^2}} d\theta^2 \tag{21}$$

respectively.

If we suppose that the immersion is not a plane, then from (21)

$$h'_2(r) \neq 0 \quad \text{and} \quad \frac{d}{dr} \left(\frac{h'_2(r)}{h'_1(r)} \right) \neq 0.$$

Otherwise, there exists a point (r_o, θ_o) where $K(r_o, \theta_o) = 0$ and, since the coefficients of the second fundamental form do not depend on θ , then $K(r_o, \theta) = 0$ for all θ . Therefore, from (11), $dg_{(r_o, \theta)} = 0$ for all θ and, since g is holomorphic, then g is constant. But it is impossible because ψ is not a plane.

Thus, we can consider the change of parameter

$$\frac{h'_2(r)}{h'_1(r)} = \frac{2t}{1+t^2},$$

whence

$$h'_2(t) = \frac{2t}{1+t^2} h'_1(t)$$

and the first and second fundamental forms of ψ (formulas (20) and (21)) become

$$\langle d\psi, d\psi \rangle = \left(\frac{1-t^2}{1+t^2} h'_1(t) \right)^2 dt^2 + h_1(t)^2 d\theta^2$$

and

$$\langle d\psi, dN \rangle = \frac{-2h'_1(t)}{1+t^2} dt^2 - \frac{2th_1(t)}{1-t^2} d\theta^2$$

respectively. Then the equation $-2aH + bK = 0$ can be rewritten as

$$a \left(\frac{-2h'_1(t)h_1(t)^2}{1+t^2} - \frac{2t(1-t^2)h_1(t)h'_1(t)^2}{(1+t^2)^2} \right) - b \frac{4th_1(t)h'_1(t)}{(1+t^2)(1-t^2)} = 0,$$

whence

$$h_1(t) = \frac{1}{2a} \left(\frac{-2bt}{1-t^2} + \frac{A(t^2-1)}{t} \right)$$

and

$$h_2(t) = \int \frac{2t}{1+t^2} h_1(t) dt = \frac{1}{2a} \left(\frac{-b(1+t^2)}{1-t^2} + 2A \ln t + \kappa \right)$$

for a constant κ , that is, ψ is up to an isometry, a piece of a LWM-catenoid of vertical axis (see Example 17).

Let us assume now that the axis of rotation is spacelike. We can suppose, up to a Lorentz transformation of \mathbf{L}^3 , that the axis of rotation is $(0, 1, 0)$, so that the immersion ψ can be written as

$$\psi(r, \theta) = (h_1(r) \sinh \theta, h_2(r), h_1(r) \cosh \theta),$$

where $h_1, h_2 : I \rightarrow \mathbf{R}$ are differential functions defined on an open interval I of \mathbf{R} , being $h_1(r) > 0$ and $h'_2(r)^2 - h'_1(r)^2 > 0$. Then the first and second fundamental forms of the immersion are

$$\langle d\psi, d\psi \rangle = (h'_2(r)^2 - h'_1(r)^2) dr^2 + h_1(r)^2 d\theta^2$$

and

$$\langle d\psi, dN \rangle = \frac{h_2''(r)h_1'(r) - h_1''(r)h_2'(r)}{\sqrt{h_2'(r)^2 - h_1'(r)^2}} dr^2 - \frac{h_1(r)h_2'(r)}{\sqrt{h_2'(r)^2 - h_1'(r)^2}} d\theta^2$$

respectively.

Reasoning as above, it can be seen that

$$\frac{d}{dr} \left(\frac{h_1'(r)}{h_2'(r)} \right) \neq 0,$$

that is, $h_1'(r)/h_2'(r)$ is a diffeomorphism. Thus, taking

$$\frac{h_1'(r)}{h_2'(r)} = \frac{-t}{\sqrt{1+t^2}} = \frac{h_1'(t)}{h_2'(t)},$$

a simple computation allows us to obtain from $-2aH + bK = 0$ the differential equation

$$a\sqrt{\frac{h_1'(t)^2}{t^2}} (t h_1(t) + (1+t^2)h_1'(t)) - b h_1'(t) \sqrt{1+t^2} = 0$$

whose solution is

$$h_1(t) = \frac{b + 2A \cos^2 t}{2a \cos t}$$

and so

$$h_2(t) = \frac{1}{2a} (b \tan t + 2At + \kappa)$$

for a constant κ . Consequently, ψ is up to an isometry, a piece of a LWM-catenoid with horizontal axis (see Example 18).

Finally, if the axis of rotation is light-like we can suppose, up to a Lorentz transformation of \mathbf{L}^3 , that the axis of rotation is $(1, 0, 1)$. Then the immersion ψ can be written as

$$\psi(x, y) = (h_1(x) + (1 - y^2)h_2(x), -2yh_2(x), h_1(x) - (1 + y^2)h_2(x)),$$

where $h_1, h_2 : I \rightarrow \mathbf{R}$, are differential functions defined on an open interval I of \mathbf{R} , being $h_2(x) > 0$, $h_1'(x) > 0$ and $h_2'(x) > 0$. Then the first and second fundamental forms of the immersion are

$$\langle d\psi, d\psi \rangle = 4h_1'(x)h_2'(x) dx^2 + 4h_2(x)^2 dy^2$$

and

$$\langle d\psi, dN \rangle = \frac{h_2(x)(h_2'(x)h_1''(x) - h_1'(x)h_2''(x))}{\sqrt{h_2(x)^2 h_1'(x)h_2'(x)}} dx^2 - 2 \frac{\sqrt{h_2(x)^2 h_1'(x)h_2'(x)}}{h_1'(x)} dy^2$$

respectively.

If ψ is not planar, performing the change

$$\frac{h_1'(x)}{h_2'(x)} = t^2 = \frac{h_1'(t)}{h_2'(t)}, \quad t < 0,$$

it can be obtain from $-2aH + bK = 0$ the differential equation

$$2t^2 h_2(t) h_2'(t)^3 (b + 2at(h_2(t) - t h_2'(t))) = 0.$$

Hence

$$h_2(t) = \frac{4At^2 - b}{4at}$$

and so

$$h_1(t) = \frac{3bt + 4At^3}{12a} + \kappa$$

for a constant κ . Consequently, ψ is, up to an isometry, a piece of an LWM-Enneper surface (see Example 19) ■

Remark 21 Observe that, among the rotation LWM-spacelike surfaces in \mathbf{L}^3 , only the planes, the hyperbolic planes and part of the LWM-catenoids with horizontal axis are complete (see Examples 17, 18 and 19). ■

5 Some Open Problems

It is well-known that every complete spacelike surface in \mathbf{L}^3 is a graph on the whole (x, y) -plane. However, the converse is not true, that is, there exist spacelike graphs on the whole (x, y) -plane which are not complete surfaces. In this sense, Cheng and Yau [4] proved that every such graph with zero mean curvature is complete.

Problem 1: Following this line, we wonder if every LWM-graph on the whole (x, y) -plane, that is, every entire solution of the equation

$$a((1 - f_y^2)f_{xx} + 2f_x f_y f_{xy} + (1 - f_x^2)f_{yy}) (1 - f_x^2 - f_y^2)^{1/2} + b(f_{xx}f_{yy} - f_{xy}^2) = 0, \quad 1 - f_x^2 - f_y^2 > 0$$

is also a complete surface.

In section 3 we have proved that the only complete ruled LWM-spacelike surfaces in the Lorentz-Minkowski space are the planes. Obviously, the hypothesis of completeness is essential. Indeed, thanks to the conformal representation for maximal surfaces, Kobayashi classified in [7] the ruled (non-complete) ones, proving that there exist four types of such surfaces.

Problem 2: To classify the ruled (non-complete) LMW-surfaces.

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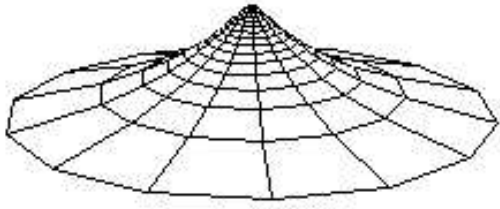
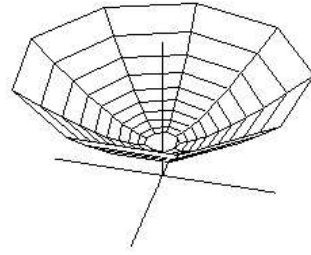


Figure 1: $H = K, K \geq 0$



$H = K, K < 0$

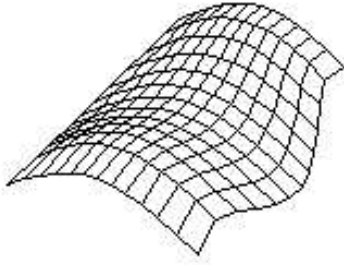
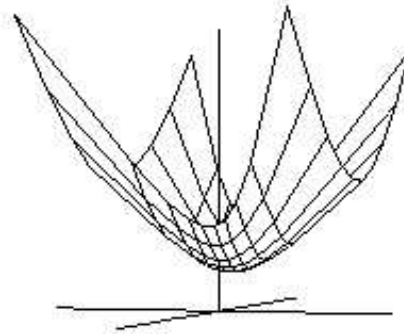


Figure 2: $H = K, K \geq 0$



$H = K, K < 0$

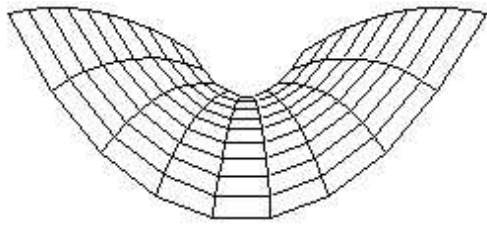
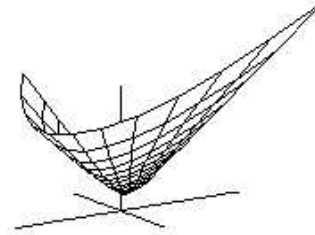


Figure 3: $H = K, K \geq 0$



$H = K, K < 0$