Some progress towards Wilf's conjecture

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Commutative Monoids

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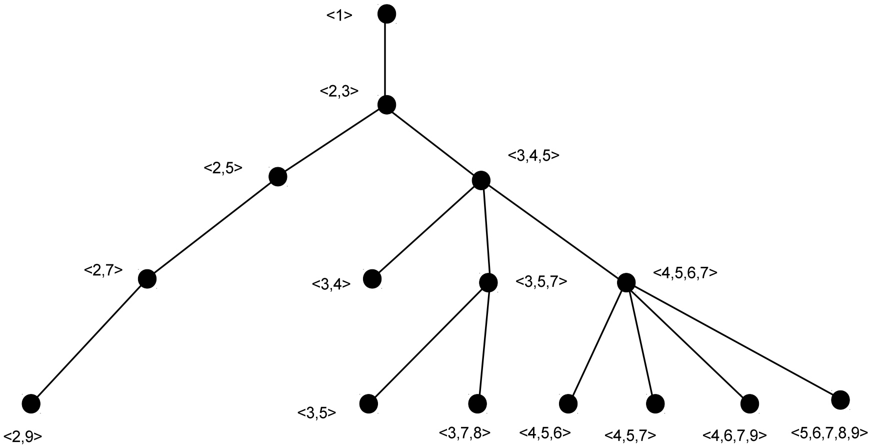

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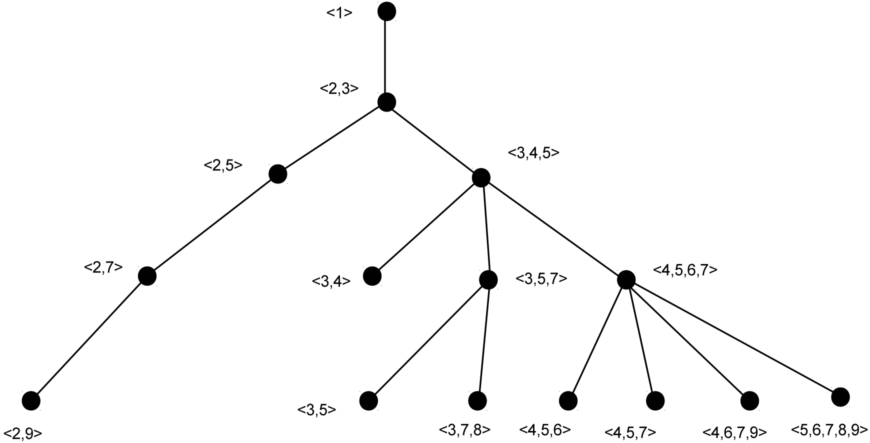

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