The Gauss map and second fundamental form of surfaces in $\mathbf{R}^3$

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Abstract

Given a surface S, a map N from S to $\mathbb{S}^2$ and a conformal structure on S, we solve the problem of existence and uniqueness of an immersion $x : S \longrightarrow \mathbb{R}^3$ with Gauss map N and such that the conformal structure on S is the induced by the second fundamental form.

1991 Mathematics Subject Classification: 53A05, 53A07, 58E20.

Key Words and Phrases: Gauss map, second fundamental form, Weierstrass representation, harmonic maps.

1 Introduction

The properties of the Gauss map on a submanifold in $\mathbb{R}^n$ and the extent to which the Gauss map determines the immersion of the submanifold have been of great interest in Differential Geometry (see [1], [7], [8], [9], [10], [13], [14], [15], [16]).

The existence and uniqueness of an immersion from a surface or a hypersurface into $\mathbb{R}^n$ with a given metric (or conformal structure) and a given Gauss map has been studied by several authors. D. A. Hoffman, R. Osserman and K. Kenmotsu ([8], [9], [10]) researched the existence and uniqueness problem for surfaces. They proved there exists a conformal non-minimal immersion from a simply-connected surface in $\mathbb{R}^n$ with a given Gauss map if and only if a set of differential equations depending on the conformal structure and the Gauss map is satisfied. The uniqueness problem for hypersurfaces was also studied by K. Abe and J. Erbacher (see [1]).

Our object in this paper is to study properties of the Gauss map of a surface immersed

*Research partially supported by DGICYT Grant No. PB97-0785

in $\mathbb{R}^3$, particularly those related to the geometry of the immersion and the conformal structure determined by its second fundamental form. The main problem considered is the existence and uniqueness of an immersion $x : S \longrightarrow \mathbb{R}^3$ from a surface S with prescribed conformal structure that yields a given Gauss map and for which the second fundamental form is a conformal metric on S.

Among the results we obtain are the following:

(A) For a simply-connected surface with non-zero Gauss curvature, the existence of such immersion is equivalent to that the Gauss map is a local diffeomorphism and a third-order differential equation involving the conformal structure and the Gauss map is satisfied. Moreover, we recover the immersion by a representation similar to the Enneper-Weierstrass formula. (Theorems 3 and 5).

(B) If the set of points where the Gauss curvature vanishes has empty interior then x is uniquely determined, up to similarities, by the Gauss map and the conformal structure given by the second fundamental form. Otherwise, if the hypothesis about the Gauss curvature is not satisfied, the immersion is, in general, non-unique. (Corollary 4 and Remark 5).

(C) A hypersurface in $\mathbb{R}^n$ with non-degenerate second fundamental form has constant Gauss-Kronecker curvature if and only if its Gauss map is harmonic from the hypersurface with the metric given by its second fundamental form. Thus, surfaces with non-zero constant Gauss curvature can be recovered from each harmonic local diffeomorphism into the unit sphere using the above mentioned formula, (Corollaries 2, 3 and Theorem 6). E. A. Ruh and J. Vilms proved in [15] similar results involving constant mean curvature and the conformal structure induced by the first fundamental form.

Some of these results will be used in [5] in order to estimate the height, area, curvature and enclosed volume of a surface with positive constant Gauss curvature in $\mathbb{R}^3$ bounding a planar curve.

2 Surfaces in $\mathbf{R}^3$

2.1 Surfaces with Positive Gauss Curvature

Let S be a smooth surface and $x : S \longrightarrow \mathbb{R}^3$ an immersion with positive Gauss curvature. Then some deleted neighborhood n_p of any point p on S lies to one side of the tangent plane T_pS to S at p. A smooth unit normal vector field $N : S \longrightarrow \mathbb{S}^2 \subseteq \mathbb{R}^3$ is obtained by assigning at each point p of S the unit normal vector to the same side of T_pS as n_p. This orients S and makes the quadratic form σ associated with the second fundamental form defined by

$$\sigma_p(v,w) =< -dN_p(v), w >, \qquad p \in S, \quad v, w \in T_pS,$$

into a positive definite metric. Here $<,>$ is the usual inner product in $\mathbb{R}^3$. Throughout §2.1, S will be considered as a Riemann surface with the conformal structure induced by σ.

Let $z = u + iv$ be a conformal parameter,

$$\begin{array}{lll} E =< x_u, x_u >, & F =< x_u, x_v >, & G =< x_v, x_v >, \\ e = \sigma(x_u, x_u), & 0 = \sigma(x_u, x_v), & e = \sigma(x_v, x_v), \end{array} \tag{1}$$

where $e > 0$ and, for instance, $x_u = \frac{\partial x}{\partial u}$. Then the Weingarten equations (see pp. 154-155 in [2] or p. 143 in [17]) state that

$$N_u = \frac{e}{EG - F^2}(-Gx_u + Fx_v), \qquad N_v = \frac{e}{EG - F^2}(Fx_u - Ex_v). \tag{2}$$

Let us denote by $g : S \longrightarrow \mathbb{C} \cup \{\infty\}$ the composition of the usual stereographic projection with N, that is,

$$g = (N_1 + iN_2)/(1 - N_3), \tag{3}$$

where $N = (N_1, N_2, N_3)$. We will also call g the **Gauss map** of the immersion. Thus we have:

Theorem 1 *Let $x : S \longrightarrow \mathbb{R}^3$ be an immersion with positive Gauss curvature, K, $g : S \longrightarrow \mathbb{C} \cup \{\infty\}$ its Gauss map and $z = u + iv$ a conformal parameter. Then*

$$\begin{aligned} \frac{\partial x_1}{\partial z} &= \frac{(1 - \overline{g}^2)g_z - (1 - g^2)\overline{g}_z}{\sqrt{K}\,(1 + g\overline{g})^2}, \\ \frac{\partial x_2}{\partial z} &= -i\frac{(1 + \overline{g}^2)g_z + (1 + g^2)\overline{g}_z}{\sqrt{K}\,(1 + g\overline{g})^2}, \\ \frac{\partial x_3}{\partial z} &= 2\frac{\overline{g}g_z - g\overline{g}_z}{\sqrt{K}\,(1 + g\overline{g})^2}, \end{aligned} \tag{4}$$

where $x = (x_1, x_2, x_3)$, $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)$ and by bar we will denote the complex conjugation.

Proof: Use (3) to get

$$\begin{aligned} \frac{\partial g}{\partial z} &= \frac{1}{(1 - N_3)^2}\left(\frac{\partial(N_1 + iN_2)}{\partial z} - N_3\frac{\partial(N_1 + iN_2)}{\partial z} + \frac{\partial N_3}{\partial z}(N_1 + iN_2)\right), \\ \frac{\partial g}{\partial \overline{z}} &= \frac{1}{(1 - N_3)^2}\left(\frac{\partial(N_1 + iN_2)}{\partial \overline{z}} - N_3\frac{\partial(N_1 + iN_2)}{\partial \overline{z}} + \frac{\partial N_3}{\partial \overline{z}}(N_1 + iN_2)\right), \end{aligned} \tag{5}$$

where $2\ \partial/\partial\overline{z} = \partial/\partial u + i\partial/\partial v = 2\ \overline{\partial/\partial z}$. Now use (1), (2),

$$N = \frac{x_u \wedge x_v}{|x_u \wedge x_v|}, \tag{6}$$

and $K = e^2/(EG - F^2)$ to express the second and third terms within brackets in (5) as products of first partials with respect to u or v of the components of x. After simplification, this gives

$$\begin{aligned} \frac{\partial g}{\partial z} &= \frac{1}{(1-N_3)^2}\left(\frac{\partial}{\partial z}(N_1 + iN_2) + \sqrt{K}\,\frac{\partial}{\partial z}(x_1 + ix_2)\right), \\ \frac{\partial g}{\partial \overline{z}} &= \frac{1}{(1-N_3)^2}\left(\frac{\partial}{\partial \overline{z}}(N_1 + iN_2) - \sqrt{K}\,\frac{\partial}{\partial \overline{z}}(x_1 + ix_2)\right). \end{aligned} \tag{7}$$

Since $< N, N >= 1$, (3) gives

$$N_1 = \frac{g + \overline{g}}{1 + g\overline{g}}, \quad N_2 = -i\frac{g - \overline{g}}{1 + g\overline{g}}, \quad N_3 = \frac{-1 + g\overline{g}}{1 + g\overline{g}}. \tag{8}$$

Then (7) can be written

$$\begin{aligned} \frac{4g_z}{(1+g\overline{g})^2} &= \left(\frac{2g}{1+g\overline{g}}\right)_z + \sqrt{K}\,(x_1 + ix_2)_z, \\ \frac{4g_{\overline{z}}}{(1+g\overline{g})^2} &= \left(\frac{2g}{1+g\overline{g}}\right)_{\overline{z}} - \sqrt{K}\,(x_1 + ix_2)_{\overline{z}}, \end{aligned}$$

or equivalently,

$$2\frac{g_z + g^2\overline{g}_z}{(1+g\overline{g})^2} = \sqrt{K}\,(x_1 + ix_2)_z, \tag{9}$$

$$2\frac{g_{\overline{z}} + g^2\overline{g}_{\overline{z}}}{(1+g\overline{g})^2} = -\sqrt{K}\,(x_1 + ix_2)_{\overline{z}}. \tag{10}$$

The two first equations of (4) are obtained from (9) and the conjugated equation of (10).

Now, since (6) gives $< x_u, N >=< x_v, N >= 0$, one has

$$x_{3u} = -\frac{N_1 x_{1u} + N_2 x_{2u}}{N_3}, \quad x_{3v} = -\frac{N_1 x_{1v} + N_2 x_{2v}}{N_3},$$

and using (8)

$$x_{3z} = \frac{g + \overline{g}}{1 - g\overline{g}} x_{1z} - i\frac{g - \overline{g}}{1 - g\overline{g}} x_{2z}. \tag{11}$$

The third equation of (4) follows from (9), (10) and (11). □

A straight computation gives us

Corollary 1 *With the above notation, the first and second fundamental forms of the immersion are given, respectively, by*

$$ds^2 = \frac{4}{K(1+g\overline{g})^2}\left(-g_z\overline{g}_z dz^2 + (g_{\overline{z}}\overline{g}_z + g_z\overline{g}_{\overline{z}})|dz|^2 - g_{\overline{z}}\overline{g}_{\overline{z}}d\overline{z}^2\right),$$
$$\sigma = 4\frac{|g_{\overline{z}}|^2 - |g_z|^2}{\sqrt{K}(1+g\overline{g})^2}|dz|^2.$$

Now, we study the structure equations of the immersion.

Theorem 2 *If $x : S \longrightarrow \mathbb{R}^3$ is an immersion with positive Gauss curvature K, then the Gauss map g satisfies:*

$$4K\left(g_{z\overline{z}} - 2g_z g_{\overline{z}}\frac{\overline{g}}{1+g\overline{g}}\right) = K_z g_{\overline{z}} + K_{\overline{z}} g_z. \tag{E}$$

Moreover, the Gauss curvature is determined, up to multiplication by positive constants, by the Gauss map.

Proof: From Theorem 1, $(x_{iz})_{\overline{z}} = (x_{i\overline{z}})_z$ $i = 1, 2, 3$ if and only if

$$\frac{8K}{1+g\overline{g}}\left((-\overline{g}+\overline{g}^3)g_z g_{\overline{z}} + (g-g^3)\overline{g}_z\overline{g}_{\overline{z}}\right) + K_z\left((1-g^2)\overline{g}_{\overline{z}} - (1-\overline{g}^2)g_{\overline{z}}\right) +$$
$$K_{\overline{z}}\left((1-g^2)\overline{g}_z - (1-\overline{g}^2)g_z\right) + 4K\left((1-\overline{g}^2)g_{z\overline{z}} - (1-g^2)\overline{g}_{z\overline{z}}\right) = 0, \tag{12}$$

$$\frac{8K}{1+g\overline{g}}\left((\overline{g}+\overline{g}^3)g_z g_{\overline{z}} + (g+g^3)\overline{g}_z\overline{g}_{\overline{z}}\right) + K_z\left((1+g^2)\overline{g}_{\overline{z}} + (1+\overline{g}^2)g_{\overline{z}}\right) +$$
$$K_{\overline{z}}\left((1+g^2)\overline{g}_z + (1+\overline{g}^2)g_z\right) - 4K\left((1+\overline{g}^2)g_{z\overline{z}} + (1+g^2)\overline{g}_{z\overline{z}}\right) = 0, \tag{13}$$

$$\frac{8K}{1+g\overline{g}}\left(-\overline{g}^2 g_z g_{\overline{z}} + g^2\overline{g}_z\overline{g}_{\overline{z}}\right) + K_z\left(g\overline{g}_{\overline{z}} - \overline{g}g_{\overline{z}}\right) + K_{\overline{z}}\left(g\overline{g}_z - \overline{g}g_z\right) +$$
$$4K\left(\overline{g}g_{z\overline{z}} - g\overline{g}_{z\overline{z}}\right) = 0. \tag{14}$$

If we take (12) minus (13) plus $2g$ times (14), then we obtain (E).

Moreover, (E) and its conjugated equation yield

$$(\log K)_z g_{\overline{z}} + (\log K)_{\overline{z}} g_z = 4\left(g_{z\overline{z}} - 2g_z g_{\overline{z}}\frac{\overline{g}}{1+g\overline{g}}\right),$$
$$(\log K)_z \overline{g}_{\overline{z}} + (\log K)_{\overline{z}} \overline{g}_z = 4\left(\overline{g}_{z\overline{z}} - 2\overline{g}_z\overline{g}_{\overline{z}}\frac{g}{1+g\overline{g}}\right).$$

Now, $|g_{\overline{z}}|^2 - |g_z|^2 > 0$ gives

$$(\log K)_z = \frac{4}{|g_{\overline{z}}|^2 - |g_z|^2}\left(\overline{g}_z g_{z\overline{z}} - g_z\overline{g}_{z\overline{z}} + 2g_z\overline{g}_z\frac{g\overline{g}_{\overline{z}} - \overline{g}g_{\overline{z}}}{1+g\overline{g}}\right). \tag{L}$$

From (L) it is clear that K is determined, up to multiplication by positive constants, by g. □

Remark 1: By Lemma 5 in [12], $K > 0$ is constant on S if and only if $K < x_z, x_z >$ is holomorphic. (Use the fact that ds^2 and σ satisfy the Codazzi-Mainardi equations on p. 235 of [2] or p. 144 of [17].) Since Theorem 3 in [12] shows that $K < x_z, x_z >$ is holomorphic on S if and only if $N : S \longrightarrow \mathbb{S}^2$ is harmonic (as defined in [3], [4] or [12]), it follows from Theorem 2 that $N : S \longrightarrow \mathbb{S}^2$ is harmonic if and only if

$$g_{z\overline{z}} - 2g_z g_{\overline{z}}\frac{\overline{g}}{1+g\overline{g}} = 0,$$

holds wherever $g \neq \infty$ on S.

Remark 2: If $x : S \longrightarrow \mathbb{R}^3$ is an immersion with constant $K > 0$, then computation based on Theorems 1 and 2, or formula (9) from [11] gives

$$\Delta^\sigma x = 2N,$$

where $\Delta^\sigma = 1/e\ \Delta$ is the Laplacian for σ and Δ is the usual Laplacian in the u,v-plane. Thus, in a similar way to constant mean curvature (see, for instance, [6]), the existence of a simply-connected surface with constant Gauss curvature $K > 0$ and boundary a Jordan curve $\Gamma \subset \mathbb{R}^3$ is equivalent to solve the following Plateau problem:
$x : \Omega \longrightarrow \mathbb{R}^3$ such that

(a) $\Delta x = 2\sqrt{K} x_u \wedge x_v$,
(b) $det(x_{uu} - x_{vv}, x_u, x_v) = 0 = det(x_{uv}, x_u, x_v)$ (conformality),
(c) $x : \partial\Omega \longrightarrow \mathbb{R}^3$ is an admissible representation of the Jordan curve Γ.

where Ω is the unit disk and det the usual determinant.

Remark 3: The equation (L) is equivalent to (E). And (E) is satisfied if and only if (12), (13) and (14) are satisfied.

Theorem 3 *Let S be a simply connected Riemann surface and $N : S \longrightarrow \mathbb{S}^2 \subseteq \mathbb{R}^3$ a differentiable map. Then, there exists an immersion $x : S \longrightarrow \mathbb{R}^3$ with Gauss map N and such that the conformal structure on S is the induced one by the second fundamental form if and only if*

$$|g_{\overline{z}}|^2 - |g_z|^2 > 0, \tag{15}$$

$$Im\left\{\frac{\partial}{\partial\overline{z}}\left(\frac{4}{|g_{\overline{z}}|^2 - |g_z|^2}\left(\overline{g}_z g_{z\overline{z}} - g_z\overline{g}_{z\overline{z}} + 2g_z\overline{g}_z\frac{g\overline{g}_{\overline{z}} - \overline{g}g_{\overline{z}}}{1+g\overline{g}}\right)\right)\right\} = 0, \tag{16}$$

where g is as in (3). Moreover, the immersion is unique, up to a similarity transformation of $\mathbb{R}^3$ and it can be recovered using the equations

$$
\begin{aligned}
x_1 &= \int Re\left\{2\frac{(1-\overline{g}^2)g_z-(1-g^2)\overline{g}_z}{\sqrt{K}\,(1+g\overline{g})^2}\,dz\right\}+c_1,\\
x_2 &= \int Re\left\{-2i\frac{(1+\overline{g}^2)g_z+(1+g^2)\overline{g}_z}{\sqrt{K}\,(1+g\overline{g})^2}\,dz\right\}+c_2,\\
x_3 &= \int Re\left\{4\frac{\overline{g}g_z-g\overline{g}_z}{\sqrt{K}\,(1+g\overline{g})^2}\,dz\right\}+c_3,
\end{aligned}
\tag{17}
$$

where

$$
\log K = \int Re\left\{\frac{8}{|g_{\overline{z}}|^2-|g_z|^2}\left(\overline{g}_z g_{z\overline{z}}-g_z\overline{g}_{z\overline{z}}+2g_z\overline{g}_z\frac{g\overline{g}_{\overline{z}}-\overline{g}g_{\overline{z}}}{1+g\overline{g}}\right)dz\right\}+\lambda,
$$

$c_1,\ c_2,\ c_3,\ \lambda$ *are real constants and the integrals are taken along a path from a fixed point to a variable point.*

Proof: If S is a Riemann surface with conformal structure given by the second fundamental form of an immersion $x : S \longrightarrow \mathbb{R}^3$, then $K > 0$, so that $\log K$ and $\partial^2(\log K)/\partial z \partial \overline{z}$ must both be real. The result follows from Theorems 1, 2 and Corollary 1.

Conversely, since S is simply connected, there exists $\varphi : S \longrightarrow \mathbb{R}$, such that $K = e^{\varphi}$, satisfying (L) if and only if (16) is satisfied. Now, from Remark 3, it is easy to check that (L) (or equivalently, (E)) is the complete integrability condition for (4).

Moreover, if $x_1, x_2 : S \longrightarrow \mathbb{R}^3$ are two immersions as above with Gauss curvature $K_1,\ K_2$, respectively, then $(\log K_1)_z = (\log K_2)_z$ and $K_1 = rK_2$ for some positive constant r. Thus $x_{2z} = \sqrt{r}x_{1z}$ and $x_2 = \sqrt{r}x_1 + c$, $c \in \mathbb{R}^3$. □

Corollary 2 *Let S be a simply connected Riemann surface. Then S can be immersed in $\mathbb{R}^3$ with constant Gauss curvature and the conformal structure on S is given by its second fundamental form if and only if there exists a harmonic local diffeomorphism from S to $\mathbb{S}^2$.*

Proof: If $x : S \longrightarrow \mathbb{R}^3$ is an immersion with constant Gauss curvature K, since S is a Riemann surface with conformal structure given by the second fundamental form, then K must be positive. Consequently, N is a local diffeomorphism and, from Remark 1, $N : S \longrightarrow \mathbb{S}^2$ must be harmonic.

Conversely, if $N : S \longrightarrow \mathbb{S}^2$ is a harmonic local diffeomorphism, then (using $-N$ for N, if necessary, to make $|g_{\overline{z}}| > |g_z|$ for the g obtained from (3)), both (15) and (16) are satisfied, so the immersion can be calculated using (17). Finally, from Theorem 3 $(\log K)_z = 0$, so that K must be a positive constant. □

2.2 Surfaces with Negative Gauss Curvature

Let S be an orientable smooth surface and $x : S \longrightarrow \mathbb{R}^3$ an immersion with negative Gauss curvature. Then σ is a Lorentz metric and S can be considered as a Lorentz surface (see [17], p. 13).

Now, we choose a unit normal vector field N on S compatible with proper σ-null coordinates (u, v), so that,

$$\sigma(x_u, x_u) = 0, \qquad f = \sigma(x_u, x_v) > 0, \qquad \sigma(x_v, x_v) = 0.$$

Any other proper σ-null coordinates $\hat{u}$, $\hat{v}$ are related to u, v by $\hat{u} = \hat{u}(u)$, $\hat{v} = \hat{v}(v)$ with $\hat{u}'(u)\,\hat{v}'(v) > 0$.

The Weingarten equations now take the form

$$N_u = \frac{f}{EG - F^2}(Fx_u - Ex_v), \qquad N_v = \frac{f}{EG - F^2}(-Gx_u + Fx_v). \tag{18}$$

Proceeding as in §2.1, use of (1), (3), (6), (18) and $K = -f^2/(EG - F^2)$ gives

$$\begin{aligned}
(x_1)_u &= -2\frac{Im((1-g^2)\overline{g}_u)}{\sqrt{-K}\,(1+g\overline{g})^2} & (x_1)_v &= 2\frac{Im((1-g^2)\overline{g}_v)}{\sqrt{-K}\,(1+g\overline{g})^2} \\
(x_2)_u &= -2\frac{Re((1+g^2)\overline{g}_u)}{\sqrt{-K}\,(1+g\overline{g})^2} & (x_2)_v &= 2\frac{Re((1+g^2)\overline{g}_v)}{\sqrt{-K}\,(1+g\overline{g})^2} \\
(x_3)_u &= -4\frac{Im(g\overline{g}_u)}{\sqrt{-K}\,(1+g\overline{g})^2} & (x_3)_v &= 4\frac{Im(g\overline{g}_v)}{\sqrt{-K}\,(1+g\overline{g})^2}
\end{aligned}$$

so that

$$\begin{aligned}
E &= -\frac{4g_u\overline{g}_u}{K\,(1+g\overline{g})^2} & F &= 2\frac{g_u\overline{g}_v + g_v\overline{g}_u}{K\,(1+g\overline{g})^2} \\
G &= -\frac{4g_v\overline{g}_v}{K\,(1+g\overline{g})^2} & f &= 2i\frac{g_u\overline{g}_v - g_v\overline{g}_u}{\sqrt{-K}\,(1+g\overline{g})^2}.
\end{aligned}$$

Setting $(x_1)_{uv} = (x_1)_{vu}$ and $(x_2)_{uv} = (x_2)_{vu}$, one obtains the following analog of Theorem 2.

Theorem 4 *Let S be an oriented surface, $x : S \longrightarrow \mathbb{R}^3$ an immersion with negative Gauss curvature K, then the Gauss map must satisfy:*

$$4K\left(g_{uv} - 2g_u g_v \frac{\overline{g}}{1+g\overline{g}}\right) = K_u g_v + K_v g_u.$$

Moreover, the Gauss curvature is determined, up to positive constants, by the Gauss map.

Remark 4: Fact 3 and Lemma 8 from [12] show that $K<0$ is constant on S if and only if $K<x_u x_u>$ and $K<x_v,x_v>$ depend only on u and v respectively. (Here again, use the fact that ds^2 and σ satisfy the Codazzi-Mainardi equations.) Lemma 6 and Theorem 4 from [12] then show that $K<0$ is constant on S if and only if $N:S\longrightarrow\mathbb{S}^2$ is harmonic. Thus $N:S\longrightarrow\mathbb{S}^2$ is harmonic on S if and only if

$$g_{uv}-2g_u g_v\frac{\overline{g}}{1+g\overline{g}}=0$$

holds wherever $g\neq\infty$.

Theorem 5 *Let S be a simply connected Lorentz surface and $N:S\longrightarrow\mathbb{S}^2\subseteq\mathbb{R}^3$ a differentiable map. Then, there exists an immersion $x:S\longrightarrow\mathbb{R}^3$ such that the structure given by the second fundamental form is the one given on S and N its Gauss map if and only if*

$$\begin{aligned}
&i(g_u\overline{g}_v-\overline{g}_u g_v)>0, \qquad \textit{and}\\
&\frac{\partial}{\partial u}\left(\frac{1}{\overline{g}_u g_v-g_u\overline{g}_v}\left(Im(g_v\overline{g}_{uv})+2|g_v|^2\frac{Im(\overline{g}g_u)}{1+g\overline{g}}\right)\right)=\\
&\qquad\qquad=\frac{\partial}{\partial v}\left(\frac{1}{\overline{g}_u g_v-g_u\overline{g}_v}\left(Im(\overline{g}_u g_{uv})+2|g_u|^2\frac{Im(g\overline{g}_v)}{1+g\overline{g}}\right)\right),
\end{aligned}$$

where g is as in (3). Moreover, the immersion is unique, up to a similarity transformation of $\mathbb{R}^3$. And it can be calculated as follows:

$$\begin{aligned}
x_1&=\int\left(-2\frac{Im((1-g^2)\overline{g}_u)}{\sqrt{-K}\,(1+g\overline{g})^2}\,du+2\frac{Im((1-g^2)\overline{g}_v)}{\sqrt{-K}\,(1+g\overline{g})^2}\,dv\right)+c_1,\\
x_2&=\int\left(-2\frac{Re((1+g^2)\overline{g}_u)}{\sqrt{-K}\,(1+g\overline{g})^2}\,du+2\frac{Re((1+g^2)\overline{g}_v)}{\sqrt{-K}\,(1+g\overline{g})^2}\,dv\right)+c_2,\\
x_3&=\int\left(-4\frac{Im(g\overline{g}_u)}{\sqrt{-K}\,(1+g\overline{g})^2}\,du+4\frac{Im(g\overline{g}_v)}{\sqrt{-K}\,(1+g\overline{g})^2}\,dv\right)+c_3,
\end{aligned}$$

where

$$\begin{aligned}
\log(-K)\;=\;&\int\left(\frac{8}{\overline{g}_u g_v-g_u\overline{g}_v}\left(Im(\overline{g}_u g_{uv})+2|g_u|^2\frac{Im(g\overline{g}_v)}{1+g\overline{g}}\right)du+\right.\\
&\left.+\frac{8}{\overline{g}_u g_v-g_u\overline{g}_v}\left(Im(g_v\overline{g}_{uv})+2|g_v|^2\frac{Im(\overline{g}g_u)}{1+g\overline{g}}\right)dv\right)+\lambda,
\end{aligned}$$

c_1, c_2, c_3, λ *are real constants and the integrals are taken along a path from a fixed point to a variable point.*

Corollary 3 *Let S be a simply connected Lorentz surface. Then S can be immersed in $\mathbb{R}^3$ with constant Gauss curvature and the conformal structure on S is given by the second fundamental form if and only if there exists a harmonic local diffeomorphism from S to $\mathbb{S}^2$. Moreover, the immersion can be calculated as in the above theorem.*

2.3 Uniqueness of the immersion.

Now we will prove the following uniqueness result.

Corollary 4 *Let S be a connected, oriented surface, $\chi_i : S \longrightarrow \mathbb{R}^3$, i=1,2, two immersions with the same Gauss map and conformal structure of the second fundamental form. If the set $S_0 = \{p \in S/\ dN_p$ is not injective$\}$ has empty interior then the immersions agree, up to a similarity of $\mathbb{R}^3$.*

Proof: Let $<,>_i$ be the induced metric on S by the immersion χ_i, that is, $<,>_i = \chi_i^*(<,>)$, $i = 1, 2$. So, since S_0 is a closed set, if $q \notin S_0$ then from theorems 3 and 5 there exists a simply connected, open neighbourhood of q on which $\chi_1 = \mu(q)\chi_2 + b(q)$, where $\mu(q) \neq 0$ and $b(q) \in \mathbb{R}^3$ are constants.

Since the interior of S_0 is empty and $<,>_1 (q) = \mu(q)^2 <,>_2 (q)$ for all $q \in S - S_0$, the above equality is true everywhere. Moreover, μ^2 is a differentiable function such that $d\mu^2 = 0$ on $S - S_0$. Therefore, μ^2 is constant and μ is constant on each connected component of $S - S_0$ and equal to r or $-r$, with $r \neq 0$.

If $p \in S_0$ then there is a neighbourhood U of p such that μ is constant on $U \cap (S - S_0)$. Otherwise, there would exist two sequences of points p_m, p_n which tend to p and $(d\chi_1)_{p_m} = r(d\chi_2)_{p_m}$, $(d\chi_1)_{p_n} = -r(d\chi_2)_{p_n}$. So, we obtain, $r(d\chi_2)_p = (d\chi_1)_p = -r(d\chi_2)_p$. Thus we can assume $\mu = r$ on S and since $d(\chi_1 - r\chi_2) = 0$ the proof is completed. □

Remark 5: If the set of points, where dN_p is not injective, has not empty interior then the corollary does not remain true. For instance, we can consider the immersions

$$\chi_1(u, v) = (2\cos u, \frac{1}{2}\sin u, v),$$
$$\chi_2(u, v) = (\frac{\cos u}{2\sqrt{\frac{1}{4}\cos^2 u + 4\sin^2 u}}, \frac{2\sin u}{\sqrt{\frac{1}{4}\cos^2 u + 4\sin^2 u}}, v),$$

with the same Gauss map and $\sigma_1 = \frac{1}{32}\sqrt{2(17 - 15\cos(2u))^3}\ \sigma_2$. But, from the expression of χ_1 and since $\chi_2(u, v) \in \mathbb{S}^1 \times \mathbb{R}$, it is clear that there does not exist a similarity φ such that $\chi_1 = \varphi \circ \chi_2$.

3 Hypersurfaces of constant Gauss-Kronecker curvature

It is known that the Gauss-Kronecker curvature of a hypersurface is zero if and only if the second fundamental form is degenerate everywhere. Now, we prove that if the second fundamental form is non-degenerate, then the Gauss-Kronecker curvature is constant if and only if the Gauss map is harmonic for the metric given by the second fundamental form.

Theorem 6 *Let M^n be an orientable n-manifold and $x : M^n \longrightarrow \mathbb{R}^{n+1}$ an immersion with non-degenerate second fundamental form. We consider M^n with the metric σ, then the Gauss-Kronecker curvature of the immersion is constant if and only if the Gauss map $N : M^n \longrightarrow \mathbb{S}^n$ is harmonic. Moreover, in that case the Laplacian of N for σ is given by $\Delta^{\sigma} N = -nHN$, where H is the mean curvature of the immersion.*

Proof: Let $E_1, \ldots, E_n$ be an orthonormal moving frame for σ in a neighbourhood of a point $p \in M^n$, that is, $\sigma(E_i, E_j) = \varepsilon_i \delta_{ij}$, with $\varepsilon_i = \pm 1$ and δ_{ij} the Kronecker delta, such that $(\nabla^{\sigma}_{E_i} E_j)(p) = 0$.

Let us calculate $< \Delta^{\sigma} N, E_j >$ at p:

$$\begin{aligned} < \Delta^{\sigma} N, E_j > &= \sum_i \varepsilon_i < E_i(E_i(N)), E_j >= \sum_i \varepsilon_i < \nabla_{E_i} \nabla_{E_i} N, E_j > \\ &= \sum_i \varepsilon_i (E_i < \nabla_{E_i} N, E_j > - < \nabla_{E_i} N, \nabla_{E_i} E_j >) \\ &= \sum_i \varepsilon_i \sigma(E_i, \nabla_{E_i} E_j), \end{aligned} \tag{19}$$

where ∇ is the Levi-Civita connection of $\mathbb{R}^{n+1}$.

If we denote by $g_{kl} =< E_k, E_l >$ and (g^{lk}) the inverse matrix of $G = (g_{kl})$, then we have

$$< \nabla_{E_i} N, E_l >= -\sigma(E_i, E_l) = -\varepsilon_i \delta_{il}, \tag{20}$$

and

$$\nabla_{E_i} N = \sum_l \varepsilon_i g^{il} E_l. \tag{21}$$

Since the Lie bracket $[E_i, E_j](p) = 0$, using Koszul formula, we obtain from (19) and (20)

$$\begin{aligned} < \Delta^{\sigma} N, E_j > &= \sum_{i,l} g^{il} < \nabla_{E_i} E_j, E_l >= \frac{1}{2} \sum_{i,l} g^{il} \left(E_i < E_j, E_l > + E_j < E_i, E_l > \right. \\ &\left. - E_l < E_i, E_j > \right) = \frac{1}{2} \sum_{i,l} g^{il} E_j(g_{il}) = \frac{1}{2} trace(G^{-1} E_j(G)). \end{aligned}$$

The Gauss-Kronecker curvature K satisfies $|K| = 1/det(G)$ and

$$E_j(\log(det(G))) = \frac{E_j(det(G))}{det(G)} = trace(G^{-1} E_j(G)),$$

where *det* denotes the usual determinant in $\mathbb{R}^n$. Thus

$$< \Delta^{\sigma} N, E_j >= -\frac{1}{2} E_j(\log |K|).$$

Therefore, N is harmonic if and only if K is constant.

Moreover, from (21)

$$\begin{aligned} < \Delta^{\sigma} N, N > &= \sum_i \varepsilon_i < E_i(E_i(N)), N > = \sum_i \varepsilon_i < \nabla_{E_i} \nabla_{E_i} N, N > \\ &= \sum_i \varepsilon_i (E_i < \nabla_{E_i} N, N > - < \nabla_{E_i} N, \nabla_{E_i} N >) \\ &= \sum_{i,l} g^{il} < E_l, \nabla_{E_i} N > = -\sum_{i,l} g^{il} \varepsilon_i \delta_{il} = -\sum_i g^{ii} \varepsilon_i = -nH. \end{aligned}$$

□

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