

Absorbing States and Elastic Interfaces in Random Media: Two Equivalent Descriptions of Self-Organized Criticality

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We elucidate a long-standing puzzle about the nonequilibrium universality classes describing self-organized criticality in sandpile models. We show that depinning transitions of linear interfaces in random media and absorbing phase transitions (with a conserved nondiffusive field) are two equivalent languages to describe sandpile criticality. This is so despite the fact that local roughening properties can be radically different in the two pictures, as explained here. Experimental implications of our work as well as promising paths for future theoretical investigations are also discussed.

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The concept of self-organized criticality (SOC) has been proposed to account for the emergence of scale invariance in nature [1]. Its main tenet is that in the presence of slow driving and fast dissipation, acting at infinitely separated time scales, many systems self-organize (without any explicit tuning of parameters) to a critical state [2]. Some archetypical examples of SOC are provided by sandpile toy models, in which grains are slowly added, locally redistributed on a fast time scale whenever an instability threshold is overcome (generating avalanches of toppling events), and finally dissipated at the open boundaries. Upon iteration, this process leads to a critical steady state.

SOC can be related to standard (nonequilibrium) critical phenomena by defining the “fixed energy ensemble” [3,4] in which driving and dissipation are switched off, so that the number of grains (or “energy”) is conserved. Using this quantity as a control parameter, a standard (i.e., non-self-organized) phase transition is observed: for large energy densities, there is a finite density of active (toppling) sites, whereas the system ends in a frozen (stable) state at low densities. It has been shown that the critical point separating these two regimes occurs at the value of the energy density at which the system self-organizes when subjected to slow-driving and boundary dissipation [4,5]. Subsequent debate has attempted to elucidate which nonequilibrium universality class stochastic sandpiles belong to. As detailed below, two alternative solutions have been proposed.

Sandpiles were first related to interfaces in random media [6]. In this language, the interface height, $h(x, t)$ is the number of times a given site x has toppled up to time t , and frozen states correspond to pinned interfaces. The resulting pinning-depinning transition was argued to fall in the quenched Edwards-Wilkinson or linear interface model (LIM) class, described by [7,8]

$$\partial_t h(x, t) = \nabla^2 h(x, t) + F + \eta(x, h), \quad (1)$$

where F is a force and $\eta(x, h)$ is a quenched white noise.

This correspondence was recently proven exact between one particular sandpile model [9] and one member of the LIM class [10] but, in general, it is only approximate, as some noise correlations need to be neglected to establish a full correspondence with Eq. (1).

Alternatively, sandpile models have been rationalized as systems exhibiting an absorbing-state phase transition [4]. Indeed, in the fixed energy ensemble, a stable configuration is one of the infinitely-many absorbing states in which the system can be trapped forever, whereas activity never ceases above the critical point. The corresponding universality class is *not* the prominent directed percolation (DP) class, but is characterized by the coupling of a DP-like activity field to a conserved, nondiffusive, auxiliary field (the energy) [4,11]. Often called C-DP (or also Manna [12]) class, it is characterized by the following set of Langevin equations:

$$\begin{aligned} \partial_t \rho &= a\rho - b\rho^2 + D_\rho \nabla^2 \rho + \omega \rho \phi + \sigma \sqrt{\rho} \eta(x, t), \\ \partial_t \phi &= D_\phi \nabla^2 \rho, \end{aligned} \quad (2)$$

where ρ is the activity field, ϕ the background energy field, and $\eta(x, t)$ a Gaussian white noise [4,11].

The validity of both of these alternative pictures has been (partially) backed by numerical measurements of critical exponents but, in general, they have not been proven to be correct so far. But, if both of the pictures are right, a remarkable consequence follows: depinning transition of LIM-class interfaces in random media and the C-DP class absorbing phase transition should be equivalent, even if they look rather different (for instance, one involves quenched disorder and the other does not). Here, we explore this issue and the more general question of whether any depinning interface universality class has an equivalent absorbing phase transition class.

Only a few works have approached the connections between these two pictures. In [13], interfaces were constructed from a DP class model, using the cumulated local activity as the interface height. Anomalously rough inter-

faces, characterized by a positive local-slope exponent κ defined by $\langle(\nabla h)^2\rangle \sim t^{2\kappa}$ [14], were found at criticality. These anomalous interfaces are not related to any known interface class. Focusing on SOC sandpiles, Alava and Muñoz [15] argued heuristically that the LIM and C-DP classes could be identified with each other [using also $h(x, t) = \int_0^t \rho(x, s) ds$ to relate Eq. (1) to Eq. (2)] although a one-to-one mapping could not be rigorously established. However, this conclusion was later challenged by Kockelkoren and Chaté (KC) [16] who found that the κ exponent takes completely different values for LIM interfaces and for interfaces constructed from models in the C-DP class. In particular, the constructed interfaces are anomalously rough below the upper critical dimension, i.e., for space dimensions $d < 4$, while LIM interfaces have $\kappa \leq 0$ (not anomalous) for $d \geq 2$. In $d = 1$, both types of interfaces are anomalously rough, but in a manifestly distinct manner, i.e., different values of κ (Fig. 1). This led KC to conclude that LIM and C-DP classes *cannot* be equivalent, even if all the other recorded “standard” critical exponents (as θ and z , see definitions below and Table I, and others) take “almost indistinguishable values” in these two problems, (which could, in principle, be attributed to a numerical coincidence [16].)

While the discrepancy in values of κ is unquestionable, simulating directly Eqs. (2) (using the method in [17]) we find all the other C-DP exponents to be indistinguishable from their counterparts in the LIM class (Table I) as well as from the corresponding values in stochastic sandpiles.

In this Letter, we show that depinning transitions of LIM interfaces and C-DP absorbing phase transitions are indeed two equivalent descriptions of SOC sandpiles in spite of the discrepancies in κ -values, which we explain. We show, using a combination of numerical results and scaling arguments, that there is a unique universality class and that

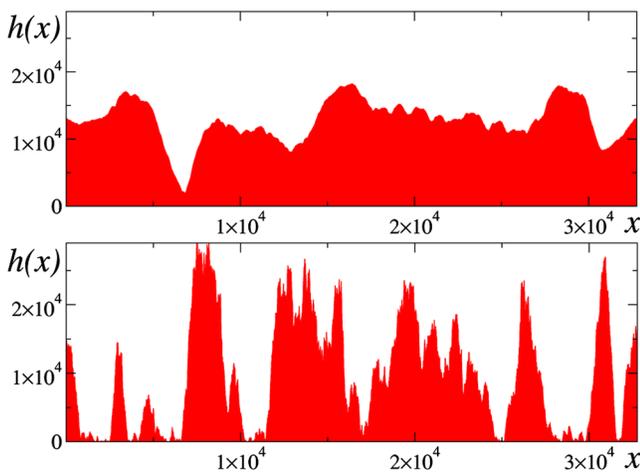


FIG. 1 (color online). Typical one-dimensional interfaces for a system of 2^{15} sites at $t = 10^5$ from flat initial conditions. Top: Leschhorn automaton [LIM class, $\kappa \approx 0.17(1)$]. Bottom: interface constructed from the Manna sandpile [12] [C-DP class, $\kappa \approx 0.43(1)$].

differences in κ -values stem from diverging *local* fluctuations, inherent to the absorbing-state picture, which do not affect other long-distance properties.

Let us start by clarifying the origin of the two possible values of κ and the scaling laws they obey by using simple scaling arguments. First, since $h = \int dt \rho(t)$ its scaling dimension is $[h] \sim t^{1-\theta}$, where θ is the density (or interface-velocity, recalling that $\partial_t h = \rho$) critical time-decay exponent: $\langle \rho(t) \rangle \sim t^{-\theta}$. This leads to $[\nabla h] \sim t^{-1/z+1-\theta}$, where $[\nabla^{-1}] \sim t^{1/z}$ defines the dynamical critical exponent z , and therefore

$$\kappa_A = 1 - \theta - \frac{1}{z} \text{ (A scaling from now on)}. \quad (3)$$

For $d = 1$, plugging the values $\theta \approx 0.13$ and $z \approx 1.42$ of the LIM or C-DP class [4,8,16,18] into this expression leads to $\kappa \approx 0.17$ which is indeed the value measured for LIM-class interfaces [8]. In higher dimensions, this scaling law yields zero (with possible logarithmic corrections in $d = 2$) or negative κ values, i.e., no anomalous scaling, as indeed observed in simulations [16].

On the other hand, assuming that interface heights at adjacent sites are asymptotically uncorrelated (this will be justified after) we have $\nabla h \sim \sqrt{h} \sim t^{(1-\theta)/2}$ and hence

$$\kappa_B = \frac{1 - \theta}{2} \text{ (B scaling)}. \quad (4)$$

KC observed that *B* scaling is verified by many interfaces constructed from microscopic models at absorbing phase transitions [16]. Our own simulations (not shown) extend this result to different sandpile models (simulated in the fixed energy ensemble): the constructed interfaces of the Manna [12], Oslo [9], and Mohanty-Dhar [19] models show *B* scaling at criticality.

To shed some light on the physical reason for the existence of two different κ values, let us consider the Leschhorn automaton, a LIM-class model showing *A* scaling. It is a discretization of Eq. (1): an integer-valued height advances at each site x following

$$h(x) \rightarrow h(x) + 1 \quad \text{iff } \nabla^2 h + F + \eta(x, h(x)) > 0, \quad (5)$$

where the (discretized) Laplacian is computed using the nearest neighbors of x , and $\eta = \pm 1$ with respective probabilities p and $1 - p$. Consider now the Manna sandpile, a C-DP class model whose local rule is as follows: if two or more grains are present at a given site, distribute two of

TABLE I. Some of the measured critical exponents of the C-DP or LIM class vs space dimension d [4,8,16–18]. The *A*- and *B*-scaling values for κ are from our own present simulations.

d	θ	z	κ_A	κ_B
1	0.13(1)	1.42(2)	0.17(1)	0.43(1)
2	0.51(2)	1.55(3)	0 ⁻	0.25(2)
3	0.77(3)	1.78(5)	<0	0.12(3)

them randomly to the nearest neighbors [4,12]. In this case, the interface $h(x)$ encoding the number of times a site has toppled since $t = 0$ shows B scaling, as said before. It can be expressed in terms of $z_{\text{in}}(x)$ and $z_{\text{out}}(x)$, the cumulated number of particles, respectively, received from and given to the nearest neighbors. For example, in $d = 1$, $z_{\text{out}}(x) = 2h(x)$ while z_{in} can be expressed as the sum of a mean flux $h(x+1) + h(x-1)$ plus a fluctuating part, $\xi(x, h(x))$, indicating stochastic deviations from this mean [6,15,16]. The toppling condition can be written as $z_0(x) + z_{\text{in}}(x) - z_{\text{out}}(x) \geq 2$ [where $z_0(x)$ is the initial number of grains] and thus, expressed in terms of the following advancement rule:

$$h(x) \rightarrow h(x) + 1 \text{ iff } \nabla^2 h - 1 + z_0(x) + \xi(x, h) > 0. \quad (6)$$

This is very similar to Eq. (5) but, as noticed in [16], there is a crucial difference: whereas in Eq. (5) $\eta(x, h(x))$ is a *bounded*, dichotomous, δ -correlated noise, the noise term $\xi(x, h(x))$ in Eq. (6) is a sum of random variables (a unit is added or subtracted for each toppling) whose amplitude, by virtue of the central limit theorem, behaves like the square root of the average of $h(x+1) + h(x-1)$, and is therefore *diverging* in time: $\langle \xi(h)^2 \rangle \sim t^{1-\theta}$. In turn, this divergence has to be compensated by the fluctuations of the Laplacian term in Eq. (6) [since the term $z_0(x)$ representing the initial condition should be irrelevant in the long-time limit and is anyhow bounded]. This is at the origin of the strong fluctuations present in the constructed interface (B scaling) but absent in the LIM class (see Fig. 1). At this point, one clearly appreciates the qualitative difference between LIM and C-DP-constructed interfaces, which occurs despite of the fact that both classes share numerically indistinguishable (standard) critical exponents, as said before. We now show that this is not the end of the story, and that we can construct A -scaling interfaces from C-DP class models, as well as modify LIM-class models to obtain B scaling.

Our first evidence showing that A scaling and B scaling can both be compatible with a unique universality class was provided by numerical integrations of Eqs. (2). Constructing an interface, as before, via $h(x, t) = \int ds \rho(x, s)$, where ρ is the continuous activity field, we obtain clear A scaling (Fig. 2), in contrast with the B scaling heretofore always observed with microscopic models. Next, mimicking microscopic models in which the interface advances by one *unit* whenever a site is active, we constructed a different interface for Eqs. (2) through $\tilde{h}(x, t) = \int_0^t ds \Theta(\rho(x, s))$, where Θ is the Heaviside step function; this new interface advances by one unit whenever there is some nonzero activity, regardless of its magnitude. Strikingly, B scaling is then observed (Fig. 2). Thus both A - and B -scaling interfaces have been constructed at the *same* absorbing phase transition point: they correspond to slightly different *observables*. We have reached a similar conclusion for the Oslo sandpile model by taking advantage of a recent result by Pruessner [10] who constructed an exact mapping between this particular sandpile and the

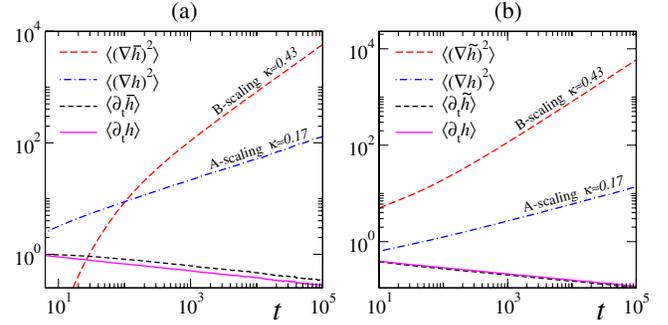


FIG. 2 (color online). Time series of interface squared-gradient and activity-density (or velocity) at criticality. (a) Equations (2) using h , [A scaling, $\kappa \approx 0.17(1)$] or \tilde{h} [B scaling, $\kappa \approx 0.43(1)$]. Parameters: $L = 2^{15}$, $b = w = \sigma^2/2 = 1$, $D = D_E = 0.25$, time-mesh 0.1; critical point $a_c = 0.86452(5)$. (b) Leschhorn automaton with $L = 2^{15}$, $F = 0$, at the critical point $p = 0.80078(5)$, and using h as in Eq. (1) [A scaling, $\kappa \approx 0.17(1)$] or \tilde{h} as in Eq. (7) [B -scaling, $\kappa \approx 0.43(1)$] with $p_c = 0.76935(5)$.

Leschhorn automaton. The local rule for $d = 1$ is as follows: distribute one grain to each nearest neighbor whenever the local height threshold is passed, this local threshold being randomly reset to be 1 or 2 grains after each toppling. To achieve an exact mapping, Pruessner showed that it is crucial to use the more symmetrical $h^\dagger(x) = h(x+1) + h(x-1)$, i.e., the accumulated number of times a given site has been charged by its neighbors rather than the accumulated activity at the site itself (the definition of h). Indeed, using $h^\dagger(x)$ for the Oslo model eliminates the diverging noise in Eq. (6) and yields A scaling in numerical simulations, whereas using $h(x)$ we observe B scaling (results not shown). We have been able to extend easily this procedure to other sandpiles as the Mohanty-Dhar one [19]. For other sandpiles with less symmetric redistribution rules as, for instance, the Manna one [12] this can be much more complicated. In this last, the 2 toppling grains at any site can go to the same neighbor. This introduces an extra noise that needs to be subtracted by an appropriate (and intricate) definition of the height variable (different from h and h^\dagger) to get rid of intrinsic local fluctuations and disentangle the hidden A scaling. These results show that a unique universality class is compatible with different values of κ , depending on microscopic details and/or the definition of the height variable; different κ exponents correspond to different, though very similar, observables. Note also that the same definition of the interface can lead to the two different types of scaling depending on the rules of the sandpile under study.

To close the loop, we now show that for standard interface depinning transitions it is also possible to generate two different κ values without affecting other exponents. Let us define $\partial_t \tilde{h}(x, t) = \partial_t h(x, t)[1 + \sigma(x, h)]$, where the quenched noise σ is 0 or 1 with probabilities p and $1 - p$, respectively, so every time the original inter-

face advances, a noise variable is added to \tilde{h} . Both interfaces are related by $\tilde{h}(x, t) = h(x, t) + \tilde{\sigma}(x, \tilde{h})$, where $\tilde{\sigma}(x, \tilde{h})$ is an accumulated noise summing up all values of $\sigma(x, h)$ at x up to height h . In terms of $\tilde{h}(x, t)$, Eq. (1) becomes

$$\partial_t \tilde{h} = \{\nabla^2[\tilde{h}(x, t) - \tilde{\sigma}(x, \tilde{h})] + \eta(x, \tilde{h})\}[1 + \sigma(x, \tilde{h})]. \quad (7)$$

Simulating Eq. (7) we observe B scaling for the \tilde{h} -interface, while removing the $\tilde{\sigma}$ -noise we readily recover A scaling for the h interface (Fig. 2), while all the other exponents coincide for both interfaces. Naïve power-counting for Eq. (7) shows that $\nabla^2 \tilde{\sigma}(x, \tilde{h})$ is an irrelevant higher-order noise, and so is the term $\sigma(x, \tilde{h})$ added to 1 [7]. Hence, upon coarse graining, Eq. (7) flows towards the standard LIM renormalization group fixed point [7], justifying that all *universal* critical exponents should coincide in Eq. (1) and (7) in accordance with our numerical results. The inclusion of the extra noise (a higher-order correction to scaling) is thus able to alter the value of κ , intensifying anomalous behavior, but not standard long-distance critical exponents, which are controlled by a unique renormalization group fixed point. The two different values of κ correspond to two different height variables, h and \tilde{h} , differing by a diverging noise. A full understanding, within the renormalization group perspective, of how local anomalous roughening properties are affected by an otherwise irrelevant noise remains a challenging task.

The fact that a given universality class can be compatible with different types of local roughening (amenable to experimental analysis) is also of interest in general studies of fluctuating interfaces [14]. Indeed, one can show that χ and χ_{loc} being, respectively, the global and local saturation roughness exponents are related by $\chi = \chi_{\text{loc}} + z\kappa$ [14], which we have numerically verified for all models studied here. Given that our results indicate that local roughening properties, as encoded by κ (or χ_{loc}), can adopt different values, depending on the presence or absence of local fluctuations, while χ , as all other nonlocal exponents, is universal, it is intriguing that experiments on the propagation of fracture cracks in wood [20] seem to lead to the opposite conclusion. See also the nice recent experiments measuring the exponents reported here for self-organized superconductors [21]. More experimental work along these lines would be most welcome, since experimental realizations of absorbing phase transitions (even for the paradigmatic DP class) are barely existing [22].

On the theoretical side, since the C-DP class and the LIM class are two faces of the same problem, both share the same upper critical dimension $d_u = 4$. This result is in contradiction with the perturbative approach for Eq. (2) in [23]. In fact, field-theoretical treatments of the LIM class demand a functional renormalization group calculation [7], and one finds that the correlator of the quenched noise

develops a cusp in the h variable. Given our results, it would be very interesting to sort out the analogue of all this in the absorbing phase transition picture, as well as attempting a nonperturbative renormalization group approach for such a case.

In summary, depinning transitions of linear interfaces in random media and absorbing phase transitions in the C-DP class are two equivalent descriptions of sandpile self-organized critical points. This clarifies the issue of universality in stochastic sandpiles and the connection of SOC to standard nonequilibrium phase transitions and opens the door to new and exciting research lines.

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