Convex polyhedron semigroups

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June 8, 2015

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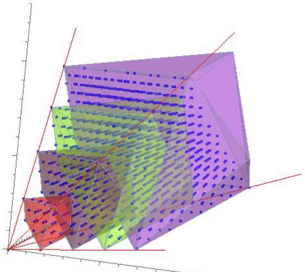
Notation

- $P = (p_1, \ldots, p_n) \in \mathbb{R}^n$ and $k \in \mathbb{R}_{\geq}$, then $kP = (kp_1, \ldots, kp_n)$
- Let $V \subset \mathbb{R}^n_{\geq}$ a finite set. Denote by $\mathbf{F} = \mathcal{H}(V)$ the convex hull of V (the smallest convex containing V).
- $\cup_{i\in\mathbb{N}} i\mathbf{F}$ is a subsemigroup of $\mathbb{R}^n_{\geq}$.
- $\mathcal{F} = (\cup_{i\in\mathbb{N}} i\mathbf{F}) \cap \mathbb{N}^n$ is a semigroup of $\mathbb{N}^n$.

If $\mathcal{F}$ is f.g., then $\mathcal{F}$ is an affine semigroups.
Inspired on proportionally modular semigroups (see [8]).
Similar to the semigroups appering in [3].



We study the case $\mathcal{F} \subset \mathbb{N}^3$.
In this work, we study:

- Conditions for $\mathcal{F}$ being finitely generated
- Algorithms to check properties associated to the semigroup ring of $\mathcal{F}$. Cohen-Macaulay, Buchsbaum properties.
- Families of Gorenstein semigroups.

Bibliography

W. Bruns, B. Ichim, T. Rmer and C. Sger
Algorithms for rational cones and affine monoids.
Available from http://www.math.uos.de/normaliz.

D. Davis
Reverse search algorithm for vertex enumeration/convex hull
Available at http://cgm.cs.mcgill.ca/~avis/C/lrs.html

J.I. García-García, M.A. Moreno-Frías, A. Sánchez-R.-Navarro, A. Vigneron-Tenorio.
Affine convex body semigroups.
Semigroup Forum vol. 87 (2) (2013), 331–350.

J.I. García-García, A. Vigneron-Tenorio.
Computing families of Cohen-Macaulay and Gorenstein rings.
Semigroup Forum vol. 88 (3) (2014), 610-620.

D. R. Grayson, M.E. Stillman.
Macaulay2, a software system for research in algebraic geometry.
Available at http://www.math.uiuc.edu/Macaulay2/

Bibliography

S. Goto, N. Suzuki, K. Watanabe.
On affine semigroup rings.
Japan. J. Math. (N.S.) 2 (1976), no. 1, 1–12.

J.C. Rosales, P.A. García-Sánchez.
On Cohen-Macaulay and Gorenstein simplicial affine semigroups.
Proc. Edinburgh Math. Soc. (2) 41 (1998), no. 3, 517–537.

J. C. Rosales, P. A. García-Sánchez, J. I. García-García, J. M. Urbano-Blanco,
Proportionally modular Diophantine inequalities,
J. Number Theory 103, 281–294 (2003).

P.A. García-Sánchez, J.C. Rosales.
On Buchsbaum simplicial affine semigroups. Pacific J. Math. 202 (2002), no. 2, 329–339.

Notation

Given $\mathcal{A} \subseteq \mathbb{R}^3_{\geq}$:

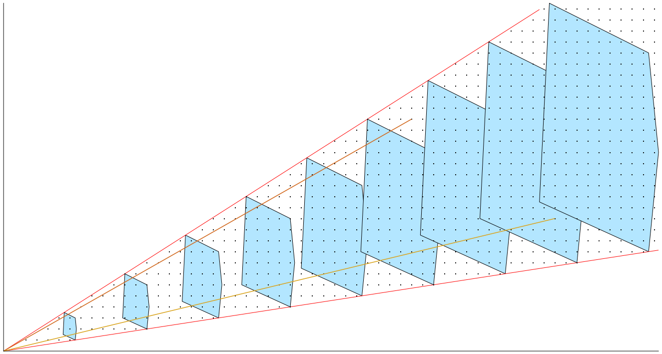
- Define the cone generated by $\mathcal{A} \subseteq \mathbb{R}^3_{\geq}$ as the set

$$L_{\mathbb{Q}_{\geq}}(\mathcal{A}) = \left\{ \sum_{i=1}^{p} q_i a_i | p \in \mathbb{N}, q_i \in \mathbb{Q}_{\geq}, a_i \in \mathcal{A} \right\}.$$

- $\mathcal{C}_{\mathcal{A}} = L_{\mathbb{Q}_{\geq}}(\mathcal{A}) \cap \mathbb{N}^3$.
- Denote by τ_i the extremal rays of $L_{\mathbb{Q}_{\geq}}(\mathcal{A})$.
- We say that a ray τ is rational if $\tau \cap (\mathbb{Q}^3 \setminus \{(0,0,0)\}) \neq \emptyset$.

Main property of $\mathcal{F}$

Does P belong to $\mathcal{F}$? Just consider the ray τ defined by P, the set $\tau \cap \mathbf{F} = \overline{AB}$. To check if $P \in \mathcal{F}$, consider $\{k \in \mathbb{N} | \frac{\mathrm{d}(P)}{\mathrm{d}(B)} \leq k \leq \frac{\mathrm{d}(P)}{\mathrm{d}(A)}\}$ where $\mathrm{d}(*)$ is the distance $\mathrm{d}(*, O)$, if this set is nonempty then $P \in \mathcal{F}$.



Is $\mathcal{F}$ finitely generated?

Theorem

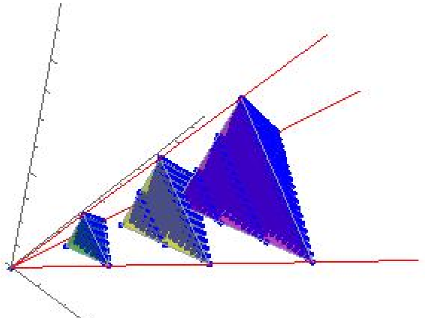
Let $\{\tau_1, \ldots, \tau_r\}$ the extremal rays of $\mathcal{C}_{\mathbf{F}}$.
The convex polyhedron semigroup $\mathcal{F}$ associated to a polyhedron $\mathbf{F}$ is finitely generated if and only if $\mathbf{F} \cap \tau_i$ contains rational points $\forall i = 1, \ldots, r$.

If this condition is satisfied, then we can move the vertices of $\mathbf{F}$ with non-rational coordinates and obtain again $\mathcal{F}$.
If all the vertices have rational coordinates, we can obtain its minimal system of generators.

With only four vertices we can obtain affine semigroups with many elements in its minimal system of generators.
If we use the defining equations of **F**, it is easy to check if an element belong or not to this semigroup.

Example

The minimal s. of gen. of the semigroup obtained from $\{(5,5,5),(12,6,6),(6,12,6),(6,6,12)\}$ has 85 elements.



Checking the Cohen-Macaulay and Buchsbaum properties

Let $(S,+)$ be a finitely generated commutative monoid. If $\mathbf{k}$ is a field, we denote by $\mathbf{k}[S]$ the semigroup ring of S over $\mathbf{k}$. Note that $\mathbf{k}[S]$ is equal to $\bigoplus_{m\in S}\mathbf{k}\chi^m$ endowed with a multiplication which is $\mathbf{k}$-linear and such that $\chi^m\cdot\chi^n=\chi^{m+n}$, m and $n\in S$.

Definition

Let $S\subset\mathbb{N}^n$ is a finitely generated semigroup with $\{n_1,\ldots,n_n\}$ is its minimal generating set.
S is a simplicial semigroup iff $L_{\mathbb{Q}_\geq}(S)$ is generated by n linearly independent generators of S.

$\mathcal{F}\subset\mathbb{N}^3$ (affine convex body semigroup) is simplicial if and only if $\mathcal{C}_\mathcal{F}$ has exactly three rays.

Theorem (Goto, Suzuki, Watanabe 76)

Let $S \subset \mathbb{N}^r$ be the affine simplicial semigroup generated by $\{n_1, \ldots, n_r, n_{r+1}, \ldots, n_{r+m}\}$ with $r = \dim(S)$. The following conditions are equivalent:

1. *S is Cohen-Macaulay.*
2. *For any $a, b \in S$ with $a + n_i = b + n_j$ $(1 \leq i \neq j \leq r)$, $a - n_j = b - n_i \in S$.*

Corollary

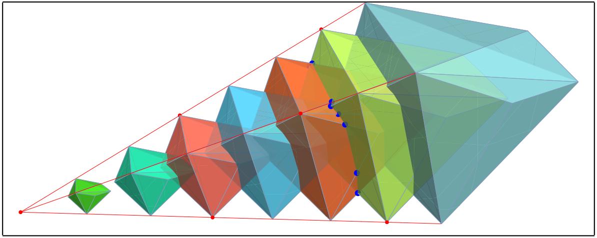
Let $\mathcal{F} \subseteq \mathbb{N}^3$ simplicial, with $\{n_1, \ldots, n_t\}$ a minimal system of generators of $\mathcal{F}$ and $n_1 \in \tau 1$, $n_2 \in \tau_2$, $n_3 \in \tau_3$. Then, the following conditions are equivalent:

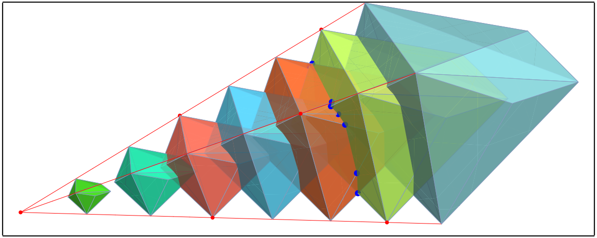
1. *$\mathcal{F}$ is Cohen-Macaulay.*
2. *For all $a \in \mathcal{C}_{\mathcal{F}} \setminus \mathcal{F}$, and every integers $i, j \in \{1, 2, 3\}$ with $i \neq j$, $\{a + n_i, a + n_i\} \not\subset \mathcal{F}$.*

$P_i \in \tau_i \cap \mathbf{F}$. P_i is the closest to the origin.
$\mathcal{F} = \langle n_1, \ldots, n_t \rangle$ is simplicial and $n_1 \in \tau_1$, $n_2 \in \tau_2$, and $n_3 \in \tau_3$.
$\mathcal{C}_{\mathcal{F}} \setminus \mathcal{F}$ can be expressed as:

$$\mathcal{C}_{\mathcal{F}} \setminus \mathcal{F} = \cup_{k \in \mathbb{N}} ((G_1 + kP_1) \cup (G_2 + kP_2) \cup (G_3 + kP_3) \cup G_{12}^k \cup G_{23}^k \cup G_{31}^k) \cup BS,$$

with BS a bounded set.
For $k \ggg 0$, the set
$((G_1 + kP_1) \cup (G_2 + kP_2) \cup (G_3 + kP_3) \cup G_{12}^k \cup G_{23}^k \cup G_{31}^k)$ contain the gaps of $\mathcal{F}$ located between $k\mathbf{F} \cap \mathcal{C}_{\mathbf{F}}$ and $(k+1)\mathbf{F} \cap \mathcal{C}_{\mathbf{F}}$.





- There exists k_i such that for all $k \geq k_i$ and every $P \in \mathbb{N}^3 \cap G_i + kP_i$, $P + n_j \in \mathcal{F}$ iff $P + (k - k_i)P_i + n_j \in \mathcal{F}$ for all $j \in \{1, 2, 3\} \setminus \{i\}$. (points to check)
- If $P \in \mathbb{N}^3 \cap G_{ij}^k$, then for all $k' \geq k$ we have $P + n_i \notin \mathcal{F}$ and $P + n_j \notin \mathcal{F}$. (good points)

Theorem

There exists a finite subset $\mathcal{A}$ of $\mathcal{C}_{\mathcal{F}}$ determined by P_i and k_i with $i = 1, 2, 3$ such that:
$\mathcal{F}$ is Cohen-Macaulay if and only if for every element P of $\mathcal{A} \setminus \mathcal{F}$, $\#(\{P + n_1, P + n_2, P + n_3\} \cap \mathcal{F}) \leq 1$.

Corollary

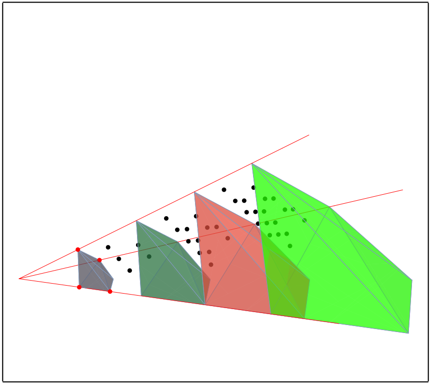
The affine semigroup associated to any tetrahedron with rational vertices is Cohen-Macaulay.

Proof.

$G_1 = G_2 = G_3 = \emptyset$. □

Example

Let **F** be the non tetrahedron polyhedron obtained from the convex hull of the vertex set
$\{\{3,3,2\},\{2,3,1\},\{1,2,3\},\{3/2,3,9/2\},\{33/16,27/8,63/16\}\}$.
The associated convex polyhedron semigroup is C-M



Buchsbaum property

Let S be a simplicial affine semigroup. $\overline{S} \subset \mathbb{N}^3$ is the affine semigroup $\{a \in \mathbb{N}^k \mid a + n_i \in S, \forall i = 1, 2, 3\}$

Theorem

[9, Theorem 5] The following conditions are equivalent:

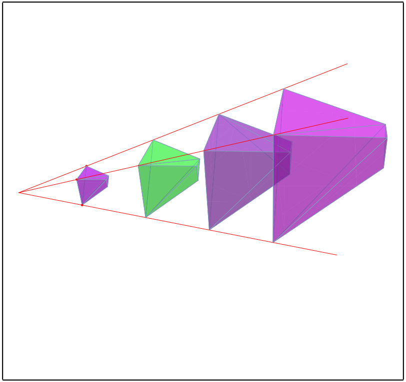
1. *S is Buchsbaum.*
2. *$\overline{S}$ is Cohen-Macaulay.*

Example

Given the vertex set of the convex polyhedron tiny

$$\mathbf{F} = \mathcal{H}(\{\{24/5, 12/5, 12/5\}, \{8/3, 16/3, 8/3\}, \{8/3, 8/3, 16/3\}, \{152/33, 152/33, 16/3\}, \{152/33, 16/3, 152/33\}, \{856/165, 68/15, 68/15\}\}),$$

This polyhedron it is not a tetrahedron, it is minimally generated by 71 elements and it is Buchsbaum.



Gorenstein property

Theorem

For a simplicial semigroup $S \subset \mathbb{N}^n$, the following conditions are equivalent:

1. *S is Gorenstein.*
2. *S is Cohen-Macaulay and $\cap_{i=1}^{n} \mathrm{Ap}(n_i)$ has a unique maximal element (with respect to the order defined by S).*

Example

The affine semigroup $\mathcal{F}$ associated to the convex hull of the vertex set $\{(4,0,0),(4+2k,0,0),(4+k,k,0),(4+k,0,1)\}$ is Gorenstein.

Thank you!