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Flat surfaces in the hyperbolic 3-space*

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Abstract. In this paper we give a conformal representation of flat surfaces in the hyperbolic 3-space using the complex structure induced by its second fundamental form. We also study some examples and the behaviour at infinity of complete flat ends.

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1. Introduction

Second order partial differential equations that arise in the context of some differential geometric problems are of special interest when they can be solved in terms of holomorphics data on a Riemann surface. One of the most famous examples is the equation of minimal surfaces in the Euclidean 3-space $\mathbb{R}^3$ whose holomorphic representation is the well-known Weierstrass-representation which plays an important role in the study of these surfaces (see [7]). In 1987 R. L. Bryant showed that a "holomorphic resolution" like the Weierstrass-representation also holds for surfaces of constant mean curvature 1 in the hyperbolic 3-space $\mathbb{H}^3$, (see [2] and [9]). Perhaps the most important key in his study is, as in the case of minimal surfaces in $\mathbb{R}^3$, that the hyperbolic Gauss map is a conformal map into $\mathbb{S}^2$, when one considers on the surface the complex structure determined by the induced metric.

Another interesting partial differential equation that could be solved using holomorphics data, (see [5], [6] and [8]), is the following Monge-Ampère type equation

$$\phi_{xx}\phi_{yy} - \phi_{xy}^2 = 1$$

which arises in the study of the second fundamental form of flat surfaces in $\mathbb{H}^3$ and some surfaces in Affine Differential Geometry.

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In this paper we start to study flat surfaces in $\mathbb{H}^3$ with the complex structure determined by its second fundamental form.

Thus in Sec. 2, we prove that the hyperbolic Gauss map on a flat surface in $\mathbb{H}^3$ with the above mentioned complex structure is a holomorphic map to $\mathbb{S}^2$ and we show that an analogue to Weierstrass' and Bryant's representations also holds for this kind of surfaces.

In Sec. 3, parallel surfaces of a flat surface and revolution flat surfaces are described using this conformal representation.

In Sec. 4, we give a new proof of the Volkov-Vladimirova's and Sasaki's theorem about the classification of complete flat surfaces in $\mathbb{H}^3$. We also show that a complete end of a flat surface is conformal to a disk minus a point. When the hyperbolic Gauss map can be extended holomorphically to this point, the end is called *regular*. Using our conformal representation we prove that regular ends can be constructed by solving an ordinary differential equation with a regular singularity.

Finally in Sec. 5, we study embedded ends and obtain a necessary and sufficient condition for a regular end to be embedded. We also prove that regular ends have the same behaviour at infinity as flat revolution surfaces.

2. Conformal representation of a flat surface in $\mathbb{H}^3$

Let $\mathbb{L}^4$ be the Minkowski 4-space endowed with linear coordinates (x_0, x_1, x_2, x_3) and the scalar product, $< .,. >$ given by the quadratic form $-x_0^2+x_1^2+x_2^2+x_3^2$. The hyperbolic 3-space, $\mathbb{H}^3$, is the simply connected Riemannian 3-manifold with sectional curvature -1, which is realized as the hyperboloid

$$\mathbb{H}^3 = \left\{(x_0, x_1, x_2, x_3) \in \mathbb{L}^4 / \ -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1, \ x_0 > 0\right\}$$

with the induced metric from $\mathbb{L}^4$.

Let $\mathbb{N}^3$ denote the positive null cone, that is

$$\mathbb{N}^3 = \left\{(x_0, x_1, x_2, x_3) \in \mathbb{L}^4 / \ -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0, \ x_0 > 0\right\}.$$

If one considers for all $v \in \mathbb{N}^3$ the halfline $[v]$ spanned by v, then the ideal boundary $\mathbb{S}^2_\infty$ of $\mathbb{H}^3$ can be regarded as the quotient of $\mathbb{N}^3$ under this action. Thus the induced metric is well-defined up to a factor and $\mathbb{S}^2_\infty$ inherits a natural conformal structure as the quotient $\mathbb{N}^3/\mathbb{R}^+$.

We consider $\mathbb{L}^4$ identified with the space of 2×2 Hermitian matrices, Herm(2), by identifying $(x_0, x_1, x_2, x_3) \in \mathbb{L}^4$ with the matrix

$$\begin{pmatrix} x_0 + x_3 & x_1 + i x_2 \\ x_1 - i x_2 & x_0 - x_3 \end{pmatrix}. \qquad (1)$$

Under this identification, one has $< m, m >= -Det(m)$, for all $m \in$ Herm(2), and the complex Lie group $\mathbf{SL}(2,\mathbb{C})$ of 2×2 complex matrices with determinant 1 acts naturally on $\mathbb{L}^4$ by the representation

$$g \cdot m = gmg^*,$$

where $g \in \mathbf{SL}(2,\mathbb{C})$, $g^* = {}^t\overline{g}$ and $m \in$ Herm(2). Consequently, $\mathbf{SL}(2,\mathbb{C})$ preserves the scalar product and orientations. The kernel of this action is $\{\pm I_2\} \subseteq \mathbf{SL}(2,\mathbb{C})$ and $\mathbf{PSL}(2,\mathbb{C}) = \mathbf{SL}(2,\mathbb{C})\ /\{\pm I_2\}$ can be regarded as the identity component of the special Lorentzian group $\mathbf{SO}(1,3)$. This action can be restricted to $\mathbb{H}^3$ as an isometric and transitive one. Thus, $\mathbb{H}^3$ is recognized as the space of unimodular positive definite 2×2 Hermitian matrices.

The space $\mathbb{N}^3$ is seen as the space of positive semi-definite 2×2 Hermitian matrices of determinant 0 and its elements can be written as $a\,{}^t\overline{a}$, where ${}^ta = (a_1, a_2)$ is a non-zero vector in $\mathbb{C}^2$ uniquely defined up to multiplication by a unimodular complex number. The map $a\,{}^t\overline{a} \longrightarrow [(a_1, a_2)] \in \mathbb{C}\mathbf{P}^1$ becomes the quotient map of $\mathbb{N}^3$ on $\mathbb{S}^2_\infty$ and identifies $\mathbb{S}^2_\infty$ with $\mathbb{C}\mathbf{P}^1$. So the natural action of $\mathbf{SL}(2,\mathbb{C})$ on $\mathbb{S}^2_\infty$ is the action of $\mathbf{SL}(2,\mathbb{C})$ on $\mathbb{C}\mathbf{P}^1$ by Möbius transformations.

Now, we denote by M a simply connected surface and $\psi : M \longrightarrow \mathbb{H}^3$ an immersion with flat induced metric $ds^2 =< d\psi, d\psi >$. Then, there exists an isothermal coordinate immersion $x + iy : M \longrightarrow \mathbb{C}$ such that

$$ds^2 = dx^2 + dy^2, \tag{2}$$

and if η is an unit normal vector field to the immersion, a straight calculation gives the following structure equations

$$\begin{aligned} \psi_{xx} &= E\eta + \psi, \\ \psi_{xy} &= F\eta, \\ \psi_{yy} &= G\eta + \psi, \\ \eta_x &= -E\psi_x - F\psi_y, \\ \eta_y &= -F\psi_x - G\psi_y, \end{aligned} \tag{3}$$

where $E,\ F$ and G are smooth functions on M and by $(\cdot)_x$ and $(\cdot)_y$ we shall denote the usual partial derivatives with respect to x and y, respectively.

Using the Gauss' and Codazzi-Mainardi's equations we have $EG - F^2 = 1$, $E_y = F_x$ and $F_y = G_x$. Hence, as M is simply connected, there exists a well-defined function ϕ on M such that $E = \phi_{xx},\ F = \phi_{xy},\ G = \phi_{yy}$ and the second fundamental form of the immersion is given by

$$d\sigma^2 = \phi_{xx}dx^2 + \phi_{yy}dy^2 + 2\phi_{xy}dx \cdot dy, \tag{4}$$

with

$$\phi_{xx}\phi_{yy} - \phi^2_{xy} = 1. \tag{5}$$

Throughout this paper we shall regard M as a Riemann surface with the conformal structure determined by the second fundamental form $d\sigma^2$.

From (5), we can choose η such that $\phi_{xx} > 0$ and consider the new coordinate immersion

$$z = u + iv = (x + \phi_x) + i(y + \phi_y). \tag{6}$$

Then, a straight computation gives

$$\begin{aligned} \psi_u &= \frac{1+\phi_{yy}}{2+\phi_{xx}+\phi_{yy}}\psi_x - \frac{\phi_{xy}}{2+\phi_{xx}+\phi_{yy}}\psi_y \\ \psi_v &= \frac{-\phi_{xy}}{2+\phi_{xx}+\phi_{yy}}\psi_x + \frac{1+\phi_{xx}}{2+\phi_{xx}+\phi_{yy}}\psi_y. \end{aligned} \tag{7}$$

Now, from (3), (4), (5), (6) and (7) we have

$$d\sigma^2 = \frac{1}{2+\phi_{xx}+\phi_{yy}}|dz|^2 \tag{8}$$

and

$$(\psi - \eta)_u = \psi_x, \qquad (\psi - \eta)_v = \psi_y. \tag{9}$$

Thus, from (2), (6), (8) and (9), $z : M \longrightarrow \mathbb{C}$ is a conformal coordinate immersion and $[\psi - \eta] : M \longrightarrow \mathbb{S}^2_\infty$ is a conformal map, which induces on M the flat Riemannian metric $|dz|^2$.

Moreover, from the above expressions, we obtain

$$\begin{aligned} (\psi - \eta)_{uu} &= \frac{1+\phi_{yy}}{2+\phi_{xx}+\phi_{yy}}\psi + \frac{1+\phi_{xx}}{2+\phi_{xx}+\phi_{yy}}\eta \\ (\psi - \eta)_{vv} &= \frac{1+\phi_{xx}}{2+\phi_{xx}+\phi_{yy}}\psi + \frac{1+\phi_{yy}}{2+\phi_{xx}+\phi_{yy}}\eta \end{aligned}$$

and by using standard notations of complex analysis, one has $4(\psi - \eta)_{z\bar{z}} = \psi + \eta$ and the immersion ψ is given by

$$\psi = \frac{1}{2}(\psi - \eta) + 2(\psi - \eta)_{z\bar{z}}. \tag{10}$$

Now, let $A, B : M \longrightarrow \mathbb{C}$ be global holomorphic functions on M such that $[\psi - \eta]$ is represented as $[(A, B)] \in \mathbb{C}\mathbf{P}^1 \equiv \mathbb{S}^2_\infty$, then

$$\psi - \eta = \lambda \begin{pmatrix} A \\ B \end{pmatrix} \left(\overline{A}, \overline{B}\right) = \lambda \begin{pmatrix} A\overline{A} & A\overline{B} \\ \overline{A}B & B\overline{B} \end{pmatrix},$$

for some positive function $\lambda \in C^\infty(M)$. Thus, from (2), (6) and (9), one gets

$$\frac{1}{2} = < (\psi - \eta)_z, (\psi - \eta)_{\bar{z}} > = \frac{1}{2}\lambda^2 \, |AB_z - BA_z|^2$$

and as $AB_z - BA_z$ does not vanish on the simply connected surface M, there exists a holomorphic function $R : M \longrightarrow \mathbb{C}$ with $R^2 = AB_z - BA_z$. Hence, we can write

$$\psi - \eta = 2\begin{pmatrix} C\overline{C} & C\overline{D} \\ \overline{C}D & D\overline{D} \end{pmatrix},$$

where $C = A/(\sqrt{2}R)$ and $D = B/(\sqrt{2}R)$. Consequently, from (10), we have the following expression for the immersion

$$\psi = \begin{pmatrix} C\overline{C} + 4C_z\overline{C_z} & C\overline{D} + 4C_z\overline{D_z} \\ \overline{C}D + 4\overline{C_z}D_z & D\overline{D} + 4D_z\overline{D_z} \end{pmatrix} \tag{11}$$

and for its unit normal

$$\eta = \begin{pmatrix} -C\overline{C} + 4C_z\overline{C_z} & -C\overline{D} + 4C_z\overline{D_z} \\ -\overline{C}D + 4\overline{C_z}D_z & -D\overline{D} + 4D_z\overline{D_z} \end{pmatrix}. \tag{12}$$

If we consider the function $f : M \longrightarrow \mathbb{C}$ defined by

$$f = \frac{\phi_{yy} - \phi_{xx} + 2i\phi_{xy}}{2 + \phi_{xx} + \phi_{yy}}, \tag{13}$$

then, from (3), (7) and (9), we obtain $(\psi + \eta)_z = f(\psi - \eta)_{\overline{z}}$ and

$$4\begin{pmatrix} C_{zz}\overline{C_z} & C_{zz}\overline{D_z} \\ \overline{C_z}D_{zz} & D_{zz}\overline{D_z} \end{pmatrix} = f\begin{pmatrix} C\overline{C_z} & C\overline{D_z} \\ \overline{C_z}D & D\overline{D_z} \end{pmatrix}.$$

As C_z and D_z cannot vanish simultaneously, we have

$$C_{zz} = \frac{1}{4}fC, \qquad D_{zz} = \frac{1}{4}fD. \tag{14}$$

Thus, from (5), (13) and (14), f is a holomorphic function which satisfies

$$|f| < 1. \tag{15}$$

Finally, from (11) and (14), the immersion ψ can be recovered as $\psi = gg^*$, where $g : M \longrightarrow \mathbf{SL}(2, \mathbb{C})$ is a holomorphic immersion given by

$$g = \begin{pmatrix} C & 2C_z \\ D & 2D_z \end{pmatrix} \tag{16}$$

such that

$$g^{-1}dg = \begin{pmatrix} 0 & f \\ 1 & 0 \end{pmatrix}\omega \tag{17}$$

and $\omega = \frac{1}{2}dz$.

Moreover, using (2), (4), (8), (13), (15) and (17), we have that the induced metric and the second fundamental form can be written, respectively, as

$$ds^2 = f\omega^2 + \overline{f}\overline{\omega}^2 + \left(1 + |f|^2\right)|\omega|^2, \qquad d\sigma^2 = \left(1 - |f|^2\right)|\omega|^2. \tag{18}$$

On the other hand, if $\widetilde{g} : M \longrightarrow \mathbf{SL}(2,\mathbb{C})$ is a holomorphic immersion with $\psi = \widetilde{g}\widetilde{g}^*$, then there exists a holomorphic map $g_0 : M \longrightarrow \mathbf{SU}(2) \subseteq \mathbf{SL}(2,\mathbb{C})$ such that $g = \widetilde{g} g_0$, where $\mathbf{SU}(2) = \{m \in \mathbf{SL}(2,\mathbb{C}) / \, mm^* = I_2\}$. Thus $(g_0)_z = 0$ and g_0 must be constant.

From the above considerations, it is not difficult to prove the following

Theorem 1. (Conformal representation)

i) Let M be a simply connected surface and $\psi : M \longrightarrow \mathbb{H}^3$ a flat immersion. If on M we consider the conformal structure determined by the second fundamental form of ψ, then there exists a holomorphic immersion $g : M \longrightarrow \mathbf{SL}(2,\mathbb{C})$ and a pair (f,ω) consisting of a holomorphic function f and a holomorphic 1-form ω on M such that:

a) $|f| < 1$ and $\omega \neq 0$ everywhere,

b) (17) is satisfied,

c) $\psi = gg^*$,

d) the induced metric and the second fundamental form of ψ are given by (18).

Moreover, g is unique up to right multiplication by a constant $g_0 \in \mathbf{SU}(2)$.

ii) Conversely, let M be a Riemann surface and $g : M \longrightarrow \mathbf{SL}(2,\mathbb{C})$ a holomorphic immersion such that $g^{-1}dg$ is as in (17) and (15) is satisfied. Then $\psi = gg^* : M \longrightarrow \mathbb{H}^3$ is a flat immersion which has induced metric and second fundamental form given by (18).

Definition 1. *The pair (f,ω) in the above Theorem will be called the* **Weierstrass data** *associated to the conformal representation of the flat immersion.*

If we consider the Weierstrass data (f,ω) and the holomorphic immersion g written in an arbitrary complex parameter ζ as $(f(\zeta), h(\zeta)d\zeta)$ and

$$g = \begin{pmatrix} C & E \\ D & F \end{pmatrix},$$

then, from (14), (16) and (17), one has that C and D are linearly independent solutions of the ordinary linear differential equation

$$X_{\zeta\zeta} - \frac{h_\zeta}{h} X_\zeta - f h^2 X = 0 \tag{19}$$

and

$$E = \frac{1}{h} C_\zeta \ , \qquad F = \frac{1}{h} D_\zeta \ .$$

Conversely, if C and D are linearly independent solutions of (19) such that

$$\frac{1}{h}(C D_\zeta - D C_\zeta) = 1,$$

then

$$g = \begin{pmatrix} C & \frac{1}{h} C_\zeta \\ D & \frac{1}{h} D_\zeta \end{pmatrix} \tag{20}$$

determines a flat immersion with Weierstrass data $(f(\zeta), h(\zeta)d\zeta)$.

Moreover, if A and B are other solutions of (19) under the previous conditions, then the flat immersion associated with A and B is, up to isometries, the immersion associated with C and D.

Theorem 2. *Let N be an oriented and connected surface and $\varphi : N \longrightarrow \mathbb{H}^3$ an immersion such that its second fundamental form is positive definite for a choice of the unit normal vector field η. If on N we consider the conformal structure induced by the second fundamental form, then the immersion $[\varphi - \eta] : N \longrightarrow \mathbb{S}^2_\infty$ is conformal if and only if either the induced metric by φ on N is flat or φ is totally umbilic.*

Proof. Let $p \in N$ and $\{E_1, E_2\}$ a smooth orthonormal basis in a neighbourhood of p that diagonalizes the metric σ associated to the second fundamental form in p, that is

$$\sigma(E_i, E_j) = h_{ij}\delta_{ij}, \qquad i, j \in \{1, 2\}.$$

Then $[\varphi - \eta]$ is a conformal map if and only if

$$\frac{(1 - h_{11})^2}{h_{11}} = \frac{(1 - h_{22})^2}{h_{22}}$$

for all $p \in N$, or equivalently

$$(h_{22} - h_{11})(1 - h_{11}h_{22}) = 0.$$

Let $K_{ext}(p) = h_{11}(p)h_{22}(p)$ be the extrinsic curvature of $p \in N$. Then the set $N' = \{p \in N / \ K_{ext}(p) \neq 1\}$ is an open subset in N and $\varphi_{|N'}$ is totally umbilic. So K_{ext} is constant on each connected component of N' and hence on its closure. Thus φ is either totally umbilic or a flat immersion.

Definition 2. *The immersions $G^+ = [\psi + \eta] : M \longrightarrow \mathbb{S}^2_\infty$ and $G^- = [\psi - \eta] : M \longrightarrow \mathbb{S}^2_\infty$ will be called* **hyperbolic Gauss maps**.

From the above definition, one has that for each point $p \in M$, the oriented normal geodesic emanating from $\psi(p)$ meets the ideal boundary $\mathbb{S}^2_\infty$ of $\mathbb{H}^3$ at $G^+(p)$ and $G^-(p)$. Moreover, if $\psi : M \longrightarrow \mathbb{H}^3$ is a flat immersion and we identify $\mathbb{S}^2_\infty$ and $\mathbb{C} \cup \{\infty\}$ with the canonical conformal structure, then from (11), (12) and (20), we can write

$$G^- = \frac{C}{D}, \qquad G^+ = \frac{dC}{dD}. \tag{21}$$

3. Some examples

1. Parallel surfaces:

Let M be a simply connected surface and $\psi : M \longrightarrow \mathbb{H}^3$ a flat immersion. By using Theorem 1, we have a conformal coordinate immersion ζ on M and Weierstrass data $(f, hd\zeta)$ such that $\psi = gg^*$ with g determined by (19) and (20).

Thus, if we take $f_t = e^{2t} f$, $\omega_t = e^{-t} hd\zeta$ and

$$g_t = \begin{pmatrix} e^{-t/2}C & \frac{1}{h}e^{t/2}C_\zeta \\ e^{-t/2}D & \frac{1}{h}e^{t/2}D_\zeta \end{pmatrix},$$

for $t \in \mathbb{R}$ such that $|f_t| < 1$, then we obtain $\psi_t = g_t g_t^* = cosh(t)\psi + sinh(t)\eta$. Therefore, the parallel immersion to ψ at a distance $|t|$ is the flat immersion ψ_t with Weierstrass data (f_t, ω_t).

2. Revolution surfaces:

We consider the half-space model of $\mathbb{H}^3$, that is, $\mathbb{R}^3_+ = \{(y_1, y_2, y_3) \in \mathbb{R}^3 / \ y_3 > 0\}$ endowed with the metric

$$d\tau^2 = \frac{1}{y_3^2}(dy_1^2 + dy_2^2 + dy_3^2).$$

By identifying $(x_0, x_1, x_2, x_3) \in \mathbb{H}^3$ with $\frac{1}{x_0+x_3}(\ x_1,\ x_2,\ 1) \in \mathbb{R}^3_+$, one can write a revolution surface in hyperbolic 3-space as

$$\psi(r, \theta) = (y_1(r)cos(\theta), y_1(r)sin(\theta), y_3(r)) \tag{22}$$

with $y_1(r) > 0$, and choose the parameter r as the arc length of the generatrix curve $(y_1(r), 0, y_3(r))$ in $\mathbb{R}^3_+$. It is easy to check that ψ is a flat immersion if and only if $y_1(r) = y_3(r)\ (ar + b)$, with $a, b \in \mathbb{R}$, non-zero simultaneously, (see [3]).

We distinguish two cases:

i) If $a = 0$, then one sees that $r + ib\theta$ is an isothermal parameter for the induced metric on ψ. From (3) and (6) one obtains that

$$z = \left(1 + \sqrt{\frac{b^2}{1+b^2}}\right) r + i\left(1 + \sqrt{\frac{1+b^2}{b^2}}\right) b\theta$$

is a conformal coordinate immersion and making an appropriate change of parameters (13) gives the following Weierstrass data associated to the immersion

$$f(\zeta) = \frac{1}{k^2}, \qquad \omega = \frac{1}{2}dz = \frac{k}{2\zeta}d\zeta,$$

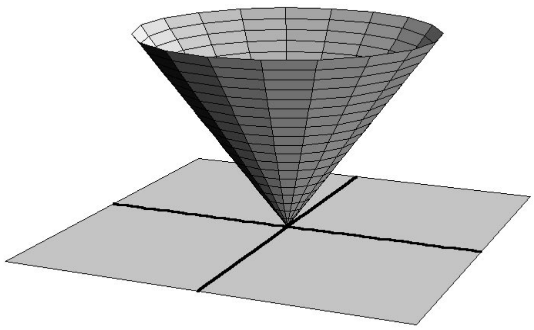

Fig. 1. $a = 0$

where $\zeta \in \mathbb{C}^*$ and $k^2 = 2b^2 + 1$. Hence, from (19) and (20), one gets

$$g(\zeta) = \frac{i\sqrt{k}}{\sqrt{2}} \begin{pmatrix} \zeta^{1/2} & \frac{1}{k}\zeta^{1/2} \\ \zeta^{-1/2} & -\frac{1}{k}\zeta^{-1/2} \end{pmatrix}$$

and the corresponding flat revolution surface is, up to isometries,

$$\psi_{0,k}(\zeta) = \left(\frac{\zeta(k^2-1)}{|\zeta|^2(k^2+1)}, \frac{2}{|\zeta|\left(k+\frac{1}{k}\right)} \right). \tag{23}$$

One can check that each of these surfaces is the set of points at a fixed distance from the geodesic $\{(0, 0, y_3) \in \mathbb{R}^3 / \ y_3 > 0\}$, (see Figure 1).

ii) If $a \neq 0$, then one finds that $\frac{1}{a}e^{log(ar+b)+ia\theta}$ is a local isothermal parameter for the induced metric on ψ. Therefore, from (3) and (6) one has

$$z = \left(1 + \sqrt{1 + \frac{1-a^2}{(ar+b)^2}}\right) \frac{1}{a} e^{log(ar+b)+ia\theta}$$

and the Weierstrass data associated to the immersion are

$$f(\zeta) = \frac{1-a^2}{\zeta^{2a}}, \qquad \omega = \frac{1}{2}dz = \frac{1}{2}\zeta^{a-1}d\zeta,$$

with $\zeta \in \mathbb{C}^*$ such that $|f(\zeta)| < 1$. Thus, from (19) and (20), we obtain

$$g(\zeta) = \frac{i}{\sqrt{2}} \begin{pmatrix} \zeta^{(a+1)/2} & (a+1)\zeta^{(1-a)/2} \\ \zeta^{(a-1)/2} & (a-1)\zeta^{-(1+a)/2} \end{pmatrix}$$

and the corresponding flat revolution surface is, up to isometries,

$$\psi_a(\zeta) = \left(\frac{\zeta(|\zeta|^{a-1} + (a^2-1)|\zeta|^{-(a+1)})}{|\zeta|^{a+1} + (a+1)^2|\zeta|^{1-a}}, \frac{2}{|\zeta|^{a+1} + (a+1)^2|\zeta|^{1-a}} \right). \tag{24}$$

We remark that the surfaces ψ_a and ψ_{-a} are isometric. In this sense, one gets a horosphere when $a = \pm 1$, (see Figures 2 and 3).

In other case, one obtains non-complete flat revolution surfaces which can be represented by the following figures:

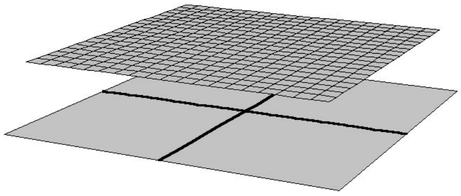

Fig. 2. $a = -1$

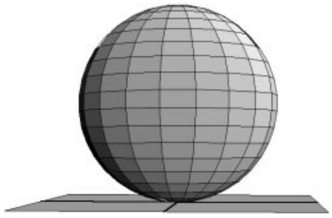

Fig. 3. $a = 1$

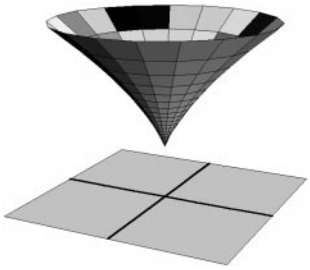

Fig. 4. $-1 < a < 0$

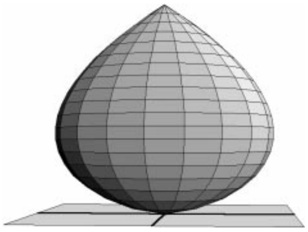

Fig. 5. $0 < a < 1$

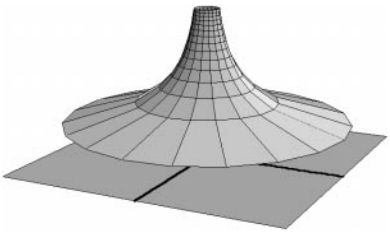

Fig. 6. $a < -1$

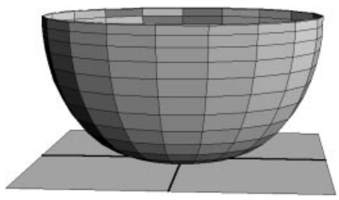

Fig. 7. $a > 1$

4. Completeness of ends

Using the above conformal representation, first we give a new proof of the well-known classification of the complete flat surfaces in $\mathbb{H}^3$, obtained by Volkov-Vladimirova and Sasaki, (see [8]).

Theorem 3. *Let M be a connected surface and* $\psi : M \longrightarrow \mathbb{H}^3$ *a complete flat immersion. Then* $\psi(M)$ *is either a horosphere or the set of points at a fixed distance from a geodesic.*

Proof. We can assume that M is simply connected, otherwise we may pass to the universal covering surface of M. Hence, with the conformal structure determined by the second fundamental form, M is biholomorphic either to the unit disk Δ or to the complex plane $\mathbb{C}$.

If (f, ω) are the Weierstrass data associated with the immersion, then (15) and (18) give

$$ds^2 \leq 4|\omega|^2 \tag{25}$$

and as $|\omega| \neq 0$, M is not of the conformal class of Δ, (see [7]). Thus, M must be biholomorphic to $\mathbb{C}$ and from (15) and (25), we can take Weierstrass data $(c, d\zeta)$, with c constant and $\zeta \in \mathbb{C}$. Using (19), (20) and the examples of Sec. 3 we can prove that $\psi(M)$ is a horosphere if $c = 0$ and $\psi(M)$ is the set of points at a fixed distance from a geodesic if $c \neq 0$.

Remark 1. Let M be a connected surface and $\psi : M \longrightarrow \mathbb{H}^3$ a flat immersion such that the Riemannian metric given by the second fundamental form $d\sigma^2$ is complete. From (15) and (18), $d\sigma^2 \leq |\omega|^2$ and using the same argument as in the proof of Theorem 3, we can also conclude that $\psi(M)$ is either a horosphere or the set of points at a fixed distance from a geodesic.

Lemma 1. *Each complete end of a flat surface in $\mathbb{H}^3$ is biholomorphic, with the conformal structure given by the second fundamental form, to a punctured disk.*

Proof. It is clear from (25), (see [7]).

Lemma 2. *Let $\Delta^* = \{\zeta \in \mathbb{C} / \, 0 < |\zeta| < 1\}$, $\psi : \Delta^* \longrightarrow \mathbb{H}^3$ a flat immersion and $(f(\zeta), \omega(\zeta))$ Weierstrass data associated with ψ. Then there exist real numbers μ and ν such that*

(A) $\quad 0 \leq \mu,\ \nu < 1, \qquad \mu + 2\nu \in \{0, 1, 2\},$

(B) $\quad f(\zeta) = \zeta^\mu f_1(\zeta)$ *and* $\omega(\zeta) = \zeta^\nu \omega_1(\zeta),$

with f_1 a single-valued holomorphic function on Δ and ω_1 a single-valued holomorphic 1*-form on Δ^*.*

Moreover, the differential equation (19) is well-defined on Δ^* and if ds^2 is complete at the origin, then ω_1 has at most a pole there.

Proof. We consider the simply connected set $\mathbb{C}^- = \{\zeta \in \mathbb{C} / \, \mathcal{R}e(\zeta) < 0\}$, then the usual exponential map is a covering map from $\mathbb{C}^-$ on Δ^* and the Weierstrass data are well-defined on $\mathbb{C}^-$, but not on Δ^* in general. However, from (18), $|f|$, $|\omega|$ and $f\omega^2$ are well-defined on Δ^*.

Hence, from these considerations, there exists a holomorphic function $g : \mathbb{C}^- \longrightarrow \mathbb{C}$ such that

$$f(e^{\zeta + 2\pi i}) = g(\zeta) f(e^{\zeta}), \qquad |g(\zeta)| = 1,$$

so, g is constant and $g(\zeta) = e^{2\pi\mu i}$, $\forall \zeta \in \mathbb{C}^-$ with $\mu \in [0, 1[$. In the same way, one has that there exists $\nu \in [0, 1[$ such that $\omega(e^{\zeta + 2\pi i}) = e^{2\pi\nu i}\omega(e^{\zeta})$ and $e^{2\pi\mu i} e^{4\pi\nu i} = 1$, that is, $\mu + 2\nu \in \{0, 1, 2\}$.

Thus, we obtain that

$$f_1 = \zeta^{-\mu} f \qquad and \qquad \omega_1 = \zeta^{-\nu} \omega$$

are single-valued on Δ^* and (19) is well-defined on Δ^*. As $|f| < 1$, one concludes that f_1 is holomorphic at the origin.

Moreover, if ds^2 is complete at the origin, using (25), we have $ds^2 \leq 4|\omega_1|^2$ and ω_1 has at most a pole in 0, see ([7]).

Proposition 1. *Let (f, ω) be a Weierstrass data on Δ^* satisfying the conditions (A) and (B) in Lemma 2. If $f\omega^2$ has at most a pole of order 2 at the origin and $\psi : \Delta^* \longrightarrow \mathbb{H}^3$ is an associated flat immersion, then the following conditions are equivalent:*
a) The immersion ψ is well-defined on Δ^*.
b) The hyperbolic Gauss map G^- is single-valued.
c) A fundamental system of solutions of (19) on Δ^* is $\{\zeta^{\lambda} a(\zeta), \zeta^{\lambda-m} b(\zeta)\}$ where a, b are holomorphic functions on Δ, non-zero at the origin, $m \in \mathbb{N}$ and λ, $\lambda - m$ are the solutions of the indicial equation associated with (19).

Proof. We can write

$$\omega = h(\zeta) d\zeta, \quad \zeta \in \Delta^*, \quad with\ h(\zeta) = \zeta^{\nu} h_1(\zeta),$$

where $\nu \in [0, 1[$ and h_1 is a holomorphic function on Δ^*.

Thus, one has

$$\frac{h'(\zeta)}{h(\zeta)} = \sum_{k=-1}^{\infty} p_k \zeta^k, \qquad f\omega^2 = \sum_{k=-2}^{\infty} q_k \zeta^k$$

where $(.)'$ denotes $\frac{d(.)}{d\zeta}$, and the indicial equation of (19) is

$$y^2 - (p_{-1} + 1) y - q_{-2} = 0. \tag{26}$$

Let λ_1, λ_2 be the solutions of (26), then a fundamental system of solutions of (19) is, (see [4]),

$$C(\zeta) = \zeta^{\lambda_1} a(\zeta), \qquad D(\zeta) = \zeta^{\lambda_2} b(\zeta) + k\, C(\zeta)\, Log(\zeta),$$

with a, b holomorphic functions on Δ which are non-zero at the origin and $k \in \mathbb{C}$, ($k \neq 0$ if $\lambda_1 = \lambda_2$).

Finally, multiplying C or D by a suitable constant, we can assume that $\frac{1}{h}(C D_\zeta - C_\zeta D) = 1$ and then (20) and (21) give $\psi = g g^*$, up to isometries, and G^- is, up to Möbius transformations, C/D. Therefore, ψ is well-defined or G^- is single-valued if and only if $\lambda_1 - \lambda_2 \in \mathbb{N}$ and $k = 0$.

Definition 3. *The positive integer m given by the above proposition is called the* **multiplicity of the end**.

Definition 4. *Let $\psi : \Delta^* \longrightarrow \mathbb{H}^3$ be a flat immersion with complete end at the origin. The end is called* **regular** *if the hyperbolic Gauss map G^- extends holomorphically across the origin.*

Theorem 4. *Let $\psi : \Delta^* \longrightarrow \mathbb{H}^3$ be a flat immersion with complete end at the origin and (f, ω) Weierstrass data associated with ψ. Then the following conditions are equivalent:*
a) The end is regular.
b) $f\omega^2$ has at most a pole with order 2 at the origin.
c) The differential equation (19) has a regular singularity at the origin.

Proof. Since the end is complete, if we write $\omega = h(\zeta)d\zeta$, then $\frac{h'(\zeta)}{h(\zeta)}$ has at most a pole of order 1 at the origin. Therefore, (see [4]), b) and c) are equivalent.

Now, we prove the equivalence between a) and b). Assume $f\omega^2$ has at most a pole with order 2 at the origin, then from the previous proposition, G^- is given, up to a Möbius transformation, by

$$G^-(\zeta) = \zeta^m \frac{a(\zeta)}{b(\zeta)}, \tag{27}$$

with $a,\ b$ holomorphic functions on Δ non-zero at the origin and $m \in \mathbb{N}$. Hence, the end is regular.

Conversely, by a straight forward calculation, we have that the Schwarzian derivative of G^- with respect to ζ is given by

$$\begin{aligned}\{G^-, \zeta\} &:= \frac{d}{d\zeta}\left(\frac{(G^-)''(\zeta)}{(G^-)'(\zeta)}\right) - \frac{1}{2}\left(\frac{(G^-)''(\zeta)}{(G^-)'(\zeta)}\right)^2 = \\ &= \frac{d}{d\zeta}\left(\frac{h'(\zeta)}{h(\zeta)}\right) - \frac{1}{2}\left(\frac{h'(\zeta)}{h(\zeta)}\right)^2 - f(\zeta)h(\zeta)^2.\end{aligned} \tag{28}$$

Thus, if G^- completes holomorphically across the origin, then $\frac{(G^-)''(\zeta)}{(G^-)'(\zeta)}$ only has at most a pole of order 1 at the origin and (28) implies b).

A similar result to the above Theorem for CMC-1 surfaces has been proved in [9].

Remark 2. From (27) the multiplicity of the end is the ramification number of G^- at the origin.

5. Embeddedness and asymptotic behaviour of ends

Let $\psi : \Delta^* \longrightarrow \mathbb{H}^3$ be a flat immersion with regular end at the origin. Up to isometries, we can assume that $G^-(0) = 0$. Thus, from (27), there exist a parameter ζ and a positive real number ε such that $G^-(\zeta) = \zeta^m$ with $\zeta \in \Delta_\varepsilon = \{z \in \mathbb{C}/ \ |z| < \varepsilon\}$.

If we write $\omega(\zeta) = h(\zeta)d\zeta$ and

$$g(\zeta) = \begin{pmatrix} C(\zeta) & \frac{1}{h(\zeta)}C'(\zeta) \\ D(\zeta) & \frac{1}{h(\zeta)}D'(\zeta) \end{pmatrix},$$

then from (21) and since $g \in \mathbf{SL}(2, \mathbb{C})$ one has $C(\zeta) = \zeta^m D(\zeta)$ and $h(\zeta) = -m\zeta^{m-1}D(\zeta)^2$, $\zeta \in \Delta^*_\varepsilon = \Delta_\varepsilon - \{0\}$.

We take $D(\zeta) = \zeta^p d(\zeta)$ where d is a holomorphic function non-zero at the origin. Then from the above considerations and by a straight forward computation we obtain

$$g(\zeta) = \begin{pmatrix} \zeta^{m+p}d(\zeta) & -\zeta^{-p}c_1(\zeta) \\ \zeta^p d(\zeta) & -\zeta^{-(m+p)}c_2(\zeta) \end{pmatrix} \tag{29}$$

where

$$c_1(\zeta) = \frac{(m+p)d(\zeta) + \zeta d'(\zeta)}{md(\zeta)^2} \quad and \quad c_2(\zeta) = \frac{pd(\zeta) + \zeta d'(\zeta)}{md(\zeta)^2} \tag{30}$$

are holomorphic functions at the origin.

Therefore, (17) gives

$$f(\zeta) = -\frac{p(m+p)d(\zeta)^2}{m^2 d(\zeta)^6}\zeta^{-2(m+2p)} + \frac{(m+2p-1)\zeta d(\zeta)d'(\zeta) + (2d'(\zeta)^2 - d(\zeta)d''(\zeta))\zeta^2}{m^2 d(\zeta)^6}\zeta^{-2(m+2p)}$$

and since (15) is satisfied

$$m + 2p \leq 0 \quad and \quad |d(0)|^2 > \frac{1}{2} \ if \ m + 2p = 0. \tag{31}$$

Moreover, if we consider the half-space model $\mathbb{R}^3_+$ of $\mathbb{H}^3$, as in Sec. 3, then ψ is given by

$$\psi(\zeta) = \left(\zeta^m \frac{|\zeta|^{2p}|d|^2 + |\zeta|^{-2(m+p)}c_1\overline{c_2}}{|\zeta|^{2(m+p)}|d|^2 + |\zeta|^{-2p}|c_1|^2}, \frac{1}{|\zeta|^{2(m+p)}|d|^2 + |\zeta|^{-2p}|c_1|^2}\right) \tag{32}$$

and writing $\psi_1 + i\psi_2$ the projection of ψ on the plane $\Pi_0 = \{(y_1, y_2, y_3) \in \mathbb{R}^3/ \ y_3 = 0\}$, one has from (30) and (32)

$$\psi_1 + i\psi_2 = \frac{\zeta^m}{|\zeta|^{2m}} \frac{|\zeta|^{2p}|d|^2 + |\zeta|^{-2(m+p)}c_1\overline{c_2}}{|\zeta|^{2p}|d|^2 + |\zeta|^{-2(m+p)}|c_1|^2} = \frac{1}{\overline{\zeta}^m}\left(1 - \frac{c_1}{\overline{d}}|\zeta|^{-2(m+2p)}A\right) \tag{33}$$

where

$$A(\zeta) = \frac{1}{|d|^2 + |c_1|^2|\zeta|^{-2(m+2p)}}. \tag{34}$$

Now, from (31), (33) and (34), one can check that $\psi_1 + i\psi_2$ is a local diffeomorphism and a proper map from a punctured neighbourhood of the origin on the exterior of a closed disk. Moreover, since the image of the loop $\gamma_\delta(t) = \delta e^{2\pi i t}$, with $t \in [0, 1]$ and $\delta > 0$ sufficiently small, winds around the origin m times, $\psi_1 + i\psi_2$ is an m-fold covering map. Thus, if γ_R is the circle centered in the origin with radius R, for R sufficiently great, then $\gamma = \psi \circ (\psi_1 + i\psi_2)^{-1} \circ \gamma_R$ is a loop in the cylinder $\{(y_1, y_2, y_3) \in \mathbb{R}^3_+ / \ y_1^2 + y_2^2 = R^2\}$ which moves around it m times and cuts itself unless $m = 1$.

From the above considerations we conclude

Theorem 5. *Let $\psi : \Delta^* \longrightarrow \mathbb{H}^3$ be a flat immersion with regular end at the origin. Then the end is embedded if and only if the multiplicity of the end is $m = 1$.*

Remark 3. If we consider a flat immersion $\psi : \Delta^* \longrightarrow \mathbb{H}^3$ with regular embedded end at the origin then, up to isometries, the end can be regarded in the half-space model of $\mathbb{H}^3$ as a graph on the exterior Ω of a bounded domain in Π_0.

Finally, we are going to study the behaviour at infinity of a regular end. First, we observe from (23), (24) and (30) that (32) is a revolution surface if and only if $d(\zeta)$ is constant (even if $m \neq 1$).

Definition 5. *Let $\psi, \ \varphi : \Delta^* \longrightarrow \mathbb{H}^3$ be flat immersions with regular ends at the origin. We say that ψ and φ have the* **same behaviour at infinity** *if and only if the hyperbolic distance between $\psi(\zeta)$ and $\varphi(\zeta)$ tends to zero when ζ tends to the origin.*

Theorem 6. *Let $\psi : \Delta^* \longrightarrow \mathbb{H}^3$ be a flat immersion with regular end at the origin. Then there exists a revolution flat immersion $\psi_R : \Delta^* \longrightarrow \mathbb{H}^3$ with end at the origin such that ψ and ψ_R have the same behaviour at infinity.*

Proof. If we take the half-space model $\mathbb{R}^3_+$ of $\mathbb{H}^3$, we can assume that ψ is given by (32) on Δ^*_ε.

Now, let ψ_R be the revolution flat surface given by

$$\psi_R(\zeta) = \left(\frac{1}{\overline{\zeta}^m} \left(1 - \frac{m+p}{m|d_0|^2} |\zeta|^{-2(m+2p)} A_0 \right), |\zeta|^{-2(m+p)} A_0 \right) \tag{35}$$

where

$$A_0 = \frac{1}{|d_0|^2 + \frac{(m+p)^2}{m^2|d_0|^2} |\zeta|^{-2(m+2p)}}$$

and $d_0 = d(0)$.

Since the hyperbolic distance between two points (y_1^1, y_2^1, y_3^1), $(y_1^2, y_2^2, y_3^2) \in \mathbb{R}^3_+$ is

$$arccosh\left(1 + \frac{(y_1^1 - y_1^2)^2 + (y_2^1 - y_2^2)^2 + (y_3^1 - y_3^2)^2}{2y_3^1 y_3^2}\right) \tag{36}$$

then, by denoting $\psi = (\psi_1, \psi_2, \psi_3)$ and $\psi_R = (\psi_{R1}, \psi_{R2}, \psi_{R3})$, we obtain

$$\frac{(\psi_1 - \psi_{R1})^2 + (\psi_2 - \psi_{R2})^2 + (\psi_3 - \psi_{R3})^2}{2\psi_3\psi_{R3}} =$$

$$= \frac{\frac{1}{|\zeta|^{2m}}\left|\frac{c_1}{\overline{d}}|\zeta|^{-2(m+2p)}A - \frac{m+p}{m|d_0|^2}|\zeta|^{-2(m+2p)}A_0\right|^2 + |\zeta|^{-4(m+p)}(A - A_0)^2}{2|\zeta|^{-4(m+p)}AA_0} =$$

$$= \frac{1}{2}\left(|\zeta|^{-2(m+2p)}\left|\frac{c_1}{\overline{d}}\sqrt{\frac{A}{A_0}} - \frac{m+p}{m|d_0|^2}\sqrt{\frac{A_0}{A}}\right|^2 + \left(\sqrt{\frac{A}{A_0}} - \sqrt{\frac{A_0}{A}}\right)^2\right).$$

And the proof follows from (30), (31), (36) and the above expression.

Remark 4. From Remark 3 a flat immersion $\psi : \Delta^* \longrightarrow \mathbb{H}^3$ with regular embedded end at the origin can be regarded as the graph of a function g on the exterior of a bounded domain in Π_0. Moreover, using (30) and (32), g can be written as

$$g(y_1, y_2) = \frac{|a_1 + g_1(y_1, y_2)|^{-2(1+p)}}{(y_1^2 + y_2^2)^{-(1+p)}}(a_2 + g_2(y_1, y_2)) \tag{37}$$

with

$$\lim_{y_1^2+y_2^2\to\infty} g_1(y_1, y_2) = 0 = \lim_{y_1^2+y_2^2\to\infty} g_2(y_1, y_2),$$

$$a_1 = 1 - \frac{2}{1 + 4|d(0)|^4}, \quad a_2 = \frac{4|d(0)|^2}{1 + 4|d(0)|^4}, \qquad if \quad 1 + 2p = 0$$

and

$$a_1 = 1, \quad a_2 = \frac{1}{|d(0)|^2}, \qquad if \quad 1 + 2p < 0.$$

Thus, if we consider (37) and the revolution flat immersion given by (35) with $m = 1$, then we obtain that the hyperbolic distance between the points in $\psi(\Delta^*)$ and $\psi_R(\Delta^*)$ with the same projection on Π_0 tends to zero at infinity.

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