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On the Ricci Curvature of Compact Spacelike Hypersurfaces in Einstein Conformally Stationary-Closed Spacetimes

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In this paper we develop an integral formula involving the Ricci and scalar curvatures of a compact spacelike hypersurface M in a spacetime $\overline{M}$ equipped with a timelike closed conformal vector field K (in short, conformally stationary-closed spacetime), and we apply it, when $\overline{M}$ is Einstein, in order to establish sufficient conditions for M to be a leaf of the foliation determined by K and to obtain some non-existence results. We also get some interesting consequences for the particular case when $\overline{M}$ is a generalized Robertson-Walker spacetime.

KEY WORDS: Einstein spacetime; conformally stationary-closed spacetime; spacelike hypersurface; Ricci curvature; generalized Robertson-Walker spacetime.

1. INTRODUCTION

The study of spacelike hypersurfaces in Lorentzian spacetimes has been recently of substantial interest from both physical and mathematical points of view. From the physical one, that interest became clear when Lichnerowicz [15] showed that the Cauchy problem of the Einstein equation with initial conditions on a maximal spacelike hypersurface (that is, with vanishing mean extrinsic curvature) has a particularly nice form, reducing to a linear differential system of first order and to a non-linear second order elliptic differential equation. We also refer the reader to

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the survey papers [12] and [16], and references therein for other reasons justifying their importance in general relativity.

From a mathematical point of view, spacelike hypersurfaces are also interesting because of their Bernstein-type properties. The problem of characterizing the maximal, or more generally, the totally umbilical hypersurfaces of different Lorentzian spacetimes has been studied by several authors in the last years. This Bernstein problem was introduced by Calabi [10] in the Lorentz-Minkowski space $\mathbf{L}^{n+1}$, who showed that for $n \leq 4$ the only complete maximal hypersurfaces are the spacelike hyperplanes. Cheng and Yau [11] extended this result to arbitrary dimension n.

When the ambient space is the de Sitter space $\mathbf{S}_1^{n+1}$, Goddard [13] conjectured that every complete spacelike hypersurface with constant mean curvature in $\mathbf{S}_1^{n+1}$ should be totally umbilical. Although this conjecture turned out to be false in its original statement, it motivated a great deal of works of several authors ([1], [17], [20]) trying to find a positive answer to the conjecture under appropriate additional hypotheses (see also [9] for an account of the subject).

As for the case of more general Lorentzian ambient spaces, in a serie of recent papers Alías, Romero and Sánchez ([5], [6],[7],[8]) have studied the uniqueness of spacelike hypersurfaces with constant mean curvature in a wide class of Lorentzian manifolds, the so called *conformally stationary spacetimes*. A such space is a manifold $\overline{M}$ endowed with a Lorentzian metric tensor $\langle , \rangle$ equipped with a timelike conformal vector field $K \in \mathcal{X}(\overline{M})$. The fact that K is conformal means that the Lie derivative of the Lorentzian metric $\langle , \rangle$ with respect to K satisfies $\mathcal{L}_K \langle , \rangle = 2\phi \langle , \rangle$ for a certain smooth function $\phi \in \mathcal{C}^\infty(\overline{M})$. In particular, when K is a Killing field (that is, $\phi \equiv 0$), then $\overline{M}$ is classically called a *stationary spacetime*.

From a purely mathematical interest, stationary spacetimes have been recently studied by different authors in order to obtain geodesic completeness of Lorentzian manifolds ([14], [21]) as well as interesting classifications results [22]. On the other hand, in [23] it is studied the geometry of stationary spacetimes from several points of view, some of them of physical interest.

The reason for the terminology *conformally stationary spacetime* is due to the fact that $\overline{M}$ endowed with the conformally related metric $\langle , \rangle^* = (1/|K|)\langle , \rangle$, where $|K| = \sqrt{-\langle K, K \rangle} > 0$ is in fact a stationary spacetime, since the timelike field K is a Killing field for $\langle , \rangle^*$ [24, Lemma 2.1].

The class of conformally stationary spacetimes includes the family of generalized Robertson-Walker spacetimes (see section 5). For a such spacetime, the conformal field K is also closed, in the sense that its metrically equivalent 1-form is closed. As it was observed by Montiel [18], if $\overline{M}$ is a conformally stationary spacetime equipped with a closed conformal field, then it is locally isometric to a generalized Robertson-Walker spacetime. For a global analogue of this assertion under the assumption of timelike geodesic completeness, see [18, Proposition

2], where the author proves that it is isometric to an appropriate quotient of a generalized Robertson-Walker spacetime.

Throughout this paper we will deal with a Lorentzian space $\overline{M}$ which admits a (globally defined) timelike closed conformal vector field K. Following the terminology introduced by Alías, Brasil and Colares in [3], we will refer to such manifold as a *conformally stationary-closed spacetime*. Observe that under this hypothesis, $\overline{M}$ admits a foliation by totally umbilical spacelike hypersurfaces with constant mean curvature by integrating the distribution orthogonal to K. In [18], Montiel classified the totally umbilical hypersurfaces with constant mean curvature of such spacetimes in terms of that foliation, under the hypothesis of the null convergence condition on the spacetime. Besides, he also obtained a uniqueness result for the case of spacelike hypersurfaces with constant scalar curvature (see also [4] for the case of spacelike hypersurfaces with constant higher order mean curvature).

Our purpose is to establish a sufficient condition for a compact spacelike hypersurface in an Einstein conformally stationary-closed spacetime $\overline{M}$ to be a leaf of this foliation in terms of a pinching condition for its Ricci curvature. With this aim, we will develop an integral formula (section 3) involving the Ricci and scalar curvatures of the hypersurface (being $\overline{M}$ non necessarily Einstein), which furthermore allows us to obtain several non-existence results. Moreover, we will obtain interesting consequences for the particular case when $\overline{M}$ is a generalized Robertson-Walker spacetime (section 5).

This work is motivated by the paper [2], where Alías studied this problem when the ambient spacetime is the de Sitter space.

2. PRELIMINARIES

Let $(\overline{M}, \langle , \rangle)$ be a $(n+1)$-dimensional $(n \geq 2)$ *conformally stationary-closed spacetime*, that is, a Lorentzian manifold which admits a timelike closed conformal vector field $K \in \mathcal{X}(\overline{M})$.

The fact that K is closed and conformal means that there exists a smooth function $\phi \in \mathcal{C}^{\infty}(\overline{M})$ satisfying

$$\overline{\nabla}_X K = \phi X \tag{1}$$

for every vector field $X \in \mathcal{X}(\overline{M})$, where $\overline{\nabla}$ stands for the Levi-Civita connection of $\overline{M}$. We will refer to ϕ as the *function associated* to K. By means of an easy calculation, we can characterize the function ϕ as

$$\phi = \frac{1}{n+1}\overline{\mathrm{div}}K$$

where $\overline{\mathrm{div}}$ denotes the divergence in $\overline{M}$, and its gradient in $\overline{M}$ is given by

$$\overline{\nabla}\phi = -\frac{K\phi}{|K|^2}K. \tag{2}$$

A smooth immersion $\psi : M \longrightarrow \overline{M}$ of an n-dimensional connected manifold M is said to be a *spacelike hypersurface* if the induced metric via ψ is a Riemannian metric on M, which, as usual, is also denoted by $\langle , \rangle$. We will denote by N the (globally defined) unitary timelike vector field in the same time-orientation as K, that is

$$\langle K, N\rangle \leq -|K| = -\sqrt{-\langle K, K\rangle} < 0$$

on M. We will refer to N as the *Gauss map* of M and we will say that M is oriented by N. We will denote by θ the *hyperbolic angle* between N and K,

$$\cosh(\theta) = \frac{-\langle K, N\rangle}{\sqrt{-\langle K, K\rangle}}$$

which is a smooth function defined on M.

On the other hand, the Gauss and Weingarten formulas for M in $\overline{M}$ are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y - \langle AX, Y\rangle N \tag{3}$$

and

$$A(X) = -\overline{\nabla}_X N \tag{4}$$

for all tangent vector fields $X, Y \in \mathcal{X}(M)$, where ∇ denotes the Levi-Civita connection of M and $A : \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$ stands for the shape operator of M in $\overline{M}$ with respect to N.

Regarding to the curvature tensor R of the hypersurface, it can be described in terms of the curvature tensor $\overline{R}$ of $\overline{M}$ and the shape operator A according the Gauss equation

$$R(X, Y)Z = \left(\overline{R}(X, Y)Z\right)^{\top} - \langle AX, Z\rangle AY + \langle AY, Z\rangle AX \tag{5}$$

for all tangent vector fields $X, Y, Z \in \mathcal{X}(M)$, where $\left(\overline{R}(X, Y)Z\right)^{\top}$ denotes the tangential component of

$$\overline{R}(X, Y)Z = \overline{\nabla}_{[X,Y]}Z - [\overline{\nabla}_X, \overline{\nabla}_Y]Z.$$

In particular, from (1) we can derive that

$$\overline{R}(X, Y)K = \frac{K\phi}{|K|^2}(\langle K, X\rangle Y - \langle K, Y\rangle X) \tag{6}$$

for every $X, Y \in \mathcal{X}(\overline{M})$.

As to the normal component of $\overline{R}(X, Y)Z$, the Codazzi equation of the hypersurface states that

$$\langle \overline{R}(X, Y)Z, N\rangle = \langle (\nabla A)(X, Y) - (\nabla A)(Y, X), Z\rangle \tag{7}$$

for all $X, Y, Z \in \mathcal{X}(M)$, where $(\nabla A)(X, Y) = (\nabla_Y A)(X)$.

From (5) it follows that the Ricci curvature of M is given by

$$\mathrm{Ric}(X, Y) = \overline{\mathrm{Ric}}(X, Y) + \langle \overline{R}(X, N)Y, N\rangle - \mathrm{trace}(A)\langle AX, Y\rangle + \langle AX, AY\rangle \tag{8}$$

for $X, Y \in \mathcal{X}(M)$, where $\overline{\mathrm{Ric}}$ stands for the Ricci curvature of the ambient spacetime $\overline{M}$. Particularly, from (6) we have

$$\overline{\mathrm{Ric}}(K, X) = -nX\phi = -n\langle \overline{\nabla}\phi, X\rangle \tag{9}$$

for all $X \in \mathcal{X}(\overline{M})$.

Therefore, the escalar curvature S of M is

$$S = \mathrm{trace}(\mathrm{Ric}) = \overline{S} + 2\overline{\mathrm{Ric}}(N, N) - n^2H^2 + \mathrm{trace}(A^2), \tag{10}$$

where $\overline{S}$ denotes the scalar curvature of $\overline{M}$ and $H = (-1/n)\mathrm{trace}(A)$ is the mean curvature of the hypersurface M. The choice of the sign $-$ in our definition of H is motivated by the fact that, in that case, the mean curvature vector is given by $\vec{H} = HN$. Therefore, $H(p) > 0$ at a point $p \in M$ if and only if $\vec{H}(p)$ is in the time-orientation determined by $N(p)$.

Coming back to the ambient space $\overline{M}$, observe that the distribution orthogonal to K determines a codimension one foliation $\mathcal{F}(K)$ in the spacetime whose leaves are totally umbilical spacelike hypersurfaces of $\overline{M}$ (see Proposition 1 in [18]). If $L \longrightarrow \overline{M}$ is a leaf of $\mathcal{F}(K)$, then the tangential component of K

$$K^\top = K + \langle K, N\rangle N \tag{11}$$

vanishes on L. Actually, the connected leaves are characterized by this condition.

Observe that the function ϕ is constant on the leaves, because it follows from (2) that

$$\nabla\phi = -\frac{K\phi}{|K|^2}K^\top = 0.$$

On the other hand, from (1) and (4) it results that

$$\phi X = \overline{\nabla}_X K = \overline{\nabla}_X(-\langle K, N\rangle N) = -X(\langle K, N\rangle)N + \langle K, N\rangle AX$$

for every vector field $X \in \mathcal{X}(L)$, so that the shape operator of L is

$$A = -\frac{\phi}{|K|}I_n.$$

Hence, we obtain from (6), (8) and (9) that the Ricci curvature of each leaf is given by

$$\begin{aligned}\mathrm{Ric}(X,X) &= \overline{\mathrm{Ric}}(X,X) - \frac{K\phi}{|K|^2}|X|^2 - (n-1)\frac{\phi^2}{|K|^2}|X|^2 \\ &= \overline{\mathrm{Ric}}(X,X) + \frac{1}{n}\frac{\overline{\mathrm{Ric}}(K,K)}{|K|^2}|X|^2 - (n-1)\frac{\phi^2}{|K|^2}|X|^2. \qquad (12)\end{aligned}$$

3. THE INTEGRAL FORMULA

In this section we will derive an integral formula for a compact spacelike hypersurface $\psi : M \longrightarrow \overline{M}$ in a conformally stationary-closed spacetime $\overline{M}$. In order to do that, we can obtain by taking covariant derivative in (11) and using (1), (3) and (4) that

$$\nabla_X K^\top = \phi X - \langle K, N\rangle AX$$

for all $X \in \mathcal{X}(M)$ and therefore

$$\mathrm{div}(K^\top) = n\phi + nH\langle K, N\rangle.$$

On the other hand, it is easy to check that the gradient of $\langle K, N\rangle$ in M is given by

$$\nabla\langle K, N\rangle = -A(K^\top), \qquad (13)$$

and from Codazzi equation (7) it follows that the laplacian of $\langle K, N\rangle$ in M is

$$\triangle\langle K, N\rangle = n\langle \nabla H, K\rangle + \overline{\mathrm{Ric}}(K^\top, N) + n\phi H + \mathrm{trace}(A^2)\langle K, N\rangle. \qquad (14)$$

Observe that from (8) we have

$$\begin{aligned}\langle AK^\top, AK^\top\rangle &= \mathrm{Ric}(K^\top, K^\top) - \overline{\mathrm{Ric}}(K^\top, K^\top) - \langle \overline{R}(K^\top, N)K^\top, N\rangle \\ &\quad - nH\langle AK^\top, K^\top\rangle,\end{aligned}$$

which jointly (13) and (14) yields

$$\begin{aligned}(1/2)\triangle\langle K, N\rangle^2 &= \langle K, N\rangle\triangle\langle K, N\rangle + |\nabla\langle K, N\rangle|^2 \\ &= n\langle \nabla H, K\rangle\langle K, N\rangle + \overline{\mathrm{Ric}}(K^\top, N)\langle K, N\rangle + nH\phi\langle K, N\rangle \\ &\quad + \mathrm{trace}(A^2)\langle K, N\rangle^2 + \mathrm{Ric}(K^\top, K^\top) - \overline{\mathrm{Ric}}(K^\top, K^\top) \\ &\quad - \langle \overline{R}(K^\top, N)K^\top, N\rangle - nH\langle AK^\top, K^\top\rangle. \qquad (15)\end{aligned}$$

Now, taking into account that

$$n\mathrm{div}\left(H\langle K, N\rangle K^\top\right) = n\langle \nabla(H\langle K, N\rangle), K^\top\rangle + nH\langle K, N\rangle\mathrm{div}(K^\top)$$

and

$$\overline{\mathrm{Ric}}(K^\top, N)\langle K, N\rangle - \overline{\mathrm{Ric}}(K^\top, K^\top) = -\overline{\mathrm{Ric}}(K^\top, K),$$

we obtain from (15)

$$\begin{aligned}
&(1/2)\triangle\langle K, N\rangle^2 - n\mathrm{div}\left(H\langle K, N\rangle K^\top\right)\\
&\quad = -n^2 H\phi\langle K, N\rangle - n^2H^2\langle K, N\rangle^2 - \overline{\mathrm{Ric}}(K^\top, K) + nH\phi\langle K, N\rangle\\
&\qquad + \mathrm{trace}(A^2)\langle K, N\rangle^2 + \mathrm{Ric}(K^\top, K^\top) - \langle \overline{R}(K^\top, N)K^\top, N\rangle.
\end{aligned}$$

Hence, using (10) and the divergence

$$\mathrm{div}(\phi K^\top) = \langle \nabla\phi, K^\top\rangle + \phi\mathrm{div}(K^\top),$$

we get that

$$\begin{aligned}
&(1/2)\triangle\langle K, N\rangle^2 - n\mathrm{div}\left(H\langle K, N\rangle K^\top\right) + (n-1)\mathrm{div}(\phi K^\top)\\
&\quad = (n-1)\langle\nabla\phi, K^\top\rangle + n(n-1)\phi^2 + \left(S - \overline{S} - 2\overline{\mathrm{Ric}}(N, N)\right)\langle K, N\rangle^2\\
&\qquad - \overline{\mathrm{Ric}}(K^\top, K) + \mathrm{Ric}(K^\top, K^\top) - \langle \overline{R}(K^\top, N)K^\top, N\rangle.
\end{aligned}$$

Finally, from (6) and (9) we can derive respectively that

$$\langle \overline{R}(K^\top, N)K^\top, N\rangle = \langle\nabla\phi, K^\top\rangle$$

and

$$\overline{\mathrm{Ric}}(K^\top, K) = -n\langle\nabla\phi, K^\top\rangle,$$

so that

$$\begin{aligned}
&(1/2)\triangle\langle K, N\rangle^2 - n\mathrm{div}\left(H\langle K, N\rangle K^\top\right) + (n-1)\mathrm{div}(\phi K^\top)\\
&\quad = 2(n-1)\langle\nabla\phi, K^\top\rangle + n(n-1)\phi^2 + \left(S - \overline{S} - 2\overline{\mathrm{Ric}}(N, N)\right)\langle K, N\rangle^2\\
&\qquad + \mathrm{Ric}(K^\top, K^\top).
\end{aligned}$$

Integrating this expression, the divergence theorem allows us to state the following result:

Theorem 1. *Let $\psi : M^n \longrightarrow \overline{M}^{n+1}$ be a compact spacelike hypersurface immersed in a conformally stationary-closed spacetime $\overline{M}$. Then*

$$\begin{aligned}
&\int_M \{2(n-1)\langle\nabla\phi, K^\top\rangle + n(n-1)\phi^2 + \left(S - \overline{S} - 2\overline{\mathrm{Ric}}(N, N)\right)\langle K, N\rangle^2\\
&\qquad + \mathrm{Ric}(K^\top, K^\top)\}\mathrm{d}V = 0
\end{aligned}$$

where $\mathrm{d}V$ *is the n-dimensional volume element of M with respect to the induced metric and the orientation given by the Gauss map in the same time-orientation as K.*

4. MAIN RESULTS

In the current section we will obtain some interesting consequences from the above integral formula when the ambient spacetime is Einstein. Thus, let us suppose that $(\overline{M}, \langle,\rangle)$ is Einstein with $\overline{\mathrm{Ric}} = \overline{c}\langle,\rangle$, $\overline{c}$ being a real constant. Then, from (9) we obtain

$$K\phi = \frac{-1}{n}\overline{\mathrm{Ric}}(K,K) = \frac{\overline{c}}{n}|K|^2$$

and therefore using (2) it results

$$\langle \nabla\phi, K^\top\rangle = -\frac{K\phi}{|K|^2}\langle K^\top, K^\top\rangle = -\frac{\overline{c}}{n}\langle K^\top, K^\top\rangle.$$

Besides,

$$-\overline{S} - 2\overline{\mathrm{Ric}}(N,N) = -\overline{c}(n+1) + 2\overline{c} = -\overline{c}(n-1)$$

so that, taking into account (11), our integral formula can be rewritten as

$$\int_M \mathrm{Ric}(K^\top, K^\top) - \frac{\overline{c}}{n}(n-1)(n+2)\langle K^\top, K^\top\rangle = \int_M \overline{c}(n-1)|K|^2 - n(n-1)\phi^2 - S\langle K, N\rangle^2 \tag{16}$$

Observe that, under the hipothesis of Einstein ambient spacetime, the Ricci curvature of the leaves of the foliation determined by K (see (12)) is given by

$$\mathrm{Ric}(X,X) = (n-1)\left(\frac{\overline{c}}{n} - \frac{\phi^2}{|K|^2}\right)|X|^2$$

for all $X \in \mathcal{X}(M)$.

Theorem 2. *Let $\psi : M^n \longrightarrow \overline{M}^{n+1}$ be a compact spacelike hypersurface immersed in an Einstein conformally stationary-closed spacetime $\overline{M}$, being $\overline{\mathrm{Ric}} = \overline{c}\langle,\rangle$ for a certain $\overline{c} > 0$, and let us denote*

$$\alpha = (n-1)\left(\frac{\overline{c}}{n} - \frac{\phi^2}{|K|^2}\right).$$

i) If $\alpha \geq 0$ on M, the hyperbolic angle θ of M is such that

$$\cosh(\theta)^2 \leq \cosh(\theta_o)^2$$

for a certain $\theta_o \geq 0$, and the Ricci curvature of M satisfies that

$$\mathrm{Ric} \leq \frac{\alpha}{\cosh(\theta_o)^2},$$

then M is a leaf of $\mathcal{F}(K)$.

ii) If $\alpha \leq 0$ on M and $\mathrm{Ric} \leq \alpha$, then M is a leaf of $\mathcal{F}(K)$.

Proof: *i)* Thanks to the hypothesis on the Ricci curvature of M we get

$$\mathrm{Ric} \leq \frac{\alpha}{\cosh(\theta_o)^2} \leq (n-1)\frac{\overline{c}}{n}$$

and therefore the first integral in (16) is non positive. Actually,

$$\mathrm{Ric}(K^\top, K^\top) - \frac{\overline{c}}{n}(n-1)(n+2)\langle K^\top, K^\top\rangle \leq -\frac{\overline{c}}{n}(n-1)(n+1) \cdot\langle K^\top, K^\top\rangle \leq 0, \tag{17}$$

with equality if and only if $K^\top = 0$ on M. Regarding to the second integral, since

$$S = \mathrm{trace}(\mathrm{Ric}) \leq \frac{n\alpha}{\cosh(\theta_o)^2}$$

we have that

$$S\langle K, N\rangle^2 \leq \frac{n\alpha}{\cosh(\theta_o)^2}\langle K, N\rangle^2 = n\alpha|K|^2\frac{\cosh(\theta)^2}{\cosh(\theta_o)^2} \leq n\alpha|K|^2 = \overline{c}(n-1)|K|^2 - n(n-1)\phi^2.$$

Consequently

$$\int_M \{\overline{c}(n-1)|K|^2 - n(n-1)\phi^2 - S\langle K, N\rangle^2\}\mathrm{d}V \geq 0$$

and the equality holds in (17), namely, $K^\top = 0$ on the hypersurface and therefore is a leaf of the foliation $\mathcal{F}(K)$.
ii) From the hypothesis on the Ricci curvature of M we obtain

$$\mathrm{Ric} \leq \alpha \leq (n-1)\frac{\overline{c}}{n},$$

so if we reason as above, the inequality (17) holds, equality arising if and only if $K^\top$ vanishes on M.

On the other hand, taking into account that $|K|^2 \leq \langle K, N\rangle^2$ on M and $\alpha \leq 0$, we can deduce that

$$S \leq n\alpha \leq n\alpha\frac{|K|^2}{\langle K, N\rangle^2}$$

and therefore

$$S\langle K, N\rangle^2 \leq n\alpha|K|^2 = \overline{c}(n-1)|K|^2 - n(n-1)\phi^2.$$

Thus

$$\int_M \{\overline{c}(n-1)|K|^2 - n(n-1)\phi^2 - S\langle K, N\rangle^2\}\mathrm{d}V \geq 0$$

and accordingly M is a leaf of $\mathcal{F}(K)$. ∎

Theorem 3. *Let $\overline{M}^{n+1}$ be an Einstein conformally stationary-closed spacetime, being $\overline{\mathrm{Ric}} = \overline{c}\langle,\rangle$ for a certain $\overline{c} > 0$. Then there not exists any compact spacelike hypersurface in $\overline{M}$ with Ricci curvature* $\mathrm{Ric} \geq (\overline{c}/n)(n-1)(n+2)$.

Proof: Indeed, if $\psi : M \longrightarrow \overline{M}$ is a compact spacelike hypersurface in $\overline{M}$ such that its Ricci curvature satisfies that $\mathrm{Ric} \geq (\overline{c}/n)(n-1)(n+2)$, we have

$$\int_M \{\mathrm{Ric}(K^\top, K^\top) - \frac{\overline{c}}{n}(n-1)(n+2)\langle K^\top, K^\top\rangle\}\mathrm{d}V \geq 0. \tag{18}$$

On the other hand, since

$$S = \mathrm{trace}(\mathrm{Ric}) \geq \overline{c}(n-1)(n+2) > 0$$

we get

$$S\langle K, N\rangle^2 \geq \overline{c}(n-1)(n+2)\langle K, N\rangle^2 \geq \overline{c}(n-1)(n+2)|K|^2,$$

so that

$$\int_M \{\overline{c}(n-1)|K|^2 - n(n-1)\phi^2 - S\langle K, N\rangle^2\}\mathrm{d}V \leq \int_M \{-\overline{c}(n+1) \cdot (n-1)|K|^2\}\mathrm{d}V < 0$$

which is not possible from (18). ■

Corollary 4. *Let $\psi : M^n \longrightarrow \overline{M}^{n+1}$ a compact spacelike hypersurface immersed in an Einstein conformally stationary-closed spacetime $\overline{M}$, being $\overline{\mathrm{Ric}} = \overline{c}\langle,\rangle$ for a certain $\overline{c} > 0$. Then the Ricci curvature of M satisfies that*

$$\min_{\substack{p \in M \\ v \in T_pM \\ |v| = 1}} \mathrm{Ric}_p(v, v) < \frac{\overline{c}}{n}(n+2)(n-1).$$

Theorem 5. *Let $\psi : M^n \longrightarrow \overline{M}^{n+1}$ a compact spacelike hypersurface immersed in an Einstein conformally stationary-closed spacetime $\overline{M}$, being $\overline{\mathrm{Ric}} = \overline{c}\langle,\rangle$ with $\overline{c} = 0$. If the Ricci curvature of M satisfies that*

$$\mathrm{Ric} \leq -(n-1)\frac{\phi^2}{|K|^2} < 0$$

then M is a leaf of $\mathcal{F}(K)$.

Proof: Since $\overline{c} = 0$, the integral formula (16) left

$$\int_M \{\mathrm{Ric}(K^\top, K^\top) + n(n-1)\phi^2 + S\langle K, N\rangle^2\}\mathrm{d}V = 0. \tag{19}$$

But from the hypothesis on the Ricci curvature we have

$$\begin{aligned}\mathrm{Ric}(K^\top, K^\top) + S\langle K, N\rangle^2 &\leq -(n-1)\frac{\phi^2}{|K|^2}\langle K^\top, K^\top\rangle - n(n-1)\frac{\phi^2}{|K|^2}\langle K, N\rangle^2 \\ &= (n-1)\frac{\phi^2}{|K|^2}|K|^2 - (n+1)(n-1)\frac{\phi^2}{|K|^2}|K|^2\cosh(\theta)^2 \\ &\leq -n(n-1)\frac{\phi^2}{|K|^2}|K|^2 = -n(n-1)\phi^2\end{aligned}$$

and therefore equality holds. Thus $\cosh(\theta)^2 = 1$, whence $K^\top = 0$ and consequently M is a leaf of $\mathcal{F}(K)$. ■

Theorem 6. *Let $\overline{M}^{n+1}$ be an Einstein conformally stationary-closed spacetime, being $\overline{\mathrm{Ric}} = \overline{c}\langle , \rangle$ with $\overline{c} = 0$. Then there not exists any compact spacelike hypersurface in $\overline{M}$ with positive Ricci curvature.*

Proof: It results immediately from (19). ■

Corollary 7. *Let $\psi : M^n \longrightarrow \overline{M}^{n+1}$ a compact spacelike hypersurface immersed in an Einstein conformally stationary-closed spacetime $\overline{M}$, being $\overline{\mathrm{Ric}} = \overline{c}\langle , \rangle$ with $\overline{c} = 0$. Then the Ricci curvature of M satisfies that*

$$\min_{\substack{p \in M \\ v \in T_pM \\ |v| = 1}} \mathrm{Ric}_p(v, v) \leq 0.$$

Theorem 8. *Let $\overline{M}^{n+1}$ be an Einstein conformally stationary-closed spacetime, being $\overline{\mathrm{Ric}} = \overline{c}\langle , \rangle$ for a certain $\overline{c} < 0$. Then there not exists any compact spacelike hypersurface in $\overline{M}$ with Ricci curvature $\mathrm{Ric} \geq 0$.*

Proof: Let us suppose that $\psi : M^n \longrightarrow \overline{M}$ is a compact spacelike hypersurface with Ricci curvature $\mathrm{Ric} \geq 0$. Then we have

$$\int_M \{\mathrm{Ric}(K^\top, K^\top) - \frac{\overline{c}}{n}(n-1)(n+2)\langle K^\top, K^\top\rangle\}\mathrm{d}V \geq 0. \tag{20}$$

On the other hand, since $\overline{c} < 0$ it follows that necessarily

$$\alpha = (n-1)\left(\frac{\overline{c}}{n} - \frac{\phi^2}{|K|^2}\right) < 0,$$

that is

$$\overline{c}(n-1)|K|^2 - n(n-1)\phi^2 < 0$$

and consequently

$$\int_M \{\overline{c}(n-1)|K|^2 - n(n-1)\phi^2 - S\langle K, N\rangle^2\}\mathrm{d}V < 0$$

in conflict with (20). ■

Corollary 9. *Let $\psi : M^n \longrightarrow \overline{M}^{n+1}$ a compact spacelike hypersurface immersed in an Einstein conformally stationary-closed spacetime $\overline{M}$, being $\overline{\mathrm{Ric}} = \overline{c}\langle , \rangle$ for a certain $\overline{c} < 0$. Then the Ricci curvature of M satisfies that*

$$\min_{\substack{p \in M \\ v \in T_pM \\ |v| = 1}} \mathrm{Ric}_p(v, v) < 0.$$

5. CASE OF GENERALIZED ROBERTSON-WALKER SPACETIMES

An interesting subclass of spacetimes equipped with a closed conformal vector field is the family of *Generalized Robertson-Walker spacetimes (GRW spacetimes)*. Recall that given (F, g) an n-dimensional ($n \geq 2$) Riemannian manifold and $I \subset \mathbf{R}$ an open interval in $\mathbf{R}$ endowed with the metric $-\mathrm{d}t^2$, the warped product manifold $\overline{M} = I \times F$ endowed with the Lorentzian metric

$$\langle , \rangle = \pi_I^*(-\mathrm{d}t^2) + f^2(\pi_I)\pi_F^*(g)$$

where $f > 0$ is a smooth function on I, and π_I and π_F denote the projections onto I and F respectively, is said to be a GRW spacetime with *base* $(I, -\mathrm{d}t^2)$, *fiber* (F, g) and *warping function* f (see [5]). Note that, in the above definition of GRW spacetime, the fiber (F, g) is not assume to be of constant sectional curvature. When this holds and $n = 3$, the GRW spacetime is a (classical) Robertson-Walker spacetime. Thus, GRW spacetimes widely extend to Robertson-Walker spacetimes and include, for instance, the de Sitter spacetime, Friedmann cosmological models and the static Einstein spacetime.

GRW spacetimes are suitable spacetimes to model universes with inhomogeneous spacelike geometry [19]. In fact, it is well-known that conformal changes of the metric of a GRW spacetime with a conformal factor which only depends on t, produce new GRW spacetimes. Even more, small deformations of the metric on the fiber of Robertson-Walker spacetimes also fit into the class of GRW spacetimes. Thus, a GRW spacetime is not necessarily spatially homogeneous, as in the classical cosmological models. Recall that spatial homogeneity seems appropriate just as a rough approach to consider the universe in the large, but not to consider it in a more accurate scale, because this assumption could not be realistic.

It is not difficult to check that, in the case of GRW spacetimes, the timelike closed conformal vector field may be chosen as

$$K = f(\pi_I)\partial/\partial_t \tag{21}$$

and so

$$\phi = f'(\pi_I). \tag{22}$$

Moreover, it is worth pointing out that the existence of a compact spacelike hypersurface in $\overline{M}$ implies the compactness of the fiber F (see, for instance, [5, Proposition 3.2]).

As is well known (see [6] for the details), a GRW spacetime $\overline{M}$ is Einstein with $\overline{\mathrm{Ric}} = \overline{c}\langle , \rangle, \overline{c} \in \mathbf{R}$, if and only if the fiber (F, g) has constant Ricci curvature c and the warping function f satisfies the differential equations

$$\frac{f''}{f} = \frac{\overline{c}}{n} \qquad \text{and} \qquad \frac{\overline{c}(n-1)}{n} = \frac{c + (n-1)(f')^2}{f^2}. \tag{23}$$

Let us consider a compact spacelike hypersurface $\psi : M \longrightarrow \overline{M}$ in an Einstein GRW spacetime $\overline{M}$ (necessarily with compact fiber). Taking into account our comments about the closed conformal vector field (21), its associated function (22), and the equations in (23), an easy computation allows us to rewrite the integral formula (16) as

$$\int_M \{\mathrm{Ric}(K^\top, K^\top) - \frac{\overline{c}}{n}(n-1)(n+2)\langle K^\top, K^\top\rangle\}\mathrm{d}V = \int_M \{nc - S\langle K, N\rangle^2\}\mathrm{d}V. \tag{24}$$

Observe that in this case, the hypothesis $K^\top = 0$ on the hypersurface implies that $\pi_I \circ \psi = t_o$ is constant on M, since

$$K^\top = -f(\pi_I \circ \psi)\nabla(\pi_I \circ \psi) \qquad \text{and} \qquad f > 0.$$

Such hypersurfaces are said to be *spacelike slices*, and they are homothetic to the fiber (F, g) with scale factor $1/f(t_o)$. In particular they have constant Ricci curvature equal to $c/f^2(t_o)$.

As a consequence of the above remarks, we can rewrite Theorems 2 and 5 as follows:

Theorem 10. *Let $\psi : M^n \longrightarrow \overline{M}^{n+1}$ be a compact spacelike hypersurface immersed in an Einstein GRW spacetime $\overline{M}$ with warping function f, being $\overline{\mathrm{Ric}} = \overline{c}\langle , \rangle$ for a certain $\overline{c} > 0$, and let us denote by c the constant Ricci curvature of the fiber.*

i) If $c \geq 0$, the hyperbolic angle θ of M is such that

$$\cosh(\theta)^2 \leq \cosh(\theta_o)^2$$

for a certain $\theta_o \geq 0$, and the Ricci curvature of M satisfies that

$$\mathrm{Ric} \leq \frac{c}{f^2 \cosh(\theta_o)^2},$$

then M is a spacelike slice.

ii) If $c \leq 0$ and $\mathrm{Ric} \leq c/f^2$, then M is a spacelike slice.

Proof: Observe that if $\overline{M}$ is an Einstein GRW spacetime, it follows from (23) that

$$\alpha = (n-1)\left(\frac{\overline{c}}{n} - \frac{\phi^2}{|K|^2}\right) = \frac{c}{f^2},$$

so that the proof results immediately from Theorem 2. ∎

Theorem 11. *Let $\psi : M^n \longrightarrow \overline{M}^{n+1}$ be a compact spacelike hypersurface immersed in an Einstein GRW spacetime $\overline{M}$ with warping function f, being $\overline{\mathrm{Ric}} = \overline{c}\langle , \rangle$ with $\overline{c} = 0$, and let us denote by c the constant Ricci curvature of the fiber. If $c < 0$ and the Ricci curvature of M satisfies that $\mathrm{Ric} \leq c/f^2$, then M is a spacelike slice.*

Theorems 3, 6 and 8, jointly their corresponding corollaries, can be enunciated for Einstein GRW spacetimes in similar terms. However, attending to the constant Ricci curvature of the fiber, Theorem 6 can be stated in the following way:

Theorem 12. *Let $\overline{M}^{n+1}$ be an Einstein GRW spacetime, being $\overline{\mathrm{Ric}} = \overline{c}\langle , \rangle$ with $\overline{c} = 0$, and let us denote by c the constant Ricci curvature of the fiber.*

i) If $c < 0$, then there not exists any compact spacelike hypersurface in $\overline{M}$ with Ricci curvature $\mathrm{Ric} \geq 0$.

ii) If $c = 0$, then there not exists any compact spacelike hypersurface in $\overline{M}$ with $\mathrm{Ric} > 0$ or $\mathrm{Ric} < 0$.

Proof: If $\psi : M^n \longrightarrow \overline{M}^{n+1}$ is a compact spacelike hypersurface in $\overline{M}$, it holds on M that (see (24))

$$\int_M \{\mathrm{Ric}(K^\top, K^\top) - nc + S\langle K, N\rangle^2\} \mathrm{d}V = 0.$$

The proof finishes easily by discussing the several cases on this formula. ∎

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