OPTIMAL ISOPERIMETRIC INEQUALITIES FOR CARTAN-HADAMARD MANIFOLDS

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Abstract. We present a new proof of the following result: consider a complete, simply connected, three-dimensional manifold, whose sectional curvatures are bounded from above by some constant $c \leqslant 0$. Then its isoperimetric profile is bounded from below by the one of Euclidean space if $c = 0$, or by the one of hyperbolic space of constant curvature c if $c < 0$.

1. Introduction

The following conjecture appeared in Aubin [2], and in Burago-Zalgaller [3] and Gromov-Lafontaine-Pansu [8]. It can also be stated in terms of Sobolev inequalities in Riemannian manifolds [9].

Conjecture 1.1. *Let M^n be a complete, simply connected, n-dimensional manifold, whose sectional curvatures satisfy inequality $K_{sec} \leqslant c \leqslant 0$, for some constant $c \leqslant 0$. Then the isoperimetric profile I_M of M^n is bounded from below by the isoperimetric profile I_c of the complete and simply connected space M^n_c whose sectional curvatures are equal to c. This implies:*

$$\text{area}(\partial\Omega) \geqslant I_c(\text{vol }\Omega), \tag{*}$$

for any domain $\Omega \subset M$ with smooth boundary. Moreover, if equality holds in (*)*, then Ω is isometric to the geodesic ball of volume* vol(Ω) *in M^n_c.*

We recall that a *Cartan-Hadamard* manifold is a complete, simply connected Riemannian manifold whose sectional curvatures are nonpositive. Such a manifold is diffeomorphic, via the exponential map at any point, to the Euclidean space of the same dimension. The *isoperimetric profile I_M* of a manifold M is the function $I_M : (0, \operatorname{vol} M) \to \mathbb{R}^+$ given by

$$I_M(V) = \inf\{\operatorname{area}\partial\Omega; \Omega \subset\subset M \text{ has smooth boundary and } \operatorname{vol}\Omega = V\}.$$

A set Ω that satisfies $\operatorname{area}\partial\Omega = I_M(\operatorname{vol}\Omega)$ is called an *isoperimetric domain.* Isoperimetric domains need no exist in Riemannian manifolds, as shown in [16, Thm. 2.16]

The above conjecture has been referred to in the literature as the Cartan-Hadamard conjecture [9, 8.2], or as Aubin conjecture [15, 17.3].

This conjecture was already proved in the two-dimensional case by A. Weil [20]. In fact it follows from the classical isoperimetric inequality for discs in surfaces with Gauss curvature bounded from above. If $K \leqslant K_0$ then:

$$L^2 \geqslant 4\pi A - K_0 A^2,$$

where A denotes the area of a set, and L its perimeter. If the surface is a plane then a classical argument (filling the holes of a region) shows that this inequality is also valid for any domain of arbitrary topological type.

Conjecture 1.1 has been proved by C. Croke [5] when $n = 4$ and $c = 0$. Croke obtained generic inequalities of the form $\operatorname{area}\partial\Omega \geqslant \beta_n\, I_0(\operatorname{vol}\Omega)$, where $\beta_n \leqslant 1$ and equality hold only for $n = 4$.

Two more reasons to believe that the above conjecture should hold are: (i) inequality (*) is true for geodesic balls in Cartan-Hadamard manifolds by classical comparison theorems, and (ii) inequality:

$$\operatorname{area}(\partial\Omega) \geqslant \varepsilon_n I_c(\operatorname{vol}\Omega),$$

holds, where $\varepsilon_n < 1$ are constants depending only on the dimension n of the manifold, see Croke [4], Hoffman-Spruck [10] (see also Michael-Simon [12]), and Burago-Zalgaller [3].

Conjecture 1.1 has been proved by B. Kleiner [11] for any $c \leqslant 0$ in dimension 3. In his proof there is a common scheme to any dimension. He only uses dimension three to prove the following Proposition, applying Gauss-Bonnet formula over the two-dimensional boundary of an isoperimetric domain

Proposition 1.2. *Let M^3 be a complete, simply connected, 3-dimensional manifold, with sectional curvatures bounded from above by a constant $c \leqslant 0$. Let Ω be a compact set with $C^{1,1}$ boundary Σ. Then*

$$\max_{\Sigma} H_\Sigma \geqslant H_c(\operatorname{area}\Sigma),$$

where H_c is the mean curvature in the model space M_c^3 of the geodesic ball of area equal to $\operatorname{area}\Sigma$.

Along these notes we shall use the terms area and volume to refer to $(n-1)$ and n-dimensional Hausdorff measures, respectively.

2. Proof of Conjecture 1.1 using Proposition 1.2

Let us see that Conjecture 1.1 is true in any dimension if the analogous of Proposition 1.2 is valid. As we said before, isoperimetric domains may not exist in a noncompact manifold. To solve this problem we shall work in geodesic balls in M.

The following result summarizes what we can say about isoperimetric domains in a manifold with boundary (in a geodesic ball in our case)

Theorem 2.1 (Existence and regularity of isoperimetric domains in manifolds with boundary)**.** *Let B^n be a compact manifold with smooth boundary ∂B^n. Let $V \in (0, \operatorname{vol} B^n)$. Then there is a domain $\Omega \subset B^n$ with boundary $\Sigma = \partial\Omega$ such that*

(i) $\operatorname{vol}\Omega = V$, $\operatorname{area}\Sigma = I_B(V)$.
(ii) *$\Sigma = \partial\Omega$ is $C^{1,1}$ in a neighborhood of ∂B.*
(iii) *There is a singular set $\Sigma_{sing} \subset \Sigma \cap \operatorname{int} B$ of Hausdorff dimension less than or equal to $n-8$ such that $(\Sigma \cap \operatorname{int} B) - \Sigma_{sing}$ is a smooth hypersurface with constant mean curvature H.*
(iv) *The mean curvature h of Σ is defined almost everywhere (except in a set of $\mathcal{H}^{n-1}$-measure zero), and we have $h \leqslant H$.*

Moreover, if Ω_n is a sequence of isoperimetric domains in B such that $\operatorname{vol}(\Omega_i) \to V$*, then* $\operatorname{area}\partial\Omega_i \to I_B(V)$.

Existence of isoperimetric domains follows from classical theorems of Geometric Measure Theory for finite perimeter sets. Regularity of $\Sigma - \Sigma_{sing}$ in the interior of B is obtained from Gonzalez, Massari, Tamanini [7]. For $C^{1,1}$ regularity near ∂B one must consult White [21], and Stredulinski-Ziemer [19]. The last line in the statement implies the continuity of the isoperimetric profile, which also follows from Gallot [6].

The proof of (iv) is obtained by taking p in the regular part of Σ and q in the intersection $\Sigma \cap \partial B$. Consider functions u, v, defined in neighborhoods of p, q, respectively, so that $\int_\Sigma u\, d\Sigma = \int_\Sigma v\, d\Sigma$. Then we get a variation that fixes the volume of Ω, and pushes Σ towards Ω near q so that the derivative of area is given by

$$\int_\Sigma nH\, u\, d\Sigma - \int_\Sigma nh\, v\, d\Sigma \geqslant 0,$$

and so

$$nH \geqslant \frac{\displaystyle\int_\Sigma nh\, v\, d\Sigma}{\displaystyle\int_\Sigma v\, d\Sigma},$$

from which the claim follows.

2.1. We now prove Conjecture 1.1. Choose a geodesic ball B that contains the domain $\Omega \subset M$. We recall that the isoperimetric profile I_B is continuous by Theorem 2.1.

Let Ω_v be an isoperimetric domain in B of volume $v = \operatorname{vol}\Omega$, and let H_v be the (constant) mean curvature of the regular part of $\Sigma_v = \partial\Omega_v$ in the interior of B.

If Proposition 1.2 is true for any dimension, then $H_v \geqslant H_c(\operatorname{area}\Sigma_v)$, and equality holds for a geodesic ball of area equal to $\operatorname{area}\Sigma_v$ in a space of constant curvature c.

Choose a deformation Ω_V with support in the regular part of Σ_v in the interior of B parameterized with respect to volume, (it is enough to consider a normal deformation uN, where $u \geqslant 0$). Then we have, for $\Delta V < 0$

$$\begin{aligned}\frac{I_B(v+\Delta V) - I_B(v)}{\Delta V} &\geqslant \frac{\operatorname{area}\partial\Omega_{v+\Delta V} - \operatorname{area}\Sigma_v}{\Delta V} \geqslant (nH_v + \varepsilon(\Delta V)) \\ &\geqslant (nH_c(I_B(v)) + \varepsilon(\Delta V)) > 0,\end{aligned}$$

what implies that I_B is strictly monotone and, so, smooth almost everywhere. Moreover, if I_B is smooth in v, then

$$I'_B(v) \geqslant nH_c(\operatorname{area}\Sigma_v). \tag{**}$$

Now we are ready to finish the proof, since translating the profile M_c^3 to left and right we obtain a foliation of the upper halfplane in $\mathbb{R}^2$, and inequality (**) follows since the function I_B meets this foliation transversally, so that the profile lies above M_c^3, since $I_B(0) = I_c(0) = 0$.

If equality holds in $I_B(v) \geqslant I_c(v)$ then we have equality of the profiles for any $V \in (0, v)$, so that I_B is smooth, equality holds in (**) for any value $V \in (0, v)$ and, so, Ω is isometric to a ball of volume v in space M_c^3 by Proposition 1.2.

3. Kleiner's proof of Proposition 1.2

Proposition 1.2 is trivial if $\Sigma \subset M^3$ is a sphere, since

$$4\pi = \int_\Sigma K\, dA = \int_\Sigma (K_{sec} + \kappa_1\kappa_2)\, dA \leqslant \int_\Sigma (c + \kappa_1\kappa_2)\, dA \leqslant (c + H^2)\operatorname{area}\Sigma,$$

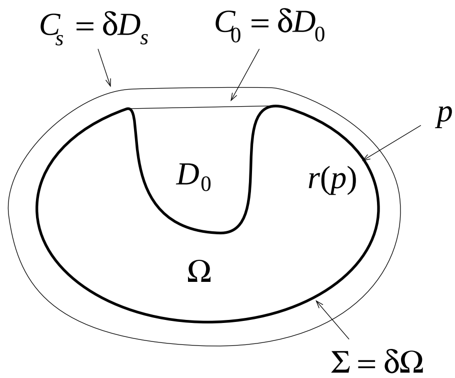
and equality holds if and only if $K_{sec} \equiv c$ over Σ and the surface is totally umbilical. In case Σ is a geodesic sphere in a space of constant curvature c, equality holds in the above inequality. This shows that

$$H \geqslant H_c(\operatorname{area}\Sigma).$$

If equality holds in the above inequality, then Σ is a totally umbilical sphere so that the sectional curvature of the tangent plane equals c and Σ has the same second fundamental form as of the sphere of area area Σ in M^3_c. It follows from Theorem 7 in [18] that the domain enclosed by Σ is a geodesic ball in M^c_3.

Let us assume now that Σ is any $C^{1,1}$ surface which encloses a domain Ω. Consider the closed convex hull D_0 of Ω. The set D_0 is convex, and compact since it is contained in a (convex) ball of M^3. Of course nothing is known about the regularity of ∂D_0, so that by using ideas of Almgren [1], we consider the domains

$$D_s = \{x \in M^3 /\ d(x, D_0) \leqslant s\}.$$



We know that

- D_s is convex,
- $r : M^3 - \operatorname{int} D_0 \to \Sigma$ is well defined and it is distance nonincreasing.
- $C_s = \partial D_s$ is homeomorphic to $\mathbb{S}^2$.

Let us call $r_s = r\big|_{C_s}$. As C_s is a $C^{1,1}$ surface, by Rademacher's Theorem (a Lipschitz function is smooth almost everywhere) its Gauss curvature and its Gauss-Kronecker curvature GK (product of principal curvatures) exist and the total curvature of C_s equals 4π. Then we have

$$\begin{aligned}
4\pi = \int_{C_s} K &= \int_{C_s} (K_{sec} + GK_{C_s}) \leqslant \int_{C_s} (c + GK_{C_s}) \\
&= \int_{r_s^{-1}(\Sigma)} (c + GK_{C_s}) + \int_{C_s - r_s^{-1}(\Sigma)} (c + GK_{C_s}) \\
&\leqslant \int_{r_s^{-1}(\Sigma)} (c + H_s^2) + c \operatorname{area}(C_s - r_s^{-1}(\Sigma)) + \int_{C_s - r_s^{-1}(\Sigma)} GK_{C_s} \\
&\leqslant \int_{r_s^{-1}(\Sigma)} (c + H_s^2) + \int_{C_s - r_s^{-1}(\Sigma)} GK_{C_s}.
\end{aligned}$$

In the first inequality we have bounded K_{sec} by c, and in the second one GK_{C_s} by the mean curvature of C_s. Equality holds in the above inequality if and only if $K_{sec} = c$ along C_s and $r_s^{-1}(\Sigma)$ is totally umbilical.

We treat now the integrals that appears in the last line.

Let us see that

$$\lim_{s\to 0}\int_{r_s^{-1}(\Sigma)}(c+H_s^2)\leqslant (c+H_0^2)\,\mathrm{area}(\Sigma\cap C_0). \tag{\#}$$

We only have to take into account that if C_s is C^2 in p (this happens for almost every $p\in C_s$) and $p\in C_s$ and $r_s(p)\in C_0\cap\Sigma$, then we have

$$2H_s(p)\leqslant 2H_0 - s\,(\mathrm{Ric}_-),$$

where Ric_- is the infimum of the Ricci curvatures in the unit sphere at every point of C_s. Passing to the limit when $s\to 0$ we have (#). The way of getting the above inequality is to apply the formula

$$\frac{d(2H_t)(p)}{dt} = -(\mathrm{Ric}(N,N)+|\sigma|^2)\leqslant -\mathrm{Ric}(N,N),$$

and integrate with respect to t between 0 y s. If equality holds in (#), then $\mathrm{area}\, C_0\cap\Sigma = \mathrm{area}\,\Sigma$, so that $\mathrm{vol}\, D_0 = \mathrm{vol}\,\Omega$, from where we conclude $D_0=\Omega$. It follows that Σ is convex.

Let us see now

$$\lim_{s\to 0}\mathrm{area}(r_s^{-1}(\Sigma)) = \mathrm{area}(C_0\cap\Sigma).$$

We use area formula for Lipschitz maps and we get

$$\int_{r_s^{-1}(\Sigma)}\mathrm{Jac}(r_s)\,dC_s = \mathrm{area}(C_0\cap\Sigma),$$

so that

$$\begin{aligned}\mathrm{area}(r_s^{-1}(\Sigma)) &= \int_{r_s^{-1}(\Sigma)}\mathrm{Jac}(r_s)\,dC_s + \int_{r_s^{-1}(\Sigma)}(1-\mathrm{Jac}(r_s))\,dC_s\\ &= \mathrm{area}(C_0\cap\Sigma) + \int_{r_s^{-1}(\Sigma)}(1-\mathrm{Jac}(r_s))\,dC_s \to \mathrm{area}(C_0\cap\Sigma),\end{aligned}$$

since $\mathrm{Jac}(r_s)$ converges uniformly to 1.

We finally see

$$\lim_{s\to 0}\int_{C_s - r_s^{-1}(\Sigma)} GK_{C_s} = 0.$$

Note first that if $p\in C_0-\Sigma$ is a smooth point, then $GK_{C_0}(p)=0$: otherwise we could push D_0 near p towards the interior of D_0 to contradict the convex hull property of D_0.

Fix now $s_0>0$ and write

$$\int_{C_s - r_s^{-1}(\Sigma)} GK_{C_s}\,dC_s = \int_{C_{s_0} - r_{s_0}^{-1}(\Sigma)} (GK_{C_s}\circ r_{s_0 s})\,\mathrm{Jac}(r_{s_0 s})\,dC_{s_0},$$

where $r_{s_0 s} = r_s^{-1}\circ r_{s_0}$. The right integral is uniformly bounded because the second fundamental form of C_s in $q = r_{s_0 s}(p)$ applied to a vector $e\in T_qC_s$ of modulus 1 equals to

$$\langle A_s(e), e\rangle = \frac{\langle J', J\rangle}{|J|^2},$$

where J is a Jacobi field along the geodesic $\gamma(s) = r_{s_0 s}(p)$ that has modulus 1 over C_{s_0}, orthogonal to C_{s_0}, and such that $J/|J| = e$. J is induced by a family of orthogonal geodesics leaving from C_{s_0}. The quantity $\langle A_s(e), e\rangle$ is bounded by the classical comparison theorems for geodesics starting from a submanifold (with sectional curvature bounded from below). We remark that the second fundamental form of C_s is uniformly bounded from above, since every point in C_s is supported by a ball of radius s.

By the above discussion, $GK_{C_s} \circ r_{s_0 s}$ converges to 0 for almost every point of $C_{s_0} - r_{s_0}^{-1}(\Sigma)$, so that the integral of GK_{C_s} converges to 0 in $C_{s_0} - r_{s_0}^{-1}(\Sigma)$.

Then we conclude

$$4\pi \leqslant (c + H_0^2)\,\text{area}\,\Sigma,$$

what implies

$$H_0 \geqslant H_c(\text{area}\,\Sigma).$$

If equality holds in the above inequality then, an analysis of the possibilities, yields

- Σ has constant mean curvature equal to the one of the geodesic ball of the same area in M_c^3.
- The sectional curvature of the tangent plane to Σ equals c.
- Σ is totally umbilical.

Then a result by Schroeder-Ziller [18] implies that Σ is a geodesic ball in M_c^3.

4. A new proof of Proposition 1.2

4.1. **The Euclidean case.** Consider first the case the case $K \leqslant 0$. Let Σ be an embedded $C^{1,1}$ compact surface. This surface has principal curvatures defined almost everywhere. Let $p \in \Sigma$ and let $d(q)$ measure the distance to p. We consider the conformal metric

$$g_\varepsilon = \rho_\varepsilon^2\, g = e^{2u_\varepsilon}\, g,$$

where

$$\rho_\varepsilon = \frac{2\varepsilon}{1+\varepsilon^2 d^2}, \qquad u_\varepsilon = \log\left(\frac{2\varepsilon}{1+\varepsilon^2 d^2}\right).$$

In case M is the Euclidean space this metric is the obtained by applying a conformal transformation to the metric of the sphere and projecting this metric orthogonally to the Euclidean space by means of the stereographic projection.

Let us see now that

$$\int_\Sigma H^2\, dA \geqslant 4\pi,$$

where equality holds if and only if Ω is flat (vanishing sectional curvatures).

By taking into account the well known relation between conformal metrics we get

$$(\#\#) \qquad e^{2u_\varepsilon} K_\varepsilon \geqslant K + e^{2u_\varepsilon},$$

where K_ε and K are the sectional curvatures of a given plane of M for the metrics g_ε and g, respectively. From now on we shall assume that they are the ones of the tangent plane to Σ.

So

$$\begin{aligned}
\int_\Sigma H^2\,dA &= \int_\Sigma (H^2+K)\,dA - \int_\Sigma K\,dA \\
&= \int_\Sigma ((H_\varepsilon^2) + K_\varepsilon)\,dA_\varepsilon - \int_\Sigma K\,dA \\
&\geqslant \int_\Sigma H_\varepsilon^2\,dA_\varepsilon + \int_\Sigma dA_\varepsilon \\
&\geqslant \int_\Sigma dA_\varepsilon,
\end{aligned}$$

where in the first equality we have used the conformal invariance of $\int (H^2 + K_{sec})\,dA$, and in the first inequality we have used inequality (##). The limit of the last integral can be computed by passing to polar (ambient) coordinates, or taking into account that it corresponds geometrically to blowing up the surface Σ at the point p with a spherical metric. So that

$$\lim_{\varepsilon\to\infty} \int_\Sigma dA_\varepsilon = 4\pi.$$

From the two last inequalities we obtain the desired estimate.

To analyze what happens when equality holds we need a more accurate estimate of the expression of the curvatures in the conformal metrics. So we write

$$e^{2u_\varepsilon} K_\varepsilon = K - \left(\frac{\varepsilon^2}{1+\varepsilon^2 d^2}\right)^2 4d^2 + \left(\frac{\varepsilon^2}{1+\varepsilon^2 d^2}\right)(\nabla^2 d^2(X,X) + \nabla^2 d^2(Y,Y)),$$

where X, Y is an orthonormal basis of the tangent plane to Σ. From this formula we get

$$\begin{aligned}
4\pi = \int_\Sigma H^2\,dA &= \int_\Sigma (H^2+K)\,dA - \int_\Sigma K\,dA \\
&= \int_\Sigma ((H_\varepsilon^2) + K_\varepsilon)\,dA_\varepsilon - \int_\Sigma K\,dA \\
&= \int_\Sigma dA_\varepsilon + \int_\Sigma \left(\frac{\varepsilon^2}{1+\varepsilon^2 d^2}\right)(\nabla^2 d^2(X,X) + \nabla^2 d^2(Y,Y) - 4)\,dA + \int_\Sigma H_\varepsilon^2\,dA_\varepsilon.
\end{aligned}$$

We already know that the first integral converges to 4π when $\varepsilon \to \infty$. So the limit of the remaining integrals is 0. Since $\nabla^2 d^2(X,X) \geqslant 2$ for any $|X| = 1$ we obtain that both integrals are positive and, in particular,

$$\nabla^2 d^2(X,X) = \nabla^2 d^2(Y,Y) = 2.$$

Standard comparison theorems in Riemannian Geometry show that, if the geodesic starting from p leaves the enclosed domain Ω in a nontangential way, then $\nabla^2 d^2 = 2\,g$ at the hitting point. Standard comparison shows that $\nabla d^2 \equiv 2\,g$ along the geodesic. Moving slightly the geodesic we get a cone so that $\nabla d^2 \equiv 2\,g$ inside this cone. Since every point in the interior of Ω can be connected with Σ by a minimizing geodesic hitting Σ orthogonally we conclude that every point inside Σ is flat and so Ω is flat.

4.2. Let us see now what happens if

$$\max H^2 \operatorname{area}(\Sigma) = 4\pi.$$

In this case, in addition to Ω flat we get that the mean curvature of the boundary is constant. For any domain of this type, Ros [17] and Montiel-Ros [14] have proved that

$$3 \operatorname{vol} \Omega \leqslant \frac{1}{H} \operatorname{area} \Sigma,$$

and equality holds if and only if Ω is isometric to a geodesic ball in Euclidean space. But the classical Minkowski formula

$$3 \operatorname{vol} \Omega = \frac{1}{H} \operatorname{area} \Sigma,$$

holds in Ω since the function $(1/2)\, d^2$ has Hessian on Ω proportional to 2 times the identity matrix. From this we conclude our proof of Proposition 1.2 in the flat case.

4.3. **The hyperbolic case $K \leqslant -1$.** In the hyperbolic case one has to consider the following family of conformal metrics

$$g_\varepsilon = \left(\frac{2\varepsilon}{(1-\varepsilon^2)+(1+\varepsilon^2)\cosh(d)}\right) g, \qquad \varepsilon > 1.$$

This family of metrics is obtained by writing the spherical metric in a disc D of $\mathbb{R}^n$ via stereographical projection in terms of the hyperbolic metric of constant curvature -1 in D. So we obtain

$$e^{2u_\varepsilon} K_\varepsilon \geqslant K + e^{2u_\varepsilon} + 1.$$

and

$$\int_\Sigma (-1+H^2)\, dA \geqslant \int_\Sigma dA_\varepsilon.$$

As in the previous case one proves that

$$\lim_{\varepsilon\to\infty} \int_\Sigma dA \to 4\pi,$$

which yields the desired estimate.

To analyze equality it is more convenient to write

$$e^{2u_\varepsilon} K_\varepsilon = K + 1 + e^{2u_\varepsilon} + \left(\frac{1+\varepsilon^2}{(1-\varepsilon^2)+(1+\varepsilon^2)\cosh(d)}\right) \times$$
$$\times \Big(\nabla^2 \cosh(d)(X,X) + \nabla^2 \cosh(d)(Y,Y) - 2\cosh(d)\Big).$$

We recall that by classical comparison theorems, when $K_{sec} \leqslant -1$ we get $\nabla^2 \cosh(d) \geqslant \cosh(d)\, \langle,\rangle$, so that the factor in the previous displayed line is nonnegative. Hence

$$4\pi = \int_\Sigma (-1+H^2)\, dA = \int_\Sigma dA_\varepsilon + \int_\Sigma H_\varepsilon^2\, dA_\varepsilon$$
$$+ \int_\Sigma \left(\frac{1+\varepsilon^2}{(1-\varepsilon^2)+(1+\varepsilon^2)\cosh(d)}\right) \times$$
$$\times \Big(\nabla^2 \cosh(d)(X,X) + \nabla^2 \cosh(d)(Y,Y) - 2\cosh(d)\Big)\, dA,$$

Letting $\varepsilon \to \infty$ and taking into account that $\lim_{\varepsilon\to\infty} \int_\Sigma dA_\varepsilon = 4\pi$ we deduce that the remaining positive integrals tend to 0 when $\varepsilon \to \infty$. In particular

$$\nabla^2 \cosh(d)(X,X) = \nabla^2 \cosh(d)(Y,Y) = \cosh(d).$$

By standard comparison theorems, and arguing as in the Euclidean case, we conclude that the metric in Ω is hyperbolic.

4.4. If

$$\max_\Sigma(-1+H^2)\,\mathrm{area}\,\Sigma = 4\pi,$$

then H is constant. Moreover, from [13, Theorem 9] we conclude, by taking inner parallels

$$\int_\Sigma \big(\cosh(d) + H\,\sinh(d)\,\langle \partial/\partial d, N\rangle\big)\,dA \geqslant 0,$$

and equality holds only when Σ is a geodesic sphere. But since the metric in Ω is hyperbolic we have $\nabla^2\cosh(d) = 2\,\langle\,,\rangle$, so that

$$\int_\Sigma \big(\cosh(d) + H\,\sinh(d)\,\langle \partial/\partial d, N\rangle\big)\,dA = 0,$$

and Proposition 1.2 also follows in the hyperbolic case.

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