The Ratliff-Rush operation on monomial ideals

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- $r(I) = \min\{\ell \mid \tilde{I} = I^{\ell+1} : I^{\ell}\}$ RR reduction number

- $\widetilde{(I^\ell)} = I^\ell$ for all $\ell \geq 1 \Leftrightarrow$ depth of the associated graded ring $G_I(R) = \oplus_{n\geq 0} I^n/I^{n+1} > 0$ (Heinzer et al., 1992)

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- I $\mathfrak{m}$-primary $\subset (R, \mathfrak{m})$ Cohen-Macaulay

  $\widetilde{(I^{\ell+1})} \subseteq I^\ell$ for all $\ell \geq 1 \Rightarrow$ the 0-th Bockstein cohomology of $G_I(R)$ vanishes (Puthenpurakal, 2012)

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  $R = k[x]/(x^2)$ and $I = \langle x \rangle$, then $\tilde{I} = \bigcup_{\ell \geq 1}^{\infty} (I^{\ell+1} : I^{\ell}) = R$,
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  $\tilde{J} = J + \langle x^2y^2 \rangle$ (in fact, $J^{\ell} = (\mathfrak{m}^4)^{\ell}$ for all $\ell \geq 2$)
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- I monomial $\Rightarrow \tilde{I}$ monomial

- What is $r(I)$?
  There are cases when
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- Heinzer et el., 1992
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  Rossi&Swanson, 2001
  Elias, 2004
  $-$, 2006 for certain classes of monomial ideals in $k[x, y]$ using numerical semigroups

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Define three sets by deleting one of the coordinates:
$X = \langle (b_j, c_j) \rangle = \{\sum_{j=0}^{n} \lambda_j (b_j, c_j) \mid \lambda_j \in \mathbb{Z}_{\geq 0}\}$,
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$(0, 0)$ belongs to each of them $\Rightarrow$ finitely generated subsemigroups of $\mathbb{Z}^2$, affine semigroups.

Lemma

Let $Z = \langle (a_j, b_j) \rangle_{j=0}^n$ *with* $a_j + b_j \leq d$ *be an affine semigroup. Assume* $(d, 0)$ *and* $(0, d)$ *belong to* Z*. Then for any two reals* $0 < \alpha < 1$ *and* $0 \leq \beta$*, there is an* L *s.t. for all* $\ell \geq L$ *we have:*

$$(r, s) \in Z \text{ and } r + s \leq \alpha \cdot d\ell + \beta$$
$$\Downarrow$$
$$(r, s) = \sum_{j=0}^n \lambda_j (a_j, b_j) \text{ where } \sum_{j=0}^n \lambda_j \leq \ell.$$

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(r, s) may be a linear combination of the (a_j, b_j)'s in different ways, we choose the one where $\lambda_j \leq d - 1$ for all $1 \leq j \leq n - 1$.
If $\lambda_j \geq d$, then $\lambda_j (a_j, b_j) = a_j (d, 0) + (\lambda_j - d)(a_j, b_j) + b_j (0, d)$.

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$$L = \frac{\beta/d + (d-1)(n-1)}{1 - \alpha}$$

The bound provided by the proof is very rough.

Example

Consider $Z = \langle (3,0),(1,0),(0,3) \rangle = \langle (a_j, b_j) \rangle$. Let $\alpha = \frac{2}{3}$, $\beta = 0$.

Then for any $\ell \geq 2$ and $(r,s) \in Z$ s. t. $r + s \leq 2\ell$ we have $(r,s) = \sum \lambda_j (a_j, b_j)$ with $\sum \lambda_j \leq \ell$.

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The bound provided in the proof is $\ell \geq 6$.

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Then there is an integer L such that for any $\ell \geq L$ we have
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$\supseteq$ may seem trivial but needs a proof, since $r + s + t = d\ell$ does not necessarily imply $(r, s) = \sum_{j=0}^{n} \lambda_j (a_j, b_j)$ for some non-negative λ_j's with $\sum_{j=0}^{n} \lambda_j = \ell$.

Definition

We define the following ideals associated to I:

$$I_Z = \langle x^r y^s z^{d-r-s} \mid (r, s) \in Z \text{ and } r + s \leq d \rangle$$
$$I_Y = \langle x^r y^{d-r-t} z^t \mid (r, t) \in Y \text{ and } r + t \leq d \rangle$$
$$I_X = \langle x^{d-s-t} y^s z^t \mid (s, t) \in X \text{ and } s + t \leq d \rangle.$$

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$$I_X = \langle x^7, x^2y^5, x^5z^2, \underline{y^5z^2}, \underline{x^3z^4}, \underline{xz^6}, y^7, y^2z^5, z^7 \rangle.$$

Proposition (II)

$I = \langle x^{a_j} y^{b_j} z^{c_j} \rangle_{j=0}^{n} \subset R$, with $a_j + b_j + c_j = d$ for all j, is $\mathfrak{m}$-primary ideal. Then for all sufficiently large ℓ we have

$$I^\ell = z^{d\ell - d} I_Z + y^{d\ell - d} I_Y + x^{d\ell - d} I_X + \langle x^3y^3, y^3z^3, x^3z^3 \rangle I_{m,\ell} \quad (1)$$

for some ideal $I_{m,\ell}$ generated by monomials of degree $d(\ell - 2)$.

Proposition (II)

$I = \langle x^{a_j} y^{b_j} z^{c_j} \rangle_{j=0}^{n} \subset R$, with $a_j + b_j + c_j = d$ for all j, is $\mathfrak{m}$-primary ideal. Then for all sufficiently large ℓ we have

$$I^{\ell} = z^{d\ell-d} I_Z + y^{d\ell-d} I_Y + x^{d\ell-d} I_X + \langle x^3y^3, y^3z^3, x^3z^3 \rangle I_{m,\ell} \quad (1)$$

for some ideal $I_{m,\ell}$ generated by monomials of degree $d(\ell-2)$.

Example

Let $I = \langle x^7, x^2y^5, y^7, z^7 \rangle$. Then $I_Y = \langle x^7, x^2y^5, \underline{x^4y^3}, \underline{x^6y}, y^7, z^7 \rangle$ and $I_X = I_Z = I$.
Then for all $\ell \geq 3$ the ideal I^{ℓ} is on the form (1).
It doesn't work for $\ell = 2$ because of the monomial x^2y^5, which gives rise to x^4y^3 and x^6y in I_Y. We have $(x^2y^5)^3 = y^{14}(x^6y)$, but cannot rewrite any monomial generator of I^2 into a product of x^6y.

Theorem

Let $I = \langle x^{a_j} y^{b_j} z^{c_j} \rangle_{j=0}^{n}$ with $a_j + b_j + c_j = d$ for all j be an $\mathfrak{m}$-primary ideal in R and I_Z, I_Y, I_X be defined as previously. Then the Ratliff-Rush ideal $\tilde{I} = I_Z \cap I_Y \cap I_X$.

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- $\supseteq$ Consider $p \in I_Z \cap I_Y \cap I_X$. Then $p \cdot x^r y^s z^t = \dots$ $= p_1 \prod (x^{a_j} y^{b_j} z^{c_j})^{\lambda_j} \in I^{\ell+1}$ for some p_1 by the lemma on the estimate of the generators of an affine semigroup element.

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- $\subseteq$ If a monomial $p \notin I_Z$, then $z^{dl} p \notin z^{dl} I_Z$ and $z^{dl} p \notin I^{l+1}$ by Proposition (II).

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- $I^{\ell+1} : I^\ell = \tilde{I}$ for all $\ell \geq 1$.

Thank you for your attention!