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Apery set of a numerical semigroup and properties of the associated graded ring of a numerical semigroup ring

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based on:


1. Setup

$g_1 < g_2 < \cdots < g_n \in \mathbb{N}, \quad \text{GCD}(g_1, \ldots, g_n) = 1$

$S = \langle g_1, \ldots, g_n \rangle := \{a_1 g_1 + \cdots + a_n g_n \mid a_i \in \mathbb{N}, i = 1, \ldots, n\}$

$R = k[[S]] := k[[t^{g_1}, \ldots, t^{g_n}]] \quad \text{(or } R := k[t^{g_1}, \ldots, t^{g_n}](t^{g_1}, \ldots, t^{g_n}))$

$R$ is a one-dimensional, local domain, with maximal ideal $m = (t^{g_1}, \ldots, t^{g_n})$ and quotient field $Q = k((t))$.

If $v : k((t)) \longrightarrow \mathbb{Z} \cup \infty$ is the natural valuation, we get

$v(R) = \{v(r) \mid r \in R \setminus \{0\}\} = S$

The associated graded ring with respect to $m$ will be denoted by

$G(m) := \bigoplus_{i \geq 0} m^i / m^{i+1}$ \quad $M := \bigoplus_{i \geq 1} m^i / m^{i+1}$
2. Problem, definition and first remarks

When is $G(m)$ a Buchsbaum ring?

**Definition.** (Stückrad-Vogel) $G(m)$ is Buchsbaum if $\mathcal{M} \cdot H^0_{\mathcal{M}} = 0$.

**Remarks.** • As $H^0_{\mathcal{M}} = (\bigcup_{k \geq 1} (0 : G(m) \mathcal{M}^k))$,

\[ G(m) \text{ is Buchsbaum} \iff \mathcal{M} \cdot (\bigcup_{k \geq 1} (0 : G(m) \mathcal{M}^k)) = 0. \]

• $G(m)$ is Buchsbaum $\iff (0 : G(m) \mathcal{M}) = (0 : G(m) \mathcal{M}^k)$, $\forall k \geq 1$.

• $(0 : G(m) \mathcal{M}) = (0) \iff (0 : G(m) \mathcal{M}^k) = (0)$, $\forall k \geq 1$.

• $G(m)$ Cohen Macaulay (C-M) $\iff (0 : G(m) \mathcal{M}) = (0)$

• $G(m)$ C-M $\implies G(m)$ Buchsbaum
3. Some references
The property for $G'(m)$ to be C-M has been studied extensively, while, not much is known about the Buchsbaum property of $G'(m)$ (except for the case that $G'(m)$ is C-M).
General case ($\mathcal{R}$ Noetherian, local ring, of dimension $d$):
- Goto, *Buchsbaum rings of maximal emb. dim.* ('82)
- Goto, *Noeth. local rings with Buchs. assoc. graded rings* ('84)

The “one-dimensional” case:

The semigroup ring case:
- Sapko, *Associated graded rings of numerical semigroup rings* (2001)
- Shen, *Tangent cone of numerical semigroup rings with small embedding dimension* (2008)
- Cortadellas-Zarzuela, *Tangent cones of numerical semigroup rings* (2009)
4. More remarks

Let $x$ be any element s.t. $v(x) = g_1$ and $\bar{x}$ its image in $G(m)$. Let $r$ be the reduction number of $m$, i.e. the minimal integer such that $m^{r+1} = x m^r$.

- $m^r / m^{r+1} \cong m^{r+i} / m^{r+i+1}$, as $R$-module, $\forall \ i \geq 1$.

- $\mathcal{M}$ is generated by $m/m^2$, hence
  \[
  (0 :_{G(m)} \mathcal{M}^i) = (0 :_{G(m)} m^i/m^{i+1})
  \]

- $(0 :_{G(m)} \mathcal{M}) \subseteq \cdots \subseteq (0 :_{G(m)} \mathcal{M}^r) = \cdots = (0 :_{G(m)} \mathcal{M}^k) = \cdots \quad (\forall k \geq r)$

- $G(m)$ is Buchsbaum (not C-M) $\iff$
  \[
  0 \neq (0 :_{G(m)} \mathcal{M}) = (0 :_{G(m)} \mathcal{M}^r)
  \]
5. Graded description of \((0 :_{G(m)} \mathcal{M}^k)\)

Recall that: \(G(m)\) is Buchsbaum \(\iff (0 :_{G(m)} \mathcal{M}) = (0 :_{G(m)} \mathcal{M}^r)\)

\[
(0 :_{G(m)} \mathcal{M}^k) = \bigoplus_{h \geq 1} \frac{(m^{h+k+1} : R \ m^k) \cap m^h}{m^{h+1}}
\]

**Remark.** For every \(h\), \((m^{h+2} : R \ m) \cap m^h \subseteq (m^{h+r+1} : R \ m^r) \cap m^h\)

This means that each direct summand of \((0 :_{G(m)} \mathcal{M})\) is contained in the corresponding direct summand of \((0 :_{G(m)} \mathcal{M}^r)\). Hence:

\(G(m)\) is Buchsbaum \(\iff (m^{h+2} : R \ m) \cap m^h = (m^{h+r+1} : R \ m^r) \cap m^h \ \forall \ h \geq 1.\)
Let $R' = B(m) = \bigcup_{n \geq 1} (m^n : Q m^n) = (m^r : Q m^r) = x^{-r}m^r$
(the last one is an equality of $R$-modules). Hence:

$$(m^{h+r+1} :_R m^r) \cap m^h = (m^{h+r+1} :_Q m^r) \cap m^h = x^{h+1}R' \cap m^h$$

(2) \hspace{1cm} (0 :_{G(m)} \mathcal{M}^r) = \bigoplus_{h \geq 1} \frac{x^{h+1}R' \cap m^h}{m^{h+1}}$

**Proposition.** (D-M-M)

(3) \hspace{1cm} (0 :_{G(m)} \mathcal{M}^r) = \bigoplus_{h=1}^{r-2} \frac{x^{h+1}R' \cap m^h}{m^{h+1}}$

(All direct summands are zero for $h \geq r - 1$, since $x^{-(h+1)}m^{h+1} = R'$, $\forall h \geq r - 1$.) Hence:

$G(m)$ is Buchsbaum $\iff (m^{h+2} :_Q m) \cap m^h = (m^{h+r+1} :_Q m^r) \cap m^h$
$\forall h = 1, \ldots, r - 2.$
The direct summands of \((0 :_{G(m)} \mathcal{M})\) and \((0 :_{G(m)} \mathcal{M}^r)\) corresponding to \(h = r - 2\), are equal:

**Lemma.**
\[
\frac{(m^{2r-1} :_Q m^r) \cap m^{r-2}}{m^{r-1}} = \frac{(m^r :_Q m) \cap m^{r-2}}{m^{r-1}}
\]

**Corollary.** (Goto) Let \(r\) be the reduction number of \(R\). If \(r \leq 3\) then \(G(m)\) is Buchsbaum.

**Corollary.** Let \(e = g_1\) be the multiplicity of \(R\). If \(e \leq 4\) then \(G(m)\) is Buchsbaum.
6. Some well-known facts

- $H \subset \mathbb{Z}$, $H \neq \emptyset$ is a relative ideal of a semigroup $S$ if $H + S \subseteq H$ and $H + s \subseteq S$ ($\exists s \in S$);
- if $H$ and $L$ are relative ideals of $S$, then $H + L$, $nH$ and $H - L := \{n \in \mathbb{Z} \mid n + L \subseteq H\}$ are relative ideals of $S$;
- $M = \{s \in S \mid s \neq 0\}$ is called the maximal ideal of $S$;
- the reduction number $r$ of $M$ is the minimal natural number such that $(r + 1)M = g_1 + rM$.

The blow up of $S = \langle g_1, g_2, \ldots, g_n \rangle$ is the numerical semigroup

$$S' = \bigcup_{n \geq 1} (nM - nM) = (rM - rM) = rM - rg_1 = \langle g_1, g_2 - g_1, \ldots, g_n - g_1 \rangle.$$
• If $I$ and $J$ are fractional ideals of $R$, then $v(I)$ and $v(J)$ are relative ideals of $S = v(R)$;

• if $I$ and $J$ are monomial ideals, then $v(I \cap J) = v(I) \cap v(J)$, $v(I^n) = n \cdot v(I)$ and $v(I :_Q J) = v(I) - v(J)$;

• if $J \subseteq I$, then $\lambda_R(I/J) = |v(I) \setminus v(J)|$. 
7. The Apery set

Fix $\bar{s} \in S$ and set $\omega_i := \min \{s \in S \mid s \equiv i \, (\text{mod} \, \bar{s})\}. \omega_0 = 0$

The Apery set of $S$ with respect of $\bar{s}$ is the set

$$\text{Ap}_{\bar{s}}(S) = \{\omega_0, \ldots, \omega_{\bar{s}-1}\}$$

We will compare the Apery sets of $S$ and $S'$, with respect to $g_1$

We fix the following notations:

$$\text{Ap}_{g_1}(S) = \{\omega_0, \ldots, \omega_{g_1-1}\}$$

$$\text{Ap}_{g_1}(S') = \{\omega'_0, \ldots, \omega'_{g_1-1}\}$$
**Definition.** (Barucci-Fröberg) For each $i = 0, 1, \ldots, g_1 - 1$ let:
- $a_i$ be the only integer such that $\omega'_i + a_i g_1 = \omega_i$;
- $b_i = \max\{l \mid \omega_i \in lM\}$.

**Remark.** $b_0 = a_0 = 0$ and $1 \leq b_i \leq a_i$.

**Theorem.** (Barucci-Fröberg)
$G(m)$ is C-M $\iff a_i = b_i$ for each $i = 0, 1, \ldots, g_1 - 1$. 
8. An example

\[ S = \langle 8, 9, 15 \rangle \ (R = k[[S]]) \]

\[ S' = \langle 1, 7, 8 \rangle = \mathbb{N} \]

\[ \text{Ap}_8(S) = \{0, 9, 18, 27, 36, 45, 30, 15\} \]

\[ \text{Ap}_8(S') = \{0, 1, 2, 3, 4, 5, 6, 7\}. \]

\[ 30 \in 2M \setminus 3M \ \Rightarrow \ b_6 = \max\{l \mid \omega_6 \in lM\} = 2 \]

\[ 30 - 6 = 3 \cdot 8 \ \Rightarrow \ a_6 = 3 \]

Hence \( G(m) \) is not C-M. We will see that \( G(m) \) is not even Buchsbaum.
9. More results using Apery sets

For each $i$ such that $a_i > b_i$, let

$$l_i = \max\{l \mid \overline{t\omega_i + lg_1} \in (0 : G(\mathfrak{m}) M^r)\}.$$  

**Remark.** $l_i$ is well defined because, from Formula (3), $\overline{t\omega_i + lg_1} \notin (0 : G(\mathfrak{m}) M^r)$ whenever $t\omega_i + lg_1 \in \mathfrak{m}^{r-1}$.

**Remark.** $l_i \leq r - 2 - b_i$. Indeed, $\omega_i \in b_i M$ hence $\omega_i + l_i g_1 \in (b_i + l_i) M$ and $b_i + l_i \leq r - 2$ by Formula (3).

**Lemma.** Let $i$ and $l_i$ as above. Then $\overline{t\omega_i + lg_1} \in (0 : G(\mathfrak{m}) M^r)$ for every $l = 0, \ldots, l_i$. 
**Lemma.** The only monomials in \((0 : G(m) \mathcal{M}^r)\) are of the form \(t^{\omega_i+lg_1}\), with \(i\) such that \(a_i > b_i\).

**Corollary.** Let \(G(m)\) be not C-M. Then

\[
(0 :_{G(m)} \mathcal{M}^r) = \langle t^{\omega_i+lg_1} | a_i > b_i, \ l = 0, \ldots, l_i \rangle_k.
\]
Proposition. (D-M-S) $G(m)$ is Buchsbaum if and only if $t^\omega_i + lg_1 \in (0 : G(m) M)$, for every $i$ such that $a_i > b_i$ and for every $l = 0 \ldots, l_i$.

Remark. Let $G(m)$ be Buchsbaum. Then $l_i < a_i - b_i$. 
If \( a_i - b_i = 1 \) for every \( i \) such that \( a_i > b_i \), then we can improve last proposition.

**Proposition.** If \( a_i - b_i = 1 \) for every \( i \) such that \( a_i > b_i \), then

\[
G(m) \text{ is Buchsbaum if and only if } t^{\omega_i} \in (0 : G(m) \mathcal{M}).
\]

In the example: \( 30 \in 2M \setminus 3M, \ 8 \in M \setminus 2M, \ 30 + 8 = 38 \in 3M \setminus 4M \Rightarrow t^{30} \notin (0 : \mathcal{M}). \)
Hence \( G(m) \) is NOT Buchsbaum.
**Remark.** Last proposition does not hold if there exists an $i$ such that $a_i - b_i \geq 2$.

Let $S = \langle 12, 19, 29, 104 \rangle$ with $r = 8$.

The only index for which $a_i > b_i$ is $i = 8$ with $a_8 = 4$ and $b_8 = 1$; moreover $\omega_8 = g_4 = 104$.

Since $\omega_8 + g_j \in 3M$ for each $g_j = 12, 19, 29, 104$, then $t\overline{\omega_8} \in (0 : G(m) M)$.

Anyway $G(m)$ is not Buchsbaum; indeed $t\overline{\omega_8 + g_1} \notin (0 : G(m) M)$ (as $\omega_8 + g_1 = 116 \in 4M \setminus 5M$ and $116 + g_1 = 128 \in 5M$), but $t\overline{\omega_8 + g_1} \in (0 : G(m) M^8)$ (as $116 + (8M \setminus 9M) \subseteq 13M$).
Our next goal is to relate the Buchsbaumness of $G(m)$ to the length $\lambda = \lambda(H^0_M)$ of the $G(m)$-module $H^0_M = (0 :_{G(m)} M^r)$.

**Lemma.** $\lambda = 1$ if and only if $(0 :_{G(m)} M^r) = G(m)x$, with $x \in (0 :_{G(m)} M)$.

**Proposition.** If $\lambda \leq 1$, then $G(m)$ is Buchsbaum.

**PROOF.** If $\lambda = 0$, then $(0 :_{G(m)} M^r) = (0)$, that is $G(m)$ is C-M. If $\lambda = 1$, then, by last lemma, $(0 :_{G(m)} M^r) = G(m)x$, with $x \in (0 :_{G(m)} M)$. This implies $(0 :_{G(m)} M^r) \subseteq (0 :_{G(m)} M)$. 
Remark. The converse of last proposition does not hold in general.

Let $S = \langle 17, 18, 21, 28, 29, 32, 33 \rangle$.

The only indexes $i$ such that $a_i > b_i$ are $i = 7, 10$ and in both cases we have that $a_i = 3 > 2 = b_i$.

Since $\omega_7 = 58, \omega_{10} = 61 \in 2M$ and $\omega_7 + g_j, \omega_{10} + g_j \in 4M$ for each $g_j = 17, 18, 21, 28, 29, 32, 33$, then $\frac{t^{\omega_7}}{t^{\omega_8}}, \frac{t^{\omega_{10}}}{t^{\omega_{10}}} \in (0 : G(m) \mathcal{M})$ and $G(m)$ is Buchsbaum.

Moreover $(0 : G(m) \mathcal{M}^r) = \langle \overline{t^{58}}, \overline{t^{61}} \rangle_{G(m)/\mathcal{M}}$, hence

$$(0) \subsetneq G(m)\overline{t^{58}} \subsetneq (0 : G(m) \mathcal{M}^r).$$
Our next aim is to relate $\lambda$ with the $l_i$’s when $G(m)$ is Buchsbaum.

**Proposition.** If $G(m)$ is Buchsbaum, then $\lambda = \sum_{i \in I}(l_i + 1)$ with $I = \{i \mid a_i > b_i\}$.

**Corollary.** If $G(m)$ is Buchsbaum, then $\lambda \leq \sum_{i \in I}(a_i - b_i)$ with $I = \{i \mid a_i > b_i\}$. Moreover, if $a_i = b_i + 1$ for every $i \in I$, then $\lambda = |I|$.
Let $s \in S$ and define $\text{ord}(s) := h$ if $s \in hM \setminus (h + 1)M$. We note that $\text{ord}(\omega_i) = b_i$.

We now introduce a partial ordering; given $u, u' \in S$, we say that $u \leq_M u'$ if $u + s = u'$ (hence $u \leq_S u'$) and $\text{ord}(u) + \text{ord}(s) = \text{ord}(u')$ for some $s \in S$. The set of maximal elements of $\text{Ap}_{g_1}(S)$ with this partial ordering is denoted with $\text{maxAp}_M(S)$.

**Remark.** $\text{maxAp}(S) \subseteq \text{maxAp}_M(S)$ and the inclusion can be strict.

**Example.** Let $S = \langle 8, 9, 15 \rangle$. Then $\text{maxAp}(S) = \{45\} \subset \{30, 45\} = \text{maxAp}_M(S)$. Note that $\text{ord}(45) = 5 > 3 = \text{ord}(30) + \text{ord}(15)$.

**Proposition.** Let $G(m)$ be Buchsbaum. Then $a_i > b_i$ implies $\omega_i \in \text{maxAp}_M(S)$. 
We recall, for each $i = 0, 1, \ldots, g_1 - 1$:

- $a_i$ is the only integer such that $\omega_i' + a_ig_1 = \omega_i$;
- $b_i = \max\{l \mid \omega_i \in lM\}$.

**Theorem.** (Barucci-Fröberg)

$G(m)$ is C-M $\iff a_i = b_i$ for each $i = 0, 1, \ldots, g_1 - 1$.

**Algorithm.**

1) compute $hM$ for $h = 1, \ldots, r + 1$,

2) find $A_{p_{g_1}}(S)$ and $A_{p_{g_1}}(S')$,

3) determine $a_i$ and $b_i$ for $i = 1, \ldots, g_1 - 1$

4) compare $a_i$ and $b_i$. If there exists $i$ such that $a_i > b_i$ then $G(m)$ is not C-M. If not, it is C-M.
Proposition. (D-M-S) \( G(m) \) is C-M if and only if \( a_i = b_i \), for those \( i \) such that \( \omega_i \in \max Ap_M(S) \).

Example. Let \( S = \langle 10, 13, 14 \rangle \). Then \( G(m) \) is C-M as \( \max Ap_M(S) = \{\omega_5 = 55, \omega_9 = 39\} \) and \( a_5 = b_5 = 4 \) and \( a_9 = b_9 = 3 \).

Remark. Last proposition does not hold in general if we only consider the elements in \( \max Ap(S) \).

Let \( S = \langle 7, 8, 9, 19 \rangle \), then \( \max Ap(S) = \{\omega_3 = 17, \omega_6 = 27\} \) and \( a_3 = b_3 = 2 \) and \( a_6 = b_6 = 3 \).

Anyway \( G(m) \) is not C-M as \( \omega_5 = 19 \) and \( a_5 = 2 > 1 = b_5 \).
13. The 3-generated case

We apply and deepen our results, when the semigroup $S$ is 3-generated. As a by-product, we give a positive answer to two conjectures raised by Sapko in 2001. These two conjectures have also been proved by Shen (2008) using completely different methods.
Remark. If an element $\omega_i$ has more than one representation as a combination of $g_2$ and $g_3$, then the representation $hg_2 + kg_3$, where $h$ is maximum, has the property that $h + k = b_i$ (this is not true if $S$ has more than 3 generators).

Theorem. (D-M-S) Assume $G(m)$ Buchsbaum. If $\omega_i = hg_2 + kg_3 \in Ap_{g_1}(S)$ such that $a_i > b_i$, then $h = 0$.

(In particular, there is at most one element $\omega_i \in Ap_{g_1}(S)$ such that $a_i > b_i$)
As a consequence we obtain a positive answer to two conjectures raised by Sapko in 2001.

**Corollary. (D-M-S)** The following condition are equivalent:

(i) \( G(m) \) Buchsbaum not C-M;

(ii) \((0 : M^r) = G(m)(tkg_3)\) for some \( k \geq 1 \) with \( tkg_3 \in (0 : G(m) M) \);

(iii) \( \lambda = 1 \)

**Corollary.** In (ii),

\[
k = \min\{j | g_2 \text{ divides } (j + 1)g_3 \text{ or } (j + 1)g_3 - g_1 \in S\}
\]
By last theorem we have that if, in the 3 generators case, $R$ is Gorenstein, then $G(m)$ is C-M if and only if it is Buchsbaum. This fact is also proved by Shen using different methods.

**Corollary.** (D-M-S) If $S$ is symmetric (that is $R$ Gorenstein), then

$$G(m) \text{ Buchsbaum} \iff G(m) \text{ C-M}$$
Remark. In the \( n \)-generated case the corollary is not true. Indeed \( S = \langle 8, 9, 12, 13, 19 \rangle \) is symmetric and the only index \( i \) for which \( a_i > b_i \) is \( i = 3 \) (in particular \( G(\mathfrak{m}) \) is not C-M); more precisely we have \( a_3 = 2, b_3 = 1 \) and \( \omega_3 = 19 \). Since \( t^{19} \in (0 : G(\mathfrak{m}) \mathcal{M}) \), then \( G(\mathfrak{m}) \) is Buchsbaum.

A numerical semigroup \( S \) is called \( M \)-pure if every element in \( \text{maxAp}_M(S) \) has the same order.

Proposition. (D-M-S) Let \( S \) be a \((n\text{-generated})\ \ M \)-pure symmetric numerical semigroup, then

\[
G(\mathfrak{m}) \ \text{Buchsbaum} \iff G(\mathfrak{m}) \ \text{C-M}
\]
As an immediate corollary of last proposition we can improve the following result of Bryant:

\[ G(\mathfrak{m}) \text{ is Gorenstein } \iff S \text{ is symmetric, } M\text{-pure and } G(\mathfrak{m}) \text{ is C-M.} \]

**Corollary.**

\[ G(\mathfrak{m}) \text{ is Gorenstein } \iff S \text{ is symmetric, } M\text{-pure and } G(\mathfrak{m}) \text{ is Buchsbaum.} \]