

# New Identities for Degrees of Syzygies in Numerical Semigroups

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**Syzygy** (from Greek,  $\sigma\upsilon\zeta\upsilon\gamma\iota\alpha$ ) - *yoke, pair, copulation, conjunction*, Oxford English Dictionary

*The notion of syzygy was introduced in algebra by J. Sylvester in Phyl. Trans., 143, 1853*

## Numerical Semigroups and Syzygies - a Standard Approach

Numerical semigroup  $S(\mathbf{d}^m)$

$$S(\mathbf{d}^m) = \left\{ s \in \mathbb{N} \cup \{0\} \mid s = \sum_{i=1}^m x_i d_i, x_i \in \mathbb{N} \cup \{0\} \right\}$$

is generated by a minimal set of integers

$$\mathbf{d}^m = \{d_1, \dots, d_m\}, \quad \gcd(d_1, \dots, d_m) = 1, \quad d_1 \geq m$$

with finite complement in  $\mathbb{N}$ ,  $\#\{\mathbb{N} \setminus S(\mathbf{d}^m)\} < \infty$

Let  $R = \mathbf{k}[X_1, \dots, X_m]$  be a polynomial ring over a field  $\mathbf{k}$  of characteristic 0 and

$$\psi_0 : R \longmapsto \mathbf{k}[z^{d_1}, \dots, z^{d_m}]$$

be the projection induced by  $X_i = z^{d_i}$ .

A semigroup ring  $k[S_m] = k[z^{d_1}, \dots, z^{d_m}]$  is a graded subring of  $R$  and has a presentation

$$k[S_m] = R/M_1, \quad M_1 = \ker(\psi_0)$$

A graded  $R$ -module  $M_1$  is minimally generated by a finite number  $\beta_1$  of binomial generators  $P_k$ ,  $1 \leq k \leq \beta_1$ ,

$$P_k(X_1, \dots, X_m) = X_1^{\varepsilon_1^k} \dots X_m^{\varepsilon_m^k} - X_1^{\varkappa_1^k} \dots X_m^{\varkappa_m^k}$$

$\beta_1 < m$  and  $\varepsilon_i^k, \varkappa_j^k \in \mathbb{N} \cup \{0\}$ , such that

$$\psi_0(P_k) = z^{\sum_i^m \varepsilon_i^k d_i} - z^{\sum_j^m \varkappa_j^k d_j} = 0$$

Associated arithmetic relations

$$\sum_i^m \varepsilon_i^k d_i = \sum_j^m \varkappa_j^k d_j, \quad i \neq j, \quad C_{1,k} = \sum_i^m \varepsilon_i^k d_i$$

are called **syzygies** of the 1st kind with degrees  $C_{1,k}$ .  $M_1$  is called the 1st syzygy module of  $k[S_m]$ . A number  $\beta_1 = \text{rank } M_1$  is called the 1st Betti number and stands for minimal number of generators of  $M_1$ .

For nonnegative integer  $r$ , the  $r + 1$ -th syzygy module  $M_{r+1}$  of  $k[S_m]$  is inductively defined as the 1st syzygy module of the  $r$ -th syzygy module  $M_r$ . The corresponding numbers  $\beta_r = \text{rank } M_r$  are called the  $r$ -th Betti numbers.

The  $r + 1$ -th syzygy degrees are coming by arithmetic relations between the  $r$ -th syzygy degrees  $C_{r,i}$  and  $d_j$

$$\sum_i^{\beta_r} \epsilon_i^k C_{r,i} + \sum_j^m \varepsilon_j^k d_j = \sum_j^{\beta_r} \nu_j^k C_{r,j} + \sum_j^m \varkappa_j^k d_j, \quad i \neq j$$

$$C_{r+1,k} = \sum_i^{\beta_r} \epsilon_i^k C_{r,i} + \sum_j^m \varepsilon_j^k d_j \quad \text{and } 1 \leq k \leq \beta_{r+1}.$$

The Hilbert syzygy theorem states that for any finitely generated graded  $R$ -module  $k[S_m]$  the  $m$ -th syzygy module  $M_m$  is free, i.e.  $k[S_m]$  admits a minimal free resolution

$$0 \rightarrow F_{m-1} \xrightarrow{\psi_{m-1}} \dots \xrightarrow{\psi_2} F_1 \xrightarrow{\psi_1} R \xrightarrow{\psi_0} k[S_m] \rightarrow 0$$

$$Im(\psi_r) = M_r = Ker(\psi_{r-1})$$

$$\beta_0(\mathbf{d}^m) - \beta_1(\mathbf{d}^m) + \dots - (-1)^m \beta_{m-1}(\mathbf{d}^m) = 0$$

## Hilbert series $H(\mathbf{d}^m; z)$

$$H(\mathbf{d}^m; z) = Q(\mathbf{d}^m; z) \cdot \prod_{i=1}^m (1 - z^{d_i})^{-1}, \quad (\star)$$

$$Q(\mathbf{d}^m; z) = 1 - Q_1(\mathbf{d}^m; z) + \dots - (-1)^m Q_{m-1}(\mathbf{d}^m; z)$$

$$Q_i(\mathbf{d}^m; z) = \sum_{j=1}^{\beta_i(\mathbf{d}^m)} z^{C_{i,j}}, \quad 1 \leq i \leq m - 1$$

All necessary cancellations of terms  $z^{C_{i,j}}$  in  $Q(\mathbf{d}^m; z)$  are already performed

## Properties of degrees of syzygies $C_{i,j}$

$$\deg Q_i(\mathbf{d}^m; z) < \deg Q_{i+1}(\mathbf{d}^m; z)$$

$$C_{i,j+1} \geq C_{i,j}, \quad C_{i+1,\beta_{i+1}} > C_{i,\beta_i}, \quad C_{i+1,1} > C_{i,1}$$

$$C_{i,j} \neq C_{i+2k-1,r}, \quad 1 \leq r \leq \beta_{i+2k-1}, \quad 1 \leq k \leq \left\lfloor \frac{m-i}{2} \right\rfloor$$

### Example # A

$$Q(\{3, 5, 7\}; z) = 1 - z^{10} - z^{12} - z^{14} + z^{17} + z^{19}$$

## Remarks

- ' One might wonder whether there is an algorithm for Hilbert functions and series which executes in polynomial time in the size of the input. The following proposition shows that such an algorithm cannot exist (unless  $P = NP$ ) '. Bayer and M. Stillman, 1992
- The values  $C_{j,i}$  are usually obtained by computer algebra systems COCOA 4.6, Macaulay2 0.9.93, SINGULAR 3.0.3 etc.
- To the best of our knowledge, even the relations between degrees of syzygies in the Hilbert series of generic numerical semigroups  $\mathbf{S}(\mathbf{d}^m)$ ,  $m \geq 3$ , are unknown except those of special generating sets  $\mathbf{d}^m$ .

# Linear Diophantine Equation (DE)

$$\sum_{i=1}^m x_i d_i = s, \quad x_i \in \mathbb{N} \cup \{0\}, \quad d_i \in \mathbb{N}, \quad d_i \leq s$$

Non-equivalent solutions

$$2x_1 + 3x_2 = 6, \quad \text{Solutions} : \{x_1 = 0, x_2 = 2; x_1 = 3, x_2 = 0\}$$

$$W(6, \{2, 3\}) = 2$$

1. How many non-equivalent solutions has DE ?

*Theory of restricted partitions*

*Sylvester's waves and generating function*

$$M(\mathbf{d}^m; z) = \sum_s W(s, \mathbf{d}^m) z^s \quad (\blacktriangle)$$

where  $W(s, \mathbf{d}^m)$  is quasipolynomial in  $s$ .

2. Does DE have at least one solution ?

*Theory of commutative semigroup rings*

*Numerical semigroup and Hilbert series*

$$H(\mathbf{d}^m; z) = \sum_s z^s, \quad s \in S(\mathbf{d}^m) \quad (\blacklozenge)$$

where  $H(\mathbf{d}^m; z)$  is rational function in  $z$ .

## Aim of the work :

*Bring together two different approaches, combinatorial (restricted partitions) and algebraic (commutative semigroup rings), and show that such merging is useful to produce new results -*

polynomial and quasipolynomial identities for degrees of syzygies in numerical semigroups

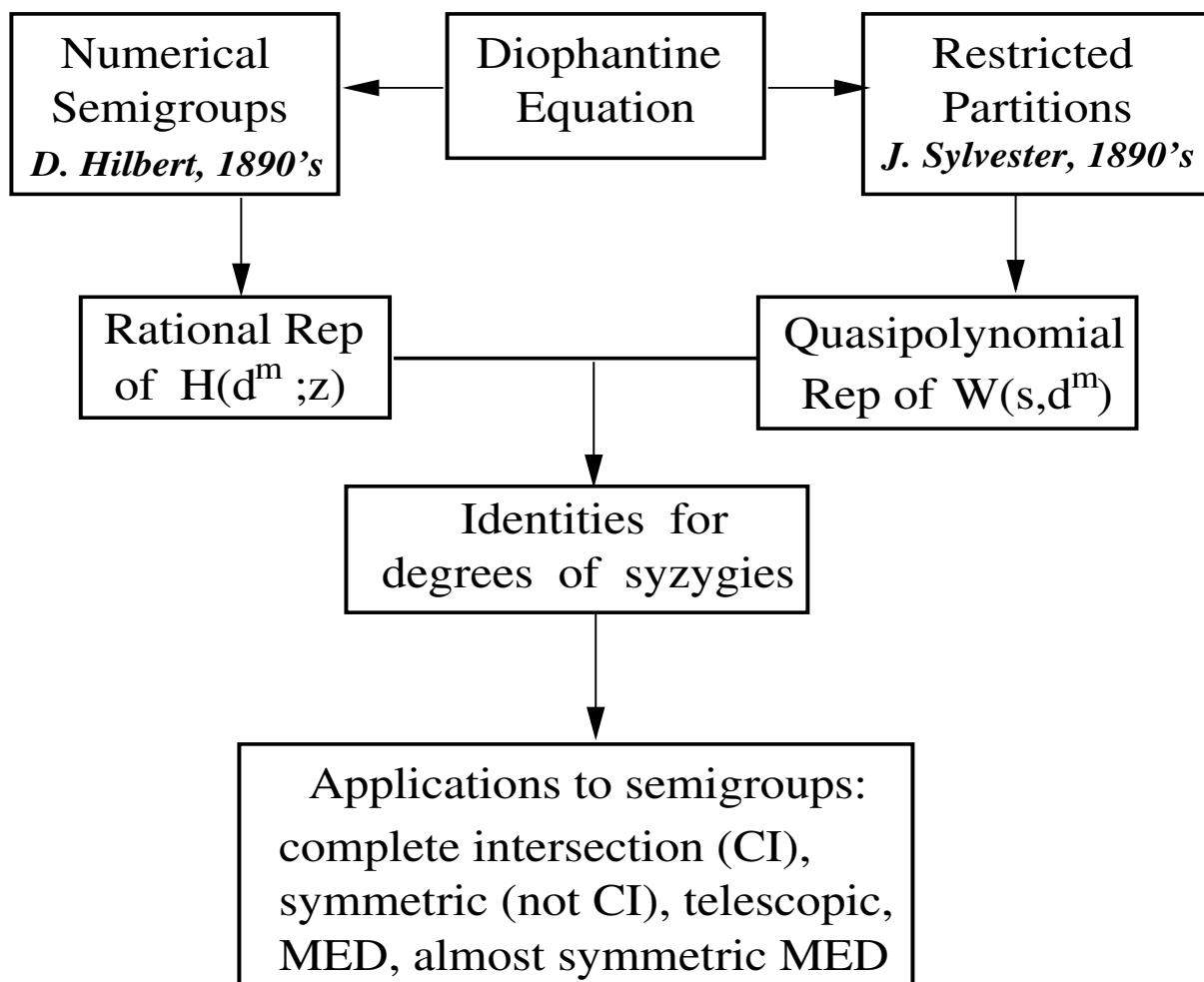


Figure 1:

# Main Results

- Theorem # 1

Let the numerical semigroup  $S(\mathbf{d}^m)$  be given with its Hilbert series  $H(\mathbf{d}^m; z)$  in accordance with (★).

Then the following polynomial identities hold,

$$\sum_{j=1}^{\beta_1(\mathbf{d}^m)} C_{j,1}^0 - \sum_{j=1}^{\beta_2(\mathbf{d}^m)} C_{j,2}^0 + \dots + (-1)^m \sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} C_{j,m-1}^0 = 1,$$

$$\sum_{j=1}^{\beta_1(\mathbf{d}^m)} C_{j,1}^r - \sum_{j=1}^{\beta_2(\mathbf{d}^m)} C_{j,2}^r + \dots + (-1)^m \sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} C_{j,m-1}^r = 0,$$

where  $1 \leq r \leq m - 2$ . However, in the case  $r = m - 1$  another polynomial identity holds

$$\sum_{j=1}^{\beta_1(\mathbf{d}^m)} C_{j,1}^{m-1} - \sum_{j=1}^{\beta_2(\mathbf{d}^m)} C_{j,2}^{m-1} + \dots + (-1)^m \sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} C_{j,m-1}^{m-1} = L_m$$

$$L_m = (-1)^m (m - 1)! \Pi_m, \quad \Pi_m = \prod_{k=1}^m d_k$$



Denote a new set  $\Xi_q(\mathbf{d}^m)$ ,  $q \geq 1$

$$\Xi_q(\mathbf{d}^m) := \{d_i \mid q \mid d_i\}, \quad \omega_q = \#\Xi_q, \quad \pi_q = \prod_{d_i \in \Xi_q} d_i$$

such that  $\Xi_1(\mathbf{d}^m) = \mathbf{d}^m$ ,  $\omega_1 = m$  and  $\pi_1 = \Pi_m$ .

## • Theorem # 2

Let the numerical semigroup  $S(\mathbf{d}^m)$  be given with its Hilbert series  $H(\mathbf{d}^m; z)$  in accordance with (★).

Then for every  $1 < q \leq \max\{d_1, \dots, d_m\}$ , and  $\gcd(n, q) = 1$ ,  $1 \leq n < q/2$ , the following quasipolynomial identities hold,

$$\begin{aligned} \sum_{j=1}^{\beta_1(\mathbf{d}^m)} C_{j,1}^r \exp\left(i \frac{2\pi n}{q} C_{j,1}\right) - \sum_{j=1}^{\beta_2(\mathbf{d}^m)} C_{j,2}^r \exp\left(i \frac{2\pi n}{q} C_{j,2}\right) + \dots + \\ (-1)^m \sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} C_{j,m-1}^r \exp\left(i \frac{2\pi n}{q} C_{j,m-1}\right) = 0, \end{aligned}$$

where  $r = 1, \dots, \omega_q - 1$ . However, in the case  $r = 0$  another trigonometric identity holds,

$$\begin{aligned} \sum_{j=1}^{\beta_1(\mathbf{d}^m)} \exp\left(i \frac{2\pi n}{q} C_{j,1}\right) - \sum_{j=1}^{\beta_2(\mathbf{d}^m)} \exp\left(i \frac{2\pi n}{q} C_{j,2}\right) + \dots + \\ (-1)^m \sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} \exp\left(i \frac{2\pi n}{q} C_{j,m-1}\right) = 1. \end{aligned}$$

# Restricted Partitions

The restricted partition function  $W(s, \mathbf{d}^m)$  is a number of partitions of  $s$  into positive integers  $\{d_1, \dots, d_m\}$ , each not greater than  $s$

$$\sum_{s=0}^{\infty} W(s, \mathbf{d}^m) z^s = \prod_{i=1}^m \frac{1}{1 - z^{d_i}} \quad (\blacksquare)$$

- Parity Properties,  $\sigma_1 = d_1 + \dots + d_m$

$$W\left(s - \frac{\sigma_1}{2}, \mathbf{d}^{2m}\right) = -W\left(-s - \frac{\sigma_1}{2}, \mathbf{d}^{2m}\right)$$

$$W\left(s - \frac{\sigma_1}{2}, \mathbf{d}^{2m+1}\right) = W\left(-s - \frac{\sigma_1}{2}, \mathbf{d}^{2m+1}\right)$$

- Frobenius number  $\mathcal{F}(\mathbf{d}^m)$  - a largest zero of  $W(s, \mathbf{d}^m)$

$$W(\mathcal{F}(\mathbf{d}^m), \mathbf{d}^m) = 0.$$

- $W(s, \mathbf{d}^m)$  is a quasipolynomial

$$W(s, \mathbf{d}^m) = \sum_{j=1}^m K_j(s, \mathbf{d}^m) s^{m-j}$$

where  $K_j(s, \mathbf{d}^m)$  is a periodic function

$$K_j(s, \mathbf{d}^m) = K_j(s + \tau_j, \mathbf{d}^m), \quad \tau_j \mid lcm(\mathbf{d}^m).$$

Due to Schur, 1935,  $K_1(s, \mathbf{d}^m) = (m-1)!^{-1} \Pi_m^{-1}$

# Example # B (*LF & Rubinstein, 2002*)

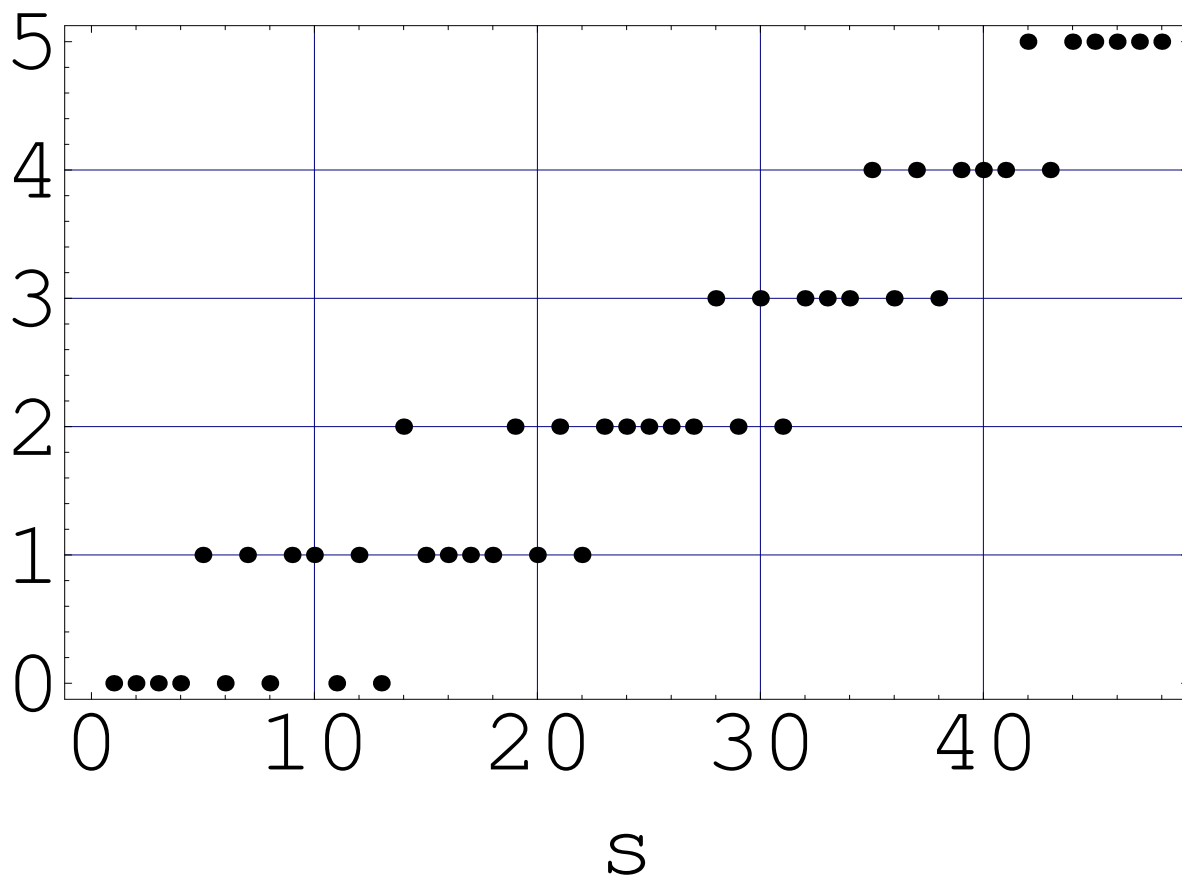


Figure 2:  $W(s, \{5, 7, 9\})$ ,  $\mathcal{F}(5, 7, 9) = 13$ .

### Example # C (Komatsu, 2003)

$$\begin{aligned}
 W(s, \{3, 5, 7\}) = & \frac{s^2}{210} + \frac{s}{14} + \frac{74}{315} + \frac{2}{9} \cos \frac{2\pi s}{3} + \\
 & \frac{8}{25} \left[ \left( \sin \frac{\pi}{5} \right)^2 \cos \frac{2\pi s}{5} + \left( \sin \frac{2\pi}{5} \right)^2 \cos \frac{4\pi s}{5} \right] - \\
 & \frac{2}{7\sqrt{7}} \left[ \sin \frac{6\pi}{7} \cos \frac{6\pi s}{7} + 2 \left( \sin \frac{\pi}{7} \right)^2 \sin \frac{2\pi s}{7} + 2 \left( \sin \frac{2\pi}{7} \right)^2 \sin \frac{4\pi s}{7} \right] \\
 & + \frac{2}{7\sqrt{7}} \left[ \sin \frac{2\pi}{7} \cos \frac{2\pi s}{7} + \sin \frac{4\pi}{7} \cos \frac{4\pi s}{7} + 2 \left( \sin \frac{3\pi}{7} \right)^2 \sin \frac{6\pi s}{7} \right]
 \end{aligned}$$

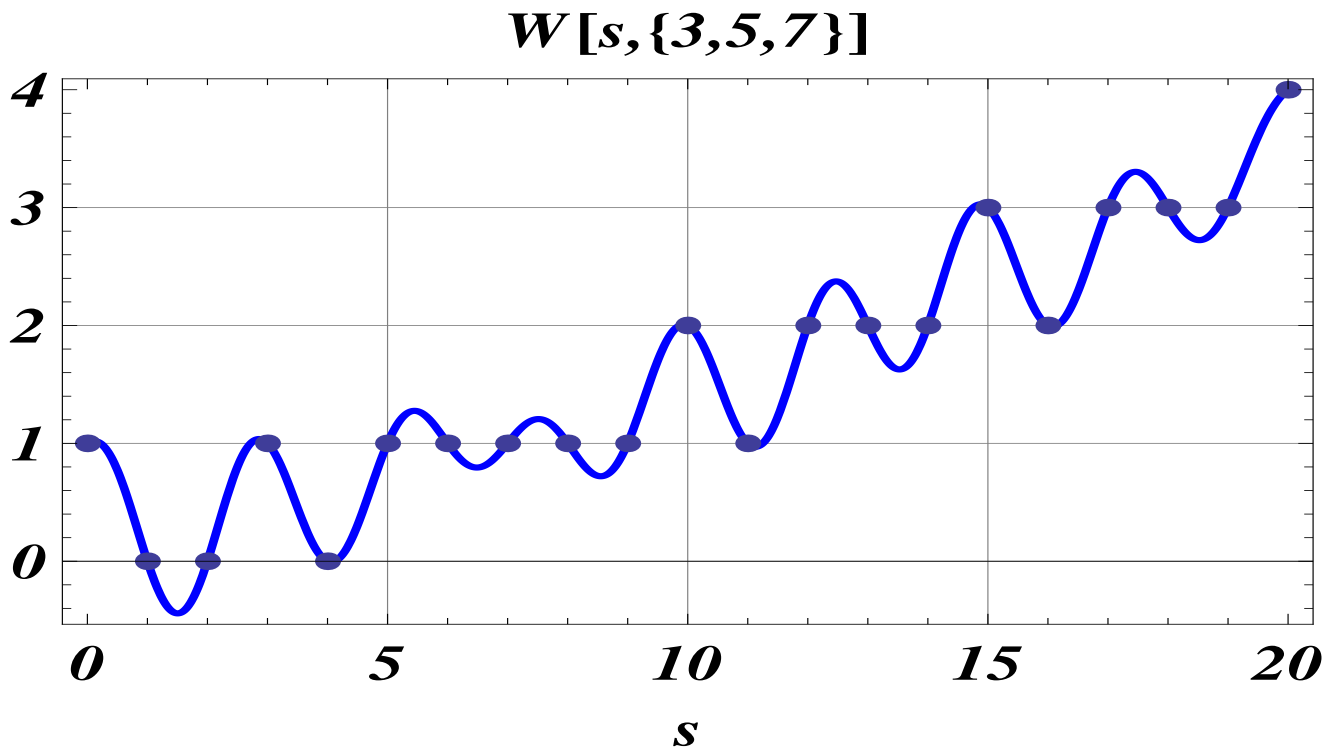


Figure 3:  $W(s, \{3, 5, 7\})$ ,  $\mathcal{F}(3, 5, 7) = 4$ .

# Sylvester Waves

- Sylvester theorem, 1892

$$W(s, \mathbf{d}^m) = \sum_{q=1, q|d_i}^{\max\{d_1, \dots, d_m\}} W_q(s, \mathbf{d}^m), \quad (\blacktriangledown)$$

where summation runs over all distinct factors of  $m$  generators. The wave  $W_q(s, \mathbf{d}^m)$  is a residue of a function  $F_q(s, z)$

$$F_q(s, z) = \sum_{\substack{1 \leq n < q \\ \gcd(n, q) = 1}} \frac{\xi_q^{-s n} e^{s z}}{\prod_{k=1}^m \left(1 - \xi_q^{d_k n} e^{-d_k z}\right)},$$

where  $\xi_q = \exp(2\pi i/q)$ .

## Umbral Representations of $W_q(s, \mathbf{d}^m)$

by Rubinstein & LF, 2006

The 1st Sylvester wave

$$W_1(s, \mathbf{d}^m) = \frac{1}{(m-1)! \pi_1} \left( s + \sigma_1 + \sum_{i=1}^m B d_i \right)^{m-1}, \quad (i)$$

where the powers  $(B d_i)^r$  are converted into  $d_i^r B_r$ , and  $B_r$  stands for Bernoulli numbers.

## The 2nd Sylvester wave

$$W_2(s, \mathbf{d}^m) = A_2 \left( s + \sigma_1 + \sum_{i=1}^{\omega_2} B d_i + \sum_{i=\omega_2+1}^m E(0) d_i \right)^{\omega_2-1}, \quad (ii)$$

$$A_2 = \frac{2^{\omega_2-m}}{(\omega_2-1)! \pi_2} \cos \pi s,$$

where the powers  $(E(0) d_i)^r$  are converted into  $d_i^r E_r(0)$ , and  $E_r(0)$  stands for Euler polynomial  $E_r(x)$  at  $x = 0$ .

## The $q$ -th Sylvester wave

$$W_q(s, \mathbf{d}^m) = \frac{1}{(\omega_q-1)! \pi_q} \sum_{\substack{1 \leq n < q \\ \gcd(n,q)=1}} \mathcal{W}_{q,n}(s, \mathbf{d}^m), \quad (iii)$$

$$\mathcal{W}_{q,n}(s, \mathbf{d}^m) = \frac{\xi_q^{-s n}}{\prod_{i=\omega_q+1}^m \left(1 - \xi_q^{d_i n}\right)} G_{q,n},$$

$$G_{q,n} = \left( s + \sigma_1 + \sum_{i=1}^{\omega_q} B d_i + \sum_{i=\omega_q+1}^m H(\xi_q^{d_i n}) d_i \right)^{\omega_q-1},$$

here  $(H(\xi_q^{d_i n}))^r = H_r(\xi_q^{d_i n})$  generalize  $E_r(0) = H_r(-1)$ .

They were introduced by Frobenius and Carlitz as the values of the function  $H_n(x)$  at  $x = \xi_q^{d_i n}$

$$\frac{1-x}{e^t-x} = 1 + \sum_{n=1}^{\infty} H_n(x) \frac{t^n}{n!}, \quad H_n\left(\frac{1}{x}\right) = (-1)^n x H_n(x).$$

## Combination of Two Generating Functions

$$\sum_{s \in S(\mathbf{d}^m)} z^s = Q(\mathbf{d}^m; z) \sum_{s \in S(\mathbf{d}^m)} W(\mathbf{d}^m; s) z^s.$$

- Master Equation

$$W(\mathbf{d}^m; s) - \sum_{j=1}^{\beta_1(\mathbf{d}^m)} W(\mathbf{d}^m; s - C_{j,1}) + \dots + (-1)^{m-1} \sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} W(\mathbf{d}^m; s - C_{j,m-1}) = 1. \quad (\clubsuit)$$

- Sketch of Proof

Substitute (i), (ii), (iii) into (▼) and subsequently insert into (♣). This give rise to the quasipolynomial Eqs in  $s^k \exp\left(\frac{2\pi n}{q}s\right)$ ,  $\gcd(n, q) = 1$ ,  $1 \leq n < q/2$ , which can be decomposed for different  $k$  and  $n, q$ .

# APPLICATIONS

## COMPLETE INTERSECTION (CI)

If  $\beta_1(\mathbf{d}^m) = m - 1$  then the semigroup is called complete intersection

$$Q(\mathbf{d}^m; z) = \prod_{j=1}^{m-1} (1 - z^{e_j}) \quad (\spadesuit)$$

- Corollary # 1

Let the complete intersection semigroup be given with its Hilbert series  $H(\mathbf{d}^m; z)$  in accordance with  $(\spadesuit)$ . Then the following identity holds,

$$\prod_{i=1}^{m-1} e_i = \prod_{i=1}^m d_i$$

*This is only one non trivial identity of  $m$  polynomials from Theorem # 1.*

- Delorme, 1976

$$Q(10, 14, 15, 21; z) = (1 - z^{30})(1 - z^{35})(1 - z^{42})$$

$$10 \cdot 14 \cdot 15 \cdot 21 = 30 \cdot 35 \cdot 42$$

*By Corollary # 1 there follows another Corollary # 2*



- **Corollary # 2**

Let the complete intersection semigroup be given with its Hilbert series  $H(\mathbf{d}^m; z)$  in accordance with (). Then the Frobenius number  $\mathcal{F}(\mathbf{d}^m)$  has the lower bound

$$\mathcal{F}(\mathbf{d}^m) \geq (m-1) \sqrt[m-1]{d_1 \cdot \dots \cdot d_m} - \sum_{i=1}^m d_i, \quad (\square)$$

*For  $m = 3$  it was already proven (LF, 2004)*

For  $m \geq 4$

$$\mathcal{F}(\{8, 9, 10, 12\}) = 23 > 22.559$$

$$\mathcal{F}(\{10, 14, 15, 21\}) = 47 > 45.991$$

$$\mathcal{F}(\{36, 54, 125, 150, 225\}) = 793 > 613.733$$

The 13-D tuple  $\mathbf{C}$  is taken from (Bermejo et al, 2007)

$$\mathbf{C} = \{304\ 920, 381\ 150, 457\ 380, 571\ 725, 97\ 911, 223\ 146, 239\ 085, \\ 159\ 390, 334\ 719, 224\ 112, 238\ 119, 252\ 126, 334\ 949\}$$

$$\mathcal{F}(\mathbf{C}) = 6\ 229\ 597 > 5\ 322\ 209.193$$

The lower bound in ( $\square$ ) is much stronger than the similar bound for  $\mathcal{F}(\mathbf{d}^m)$  in the case of nonsymmetric numerical semigroup (Killingbergtrø, 2000)

$$\mathcal{F}(\mathbf{d}^m) \geq \sqrt[m-1]{(m-1)!} \sqrt[m-1]{d_1 \cdot \dots \cdot d_m} - \sum_{i=1}^m d_i.$$

- **Telescopic Semigroups**

This is a most simple case of CI

For given numerical semigroup  $S(\mathbf{d}^m)$  with generators  $\{d_1, \dots, d_m\}$ ,  $\gcd(d_1, \dots, d_m) = 1$ , (not necessarily in increasing order), let us denote

$g_k = \gcd(d_1, \dots, d_k) > 1$  and  $S_k = S\left(\left\{\frac{d_1}{g_k}, \frac{d_2}{g_k}, \dots, \frac{d_k}{g_k}\right\}\right)$  for  $1 \leq k < m$ , and  $g_1 = d_1$ ,  $g_m = 1$ . Then  $S(\mathbf{d}^m)$  is said to be telescopic iff  $\frac{d_k}{g_k} \in S_{k-1}$  for all  $2 \leq k \leq m$ .

The degrees of syzygies read

$$e_1 = \text{lcm}(d_1, d_2), \quad e_j = d_{j+1} \frac{g_j}{g_{j+1}}, \quad 1 < j < m$$

They satisfy Theorem # 1 and Corollary # 1

- **Kraft, 1985**

$$Q(8, 9, 10, 12; z) = (1 - z^{18})(1 - z^{20})(1 - z^{24})$$

$$24 = \text{lcm}(8, 12), \quad 20 = 10 \frac{\gcd(8, 12)}{\gcd(8, 12, 10)}, \quad 18 = 9 \gcd(8, 12, 10)$$

$$q = 3, \quad \omega_3 = 2, \quad r = 1, \quad n = 1, \quad \beta_1 = \beta_2 = 3, \quad \beta_3 = 1$$

$$18 + 20 \exp\left(i \frac{40\pi}{3}\right) + 24 - 38 \exp\left(i \frac{76\pi}{3}\right) - 44 \exp\left(i \frac{88\pi}{3}\right) -$$

$$42 + 62 \exp\left(i \frac{124\pi}{3}\right) \equiv 0.$$

# Symmetric Semigroups $S(\mathbf{d}^m)$

For short, denote

$$\deg Q(\mathbf{d}^m; z) = \mathcal{Q}_m, \quad \mathcal{F}(\mathbf{d}^m) = \mathcal{F}_m = \mathcal{Q}_m - \sigma_1$$

A semigroup is called *symmetric* if for any integer  $s$  holds: if  $s \in S(\mathbf{d}^m)$  then  $\mathcal{F}_m - s \notin S(\mathbf{d}^m)$ .

For different parities of edim we have

$$\mathcal{Q}_{2m}^k - \sum_{r=1}^{m-1} (-1)^r \sum_{j=1}^{\beta_r(\mathbf{d}^{2m})} \left\{ C_{j,r}^k - [\mathcal{Q}_{2m} - C_{j,r}]^k \right\} = L_{2m}$$

$$L_{2m} = (2m - 1)! \Pi_{2m} \delta_{k,2m-1}, \quad 1 \leq k \leq 2m - 1$$

$$\mathcal{Q}_{2m+1}^k + \sum_{r=1}^{m-1} (-1)^r \sum_{j=1}^{\beta_r(\mathbf{d}^{2m+1})} \left\{ C_{j,r}^k + [\mathcal{Q}_{2m+1} - C_{j,r}]^k \right\} +$$

$$(-1)^m \sum_{j=1}^{\beta_m(\mathbf{d}^{2m+1})} C_{j,m}^k + L_{2m+1} = 0$$

$$L_{2m+1} = (2m)! \Pi_{2m+1} \delta_{k,2m}, \quad 1 \leq k \leq 2m$$

- **Symmetric Semigroups  $S(\mathbf{d}^4)$  (not CI)**

*In 1975, Bresinsky has shown that symmetric semigroups  $S(\mathbf{d}^4)$ , which are not CI, have always  $\beta_1(\mathbf{d}^4) = 5$ . Denote invariants  $I_k = \sum_{j=1}^5 C_{j,1}^k$  and get*

$$8I_3 - 6I_2I_1 + I_1^3 = 24\Pi_4, \quad \mathcal{Q}_4 = \frac{1}{2} I_1$$

- **Delorme, 1976**

$$Q(5, 6, 7, 8; z) = 1 - z^{12} - z^{13} - z^{14} - z^{15} - z^{16} + z^{19} + z^{20} + z^{21} + z^{22} + z^{23} - z^{35}$$

$$I_1 = 70, \quad I_2 = 990, \quad I_3 = 14140, \quad \Pi_4 = 1680$$

- **LF, 2009**

$$Q(8, 13, 15, 17; z) = 1 - z^{30} - z^{32} - z^{34} - z^{39} - z^{41} + z^{47} + z^{49} + z^{54} + z^{56} + z^{58} - z^{88}$$

$$I_1 = 176, \quad I_2 = 6282, \quad I_3 = 227312, \quad \Pi_4 = 26520$$

$$\begin{aligned} q = 13, \quad \omega_{13} = 1, \quad r = 0, \quad n = 5, \quad \beta_1 = \beta_2 = 5, \quad \beta_3 = 1 \\ \exp\left(5i\frac{60\pi}{13}\right) + \exp\left(5i\frac{64\pi}{13}\right) + \exp\left(5i\frac{68\pi}{13}\right) + \exp\left(5i\frac{78\pi}{13}\right) + \\ \exp\left(5i\frac{82\pi}{13}\right) - \exp\left(5i\frac{94\pi}{13}\right) - \exp\left(5i\frac{98\pi}{13}\right) - \exp\left(5i\frac{108\pi}{13}\right) \\ - \exp\left(5i\frac{112\pi}{13}\right) - \exp\left(5i\frac{116\pi}{13}\right) + \exp\left(5i\frac{176\pi}{13}\right) \equiv 1. \end{aligned}$$

- **Symmetric Semigroups  $S(d^5)$  (not CI)**

For symmetric semigroups  $S(d^m)$  (not CI),  $m \geq 5$ , the problem of admissible values of  $\beta_1$  is still open. Denote  $J_{r,k} = \sum_{j=1}^{\beta_r} C_{j,r}^k$  where  $r = 1, 2$  and  $1 \leq k \leq 4$ , and get

1.  $J_{2,1} = (\beta_1 - 1)Q_5$ ,
2.  $J_{2,1}(2J_{1,1} - J_{2,1}) + (\beta_1 - 1)(J_{2,2} - 2J_{1,2}) = 0$ ,
3.  $J_{2,1}^2(3J_{1,1} - J_{2,1}) - 3(\beta_1 - 1)J_{1,2}J_{2,1} + (\beta_1 - 1)^2 J_{2,3} = 0$ ,
4.  $J_{2,1}^3(4J_{1,1} - J_{2,1}) - 6(\beta_1 - 1)J_{1,2}J_{2,1}^2 + 4(\beta_1 - 1)^2 J_{1,3}J_{2,1} + (\beta_1 - 1)^3 (J_{2,4} - 2J_{1,4} - 24\Pi_5) = 0$ .

- **Bresinsky 1979, Eisenbud's computer program**

$$\begin{aligned}
 Q(19, 23, 29, 31, 37; z) = & 1 - z^{60} - z^{69} - z^{75} - \\
 & z^{77} - z^{81} - z^{85} - z^{87} - z^{93} - z^{95} + z^{98} - z^{99} + \\
 & z^{100} - z^{103} + z^{104} - z^{105} + 2z^{106} + z^{108} + z^{110} - \\
 & z^{111} + z^{112} + z^{114} + z^{116} + 2z^{118} + 2z^{122} + z^{124} + \\
 & z^{126} + z^{128} - z^{129} + z^{130} + z^{132} + 2z^{134} - z^{135} + \\
 & z^{136} - z^{137} + z^{140} - z^{141} + z^{142} - z^{145} - z^{147} - z^{153} - \\
 & z^{155} - z^{159} - z^{163} - z^{165} - z^{171} - z^{180} + z^{240}
 \end{aligned}$$

$$\begin{aligned}
\beta_1 &= 13, & \beta_2 &= 24, & \beta_3 &= 13, & \Pi_5 &= 14535931 \\
J_{1,1} &= 1140, & J_{1,2} &= 102700, & J_{1,3} &= 9477000, \\
& & J_{1,4} &= 893181868, & Q_5 &= 240 \\
J_{2,1} &= 2880, & J_{2,2} &= 349400, & J_{2,3} &= 42840000, \\
& & J_{2,4} &= 5306106080
\end{aligned}$$

Verify the 4th identity:

$$\begin{aligned}
&2880^3(4 \cdot 1140 - 2880) - 6 \cdot 12 \cdot 102700 \cdot 2880^2 + \\
&4 \cdot 12^2 \cdot 9477000 \cdot 2880 + 12^3(5306106080 - \\
&2 \cdot 893181868 - 24 \cdot 14535931) \equiv 0
\end{aligned}$$

# Nonsymmetric Semigroups $S(d^3)$

- Hilbert Series  $H(d^3; z)$ , (Herzog, 1970)

$$Q(d^3; z) = 1 - z^{e_1} - z^{e_2} - z^{e_3} + z^{q_1} + z^{q_2}$$

By Theorem # 1 it follows

$$\begin{aligned} e_1 + e_2 + e_3 &= q_1 + q_2, \\ e_1^2 + e_2^2 + e_3^2 &= q_1^2 + q_2^2 - 2d_1d_2d_3 \end{aligned}$$

that gives (Rosales & García-Sánchez, 2004, LF, 2004)

$$\begin{aligned} q_{1,2} &= \frac{1}{2} \left[ (e_1 + e_2 + e_3) \pm \sqrt{\Delta} \right] \\ \Delta &= e_1^2 + e_2^2 + e_3^2 - 2(e_1e_2 + e_2e_3 + e_3e_1) + 4d_1d_2d_3 \end{aligned}$$

By Theorem # 2 we have a set of non algebraic Eqs in accordance with Curtis' theorem (1990)

$$\xi_q^{ne_1} + \xi_q^{ne_2} + \xi_q^{ne_3} = \xi_q^{nq_1} + \xi_q^{nq_2} + 1,$$

$$\xi_q = \exp\left(\frac{2\pi i}{q}\right), \quad q \mid d_i, \quad \gcd(n, q) = 1.$$

## Pseudosymmetric Semigroups $\mathbf{S}(\mathbf{d}^3)$

Semigroup is pseudosymmetric if  $\mathcal{F}_m$  is even and the only integer such  $s \in \mathbb{N} \setminus \mathbf{S}(\mathbf{d}^m)$  and  $\mathcal{F}_m - s \notin \mathbf{S}(\mathbf{d}^m)$  is  $s = \mathcal{F}_m/2$ .

- Degrees of Syzygies (LF, 2007)

$$e_1 = \frac{1}{2}(\mathcal{Q}_3 + \sigma_1 - 2d_1), \quad q_1 = \frac{1}{2}(\mathcal{Q}_3 + \sigma_1)$$

$$e_2 = \frac{1}{2}(\mathcal{Q}_3 + \sigma_1 - 2d_2), \quad q_2 = \mathcal{Q}_3$$

$$e_3 = \frac{1}{2}(\mathcal{Q}_3 + \sigma_1 - 2d_3)$$

By Theorem # 1 we have

$$\mathcal{Q}_3 = \sqrt{(d_1 + d_2 + d_3)^2 - 4(d_1d_2 + d_2d_3 + d_3d_1) + 4d_1d_2d_3}$$

was get by Rosales & García-Sánchez, 2005, LF, 2007