Shifted numerical semigroups and their tangent cones

Santiago Zarzuela

University of Barcelona

Special session on Commutative Monoids

Joint Meeting of the American, European and Portuguese Mathematical Societies

June 10th - 13th, 2015

Porto, Portugal

Based on joint work in progress with

Reheleh Jafari (IPM, Tehran)

- The conjecture of Herzog-Srinivasan.
- Homogeneous semigroups and semigroups of homogeneous type.
- Asymptotic behavior under shifting.

The conjecture of Herzog-Srinivasan

• Let $\underline{\mathbf{m}} := 0 < m_1 < \cdots < m_d$ be a family of positive integers.

• Let $S = \langle m_1, \ldots, m_d \rangle \subseteq \mathbb{N}$ be the semigroup the generated by the family $\underline{\mathbf{m}}$.

• Let K be a field and $K[S] = K[t^{m_1}, \ldots, t^{m_d}] \subseteq K[t]$ be the semigroup ring defined by $\underline{\mathbf{m}}$.

We may consider the presentation:

$$0 \longrightarrow I(S) \longrightarrow K[t_1, \ldots, t_d] \xrightarrow{\varphi} K[S] \longrightarrow 0$$

given by $\varphi(t_i) = t^{m_i}$.

- We call the Betti numbers of $I(S)$ as the Betti numbers of S.

• Let $\underline{\mathbf{m}} := 0 < m_1 < \cdots < m_d$ be a family of positive integers.

• Let $S = \langle m_1, \ldots, m_d \rangle \subseteq \mathbb{N}$ be the semigroup the generated by the family $\underline{\mathbf{m}}$.

• Let K be a field and $K[S] = K[t^{m_1}, \ldots, t^{m_d}] \subseteq K[t]$ be the semigroup ring defined by $\underline{\mathbf{m}}$.

We may consider the presentation:

$$0 \longrightarrow I(S) \longrightarrow K[t_1, \ldots, t_d] \xrightarrow{\varphi} K[S] \longrightarrow 0$$

given by $\varphi(t_i) = t^{m_i}$.

- We call the Betti numbers of $I(S)$ as the Betti numbers of S.

• Let $\underline{\mathbf{m}} := 0 < m_1 < \cdots < m_d$ be a family of positive integers.

• Let $S = \langle m_1, \ldots, m_d \rangle \subseteq \mathbb{N}$ be the semigroup the generated by the family $\underline{\mathbf{m}}$.

• Let K be a field and $K[S] = K[t^{m_1}, \ldots, t^{m_d}] \subseteq K[t]$ be the semigroup ring defined by $\underline{\mathbf{m}}$.

We may consider the presentation:

$$0 \longrightarrow I(S) \longrightarrow K[t_1, \ldots, t_d] \xrightarrow{\varphi} K[S] \longrightarrow 0$$

given by $\varphi(t_i) = t^{m_i}$.

- We call the Betti numbers of $I(S)$ as the Betti numbers of S.

• Let $\underline{\mathbf{m}} := 0 < m_1 < \cdots < m_d$ be a family of positive integers.

• Let $S = \langle m_1, \ldots, m_d \rangle \subseteq \mathbb{N}$ be the semigroup the generated by the family $\underline{\mathbf{m}}$.

• Let K be a field and $K[S] = K[t^{m_1}, \ldots, t^{m_d}] \subseteq K[t]$ be the semigroup ring defined by $\underline{\mathbf{m}}$.

We may consider the presentation:

$$0 \longrightarrow I(S) \longrightarrow K[t_1, \ldots, t_d] \xrightarrow{\varphi} K[S] \longrightarrow 0$$

given by $\varphi(t_i) = t^{m_i}$.

- We call the Betti numbers of $I(S)$ as the Betti numbers of S.

• Let $\underline{\mathbf{m}} := 0 < m_1 < \cdots < m_d$ be a family of positive integers.

• Let $S = \langle m_1, \ldots, m_d \rangle \subseteq \mathbb{N}$ be the semigroup the generated by the family $\underline{\mathbf{m}}$.

• Let K be a field and $K[S] = K[t^{m_1}, \ldots, t^{m_d}] \subseteq K[t]$ be the semigroup ring defined by $\underline{\mathbf{m}}$.

We may consider the presentation:

$$0 \longrightarrow I(S) \longrightarrow K[t_1, \ldots, t_d] \xrightarrow{\varphi} K[S] \longrightarrow 0$$

given by $\varphi(t_i) = t^{m_i}$.

- We call the Betti numbers of $I(S)$ as the Betti numbers of S.

- For any $j \geq 0$ we consider the shifted family

$$\underline{\mathbf{m}} + j := 0 < m_1 + j < \cdots < m_d + j$$

and the semigroup

$$S + j := \langle m_1 + j, \ldots, m_d + j \rangle$$

that we call the j-th shifting of S.

Conjecture (by J. Herzog and H. Srinivasan):

The Betti numbers of $S + j$ are eventually periodic on j with period $m_d - m_1$.

- For any $j \geq 0$ we consider the shifted family

$$\underline{\mathbf{m}} + j := 0 < m_1 + j < \cdots < m_d + j$$

and the semigroup

$$S + j := \langle m_1 + j, \ldots, m_d + j \rangle$$

that we call the j-th shifting of S.

Conjecture (by J. Herzog and H. Srinivasan):

The Betti numbers of $S + j$ are eventually periodic on j with period $m_d - m_1$.

Remark:

- If we start with S a numerical semigroup, it may happen that $S+j$ is not anymore a numerical semigroup.

For instance, let $S = \langle 3, 5 \rangle$: then $S+1 = \langle 4, 6 \rangle$.

- Also, we may start with a family which is a minimal system of generators of S but the shifted family is not anymore a minimal system of generators of $S+j$.

For instance, $S = \langle 3, 5, 7 \rangle$: then $S+1 = \langle 4, 6, 8 \rangle = \langle 4, 6 \rangle$.

- But if S is minimally generated by $m_1, \ldots, m_d$ then $S+j$ is minimally generated by $m_1+j, \ldots, m_d+j$ for any $j > m_d - 2m_1$.

Remark:

- If we start with S a numerical semigroup, it may happen that $S+j$ is not anymore a numerical semigroup.

For instance, let $S=\langle 3,5\rangle$: then $S+1=\langle 4,6\rangle$.

- Also, we may start with a family which is a minimal system of generators of S but the shifted family is not anymore a minimal system of generators of $S+j$.

For instance, $S=\langle 3,5,7\rangle$: then $S+1=\langle 4,6,8\rangle=\langle 4,6\rangle$.

- But if S is minimally generated by $m_1,\ldots,m_d$ then $S+j$ is minimally generated by $m_1+j,\ldots,m_d+j$ for any $j>m_d-2m_1$.

Remark:

- If we start with S a numerical semigroup, it may happen that $S + j$ is not anymore a numerical semigroup.

For instance, let $S = \langle 3, 5 \rangle$: then $S + 1 = \langle 4, 6 \rangle$.

- Also, we may start with a family which is a minimal system of generators of S but the shifted family is not anymore a minimal system of generators of $S + j$.

For instance, $S = \langle 3, 5, 7 \rangle$: then $S + 1 = \langle 4, 6, 8 \rangle = \langle 4, 6 \rangle$.

- But if S is minimally generated by $m_1, \ldots, m_d$ then $S + j$ is minimally generated by $m_1 + j, \ldots, m_d + j$ for any $j > m_d - 2m_1$.

The conjecture holds for:

- $d = 3$ (A. V. Jayanthan and H. Srinivasan, 2013).

- Arithmetic sequences (P. Gimenez, I. Senegupta, and H. Srinivasan, 2013).

- In general (Thran Vu, 2014).

Namely, there exists positive value N such that for any $j > N$ the Betti numbers of $S + j$ are periodic with period $m_d - m_1$.

Remark:

The bound N depends on the Castelnuovo-Mumford regularity of $J(S)$, the ideal generated by the homogeneous elements in $I(S)$.

The conjecture holds for:

- $d = 3$ (A. V. Jayanthan and H. Srinivasan, 2013).
- Arithmetic sequences (P. Gimenez, I. Senegupta, and H. Srinivasan, 2013).
- In general (Thran Vu, 2014).

Namely, there exists positive value N such that for any $j > N$ the Betti numbers of $S + j$ are periodic with period $m_d - m_1$.

Remark:

The bound N depends on the Castelnuovo-Mumford regularity of $J(S)$, the ideal generated by the homogeneous elements in $I(S)$.

The conjecture holds for:

• $d = 3$ (A. V. Jayanthan and H. Srinivasan, 2013).

• Arithmetic sequences (P. Gimenez, I. Senegupta, and H. Srinivasan, 2013).

• In general (Thran Vu, 2014).

Namely, there exists positive value N such that for any $j > N$ the Betti numbers of $S + j$ are periodic with period $m_d - m_1$.

Remark:

The bound N depends on the Castelnuovo-Mumford regularity of $J(S)$, the ideal generated by the homogeneous elements in $I(S)$.

The conjecture holds for:

- $d = 3$ (A. V. Jayanthan and H. Srinivasan, 2013).
- Arithmetic sequences (P. Gimenez, I. Senegupta, and H. Srinivasan, 2013).
- In general (Thran Vu, 2014).

Namely, there exists positive value N such that for any $j > N$ the Betti numbers of $S + j$ are periodic with period $m_d - m_1$.

Remark:

The bound N depends on the Castelnuovo-Mumford regularity of $J(S)$, the ideal generated by the homogeneous elements in $I(S)$.

• Let $I^*(S)$ be the initial ideal of $I(S)$, that is, the ideal generated by the initial forms of the elements of $I(S)$.

- $I^*(S) \subset K[t_1, \ldots, t_d]$ is an homogeneous ideal. It is the definition ideal of the tangent cone of S: $G(S)$.

As a consequence of the work of Vu, J. Herzog and D. I. Stamate, 2014, have shown that for any $j > N$,

$$\beta_i(I(S+j)) = \beta_i(I^*(S+j)) \text{ for all } i \geq 0$$

In particular, for any $j > N$, $G(S)$ is Cohen-Macaulay.

• Let $I^*(S)$ be the initial ideal of $I(S)$, that is, the ideal generated by the initial forms of the elements of $I(S)$.

- $I^*(S) \subset K[t_1, \ldots, t_d]$ is an homogeneous ideal. It is the definition ideal of the tangent cone of S: $G(S)$.

As a consequence of the work of Vu, J. Herzog and D. I. Stamate, 2014, have shown that for any $j > N$,

$$\beta_i(I(S+j)) = \beta_i(I^*(S+j)) \text{ for all } i \geq 0$$

In particular, for any $j > N$, $G(S)$ is Cohen-Macaulay.

Our purpose is to reverse this fact, and also to provide a bound which only depends on the initial data of the family $\underline{\mathbf{m}}$.

- We will give a condition on the Apéry set of S with respect to its multiplicity, that jointly with the Cohen-Macaulay property of $G(S)$ will imply the above equality of the Betti numbers.

- Then, we will show that these two conditions eventually hold for $S + j$, with a bound L that we can easily compute in terms $m_1, \ldots, m_d$.

Moreover, this bound will only depend on the class of the shifted semigroups.

Our purpose is to reverse this fact, and also to provide a bound which only depends on the initial data of the family $\underline{\mathbf{m}}$.

- We will give a condition on the Apéry set of S with respect to its multiplicity, that jointly with the Cohen-Macaulay property of $G(S)$ will imply the above equality of the Betti numbers.

- Then, we will show that these two conditions eventually hold for $S + j$, with a bound L that we can easily compute in terms $m_1, \ldots, m_d$.

Moreover, this bound will only depend on the class of the shifted semigroups.

Our purpose is to reverse this fact, and also to provide a bound which only depends on the initial data of the family $\underline{\mathbf{m}}$.

- We will give a condition on the Apéry set of S with respect to its multiplicity, that jointly with the Cohen-Macaulay property of $G(S)$ will imply the above equality of the Betti numbers.

- Then, we will show that these two conditions eventually hold for $S + j$, with a bound L that we can easily compute in terms $m_1, \ldots, m_d$.

Moreover, this bound will only depend on the class of the shifted semigroups.

Our purpose is to reverse this fact, and also to provide a bound which only depends on the initial data of the family $\underline{\mathbf{m}}$.

- We will give a condition on the Apéry set of S with respect to its multiplicity, that jointly with the Cohen-Macaulay property of $G(S)$ will imply the above equality of the Betti numbers.

- Then, we will show that these two conditions eventually hold for $S + j$, with a bound L that we can easily compute in terms $m_1, \ldots, m_d$.

Moreover, this bound will only depend on the class of the shifted semigroups.

Homogeneous semigroups and semigroups of homogeneous type

• Let $\mathbf{a} = (a_1, \ldots, a_d)$ a vector of non-negative integers. Then we define the total order of $\mathbf{a}$ as $|\mathbf{a}| = \sum_{i=1}^{d} a_i$. We also set $s(\mathbf{a}) = \sum_{i=1}^{d} a_i m_i \in S$.

• Given an expression of an element $s \in S$: $s = \sum_{i=1}^{d} a_i m_i$, we call the vector $\mathbf{a} = (a_1, \ldots, a_d)$ a factorization of s.

Then, we define the order of s as the maximum total order among the factorizations of s.

• An expression of s is then said to be maximal if the total order of its factorization is the order of s.

A factorization of an element whose total order is maximal is called a maximal factorization.

• Let $\mathbf{a} = (a_1, \ldots, a_d)$ a vector of non-negative integers. Then we define the total order of $\mathbf{a}$ as $|\mathbf{a}| = \sum_{i=1}^{d} a_i$. We also set $s(\mathbf{a}) = \sum_{i=1}^{d} a_i m_i \in S$.

• Given an expression of an element $s \in S$: $s = \sum_{i=1}^{d} a_i m_i$, we call the vector $\mathbf{a} = (a_1, \ldots, a_d)$ a factorization of s.

Then, we define the order of s as the maximum total order among the factorizations of s.

• An expression of s is then said to be maximal if the total order of its factorization is the order of s.

A factorization of an element whose total order is maximal is called a maximal factorization.

• Let $\mathbf{a} = (a_1, \ldots, a_d)$ a vector of non-negative integers. Then we define the total order of $\mathbf{a}$ as $|\mathbf{a}| = \sum_{i=1}^{d} a_i$. We also set $s(\mathbf{a}) = \sum_{i=1}^{d} a_i m_i \in S$.

• Given an expression of an element $s \in S$: $s = \sum_{i=1}^{d} a_i m_i$, we call the vector $\mathbf{a} = (a_1, \ldots, a_d)$ a factorization of s.

Then, we define the order of s as the maximum total order among the factorizations of s.

• An expression of s is then said to be maximal if the total order of its factorization is the order of s.

A factorization of an element whose total order is maximal is called a maximal factorization.

• Let $\mathbf{a} = (a_1, \ldots, a_d)$ a vector of non-negative integers. Then we define the total order of **a** as $|\mathbf{a}| = \sum_{i=1}^{d} a_i$. We also set $s(\mathbf{a}) = \sum_{i=1}^{d} a_i m_i \in S$.

• Given an expression of an element $s \in S$: $s = \sum_{i=1}^{d} a_i m_i$, we call the vector $\mathbf{a} = (a_1, \ldots, a_d)$ a factorization of s.

Then, we define the order of s as the maximum total order among the factorizations of s.

• An expression of s is then said to be maximal if the total order of its factorization is the order of s.

A factorization of an element whose total order is maximal is called a maximal factorization.

• A subset $T \subset S$ is said to be homogeneous if all the expressions of elements in T are maximal.

Definition

We then say that S is homogeneous if the Apéry set $\mathrm{AP}(S, e)$ is homogeneous, where $e = m_1$ is the multiplicity of S.

• If $d = 2$ then S is homogeneous.

• If $e = d$ (maximal embedding dimension) or $e = d - 1$ (almost maximal embedding dimension) then S is homogeneous.

• If S is generated by a generalized arithmetic sequence then S is homogeneous.

• A subset $T \subset S$ is said to be homogeneous if all the expressions of elements in T are maximal.

Definition

We then say that S is homogeneous if the Apéry set $\mathrm{AP}(S, e)$ is homogeneous, where $e = m_1$ is the multiplicity of S.

• If $d = 2$ then S is homogeneous.

• If $e = d$ (maximal embedding dimension) or $e = d - 1$ (almost maximal embedding dimension) then S is homogeneous.

• If S is generated by a generalized arithmetic sequence then S is homogeneous.

• A subset $T \subset S$ is said to be homogeneous if all the expressions of elements in T are maximal.

Definition

We then say that S is homogeneous if the Apéry set $\mathrm{AP}(S, e)$ is homogeneous, where $e = m_1$ is the multiplicity of S.

• If $d = 2$ then S is homogeneous.

• If $e = d$ (maximal embedding dimension) or $e = d - 1$ (almost maximal embedding dimension) then S is homogeneous.

• If S is generated by a generalized arithmetic sequence then S is homogeneous.

- Remember that the defining ideal $I(S)$ may be generated by binomials of the form $x^{\mathbf{a}} - x^{\mathbf{b}}$.

For those binomials we have that $s(\mathbf{a}) = s(\mathbf{b})$ and so both $\mathbf{a}$ and $\mathbf{b}$ provide factorizations of the same element $s \in S$.

- Also that a family of elements of $I(S)$ such that their initial forms generate $I^*(S)$ is called a standard basis.

Any standard basis is system of generators of $I(S)$ (but not conversely).

- Remember that the defining ideal $I(S)$ may be generated by binomials of the form $x^{\mathbf{a}} - x^{\mathbf{b}}$.

For those binomials we have that $s(\mathbf{a}) = s(\mathbf{b})$ and so both **a** and **b** provide factorizations of the same element $s \in S$.

- Also that a family of elements of $I(S)$ such that their initial forms generate $I^*(S)$ is called a standard basis.

Any standard basis is system of generators of $I(S)$ (but not conversely).

- Remember that the defining ideal $I(S)$ may be generated by binomials of the form $x^{\mathbf{a}} - x^{\mathbf{b}}$.

For those binomials we have that $s(\mathbf{a}) = s(\mathbf{b})$ and so both **a** and **b** provide factorizations of the same element $s \in S$.

- Also that a family of elements of $I(S)$ such that their initial forms generate $I^*(S)$ is called a standard basis.

Any standard basis is system of generators of $I(S)$ (but not conversely).

Proposition (1)

The following are equivalent:

(1) *S is homogeneous and $G(S)$ is Cohen-Macaulay.*

(2) *There exists a minimal set of generators E for $I(S)$ such that for all $x^{\mathbf{a}} - x^{\mathbf{b}} \in E$ with $|\mathbf{a}| > |\mathbf{b}|$, we have $a_1 \neq 0$.*

(3) *There exists a minimal set of generators E for $I(S)$ which is a standard basis and for all $x^{\mathbf{a}} - x^{\mathbf{b}} \in E$ with $|\mathbf{a}| > |\mathbf{b}|$, we have $a_1 \neq 0$.*

Proposition (1)

The following are equivalent:

1. *S is homogeneous and $G(S)$ is Cohen-Macaulay.*
2. *There exists a minimal set of generators E for $I(S)$ such that for all $x^{\mathbf{a}} - x^{\mathbf{b}} \in E$ with $|\mathbf{a}| > |\mathbf{b}|$, we have $a_1 \neq 0$.*
3. *There exists a minimal set of generators E for $I(S)$ which is a standard basis and for all $x^{\mathbf{a}} - x^{\mathbf{b}} \in E$ with $|\mathbf{a}| > |\mathbf{b}|$, we have $a_1 \neq 0$.*

Proposition (1)

The following are equivalent:

(1) *S is homogeneous and $G(S)$ is Cohen-Macaulay.*

(2) *There exists a minimal set of generators E for $I(S)$ such that for all $x^{\mathbf{a}} - x^{\mathbf{b}} \in E$ with $|\mathbf{a}| > |\mathbf{b}|$, we have $a_1 \neq 0$.*

(3) *There exists a minimal set of generators E for $I(S)$ which is a standard basis and for all $x^{\mathbf{a}} - x^{\mathbf{b}} \in E$ with $|\mathbf{a}| > |\mathbf{b}|$, we have $a_1 \neq 0$.*

Remark:

Condition (2) in the above proposition plays a fundamental role in Vu's results.

Definition

We say that S is of homogeneous type if $\beta_i(S) = \beta_i(G(S))$ for all $i \geq 0$.

Proposition (2)

Let S be a homogeneous semigroup such that $G(S)$ is Cohen-Macaulay. Then S is of homogenous type.

Remark:

Condition (2) in the above proposition plays a fundamental role in Vu's results.

Definition

We say that S is of homogeneous type if $\beta_i(S) = \beta_i(G(S))$ for all $i \geq 0$.

Proposition (2)

Let S be a homogeneous semigroup such that $G(S)$ is Cohen-Macaulay. Then S is of homogenous type.

Remark:

Condition (2) in the above proposition plays a fundamental role in Vu's results.

Definition

We say that S is of homogeneous type if $\beta_i(S) = \beta_i(G(S))$ for all $i \geq 0$.

Proposition (2)

Let S be a homogeneous semigroup such that $G(S)$ is Cohen-Macaulay. Then S is of homogenous type.

Example (3)

Let $S := \langle 15, 21, 28 \rangle$. Then S is of homogeneous type. The defining ideal is generated by a standard basis:

$$I(S) = (x_2^4 - x_3^3, x_1^7 - x_2^5)$$

but it is not homogeneous:

$$3 \times 28 = 4 \times 21 = 84 \in \mathrm{AP}(S, 15)$$

Asymptotic behavior under shifting

- Let $n_i := m_d - m_{d-i}$, for all $1 \leq i \leq d-1$.
- Let $g := gcd(n_1, \ldots, n_{d-1})$ and $T := \langle \frac{n_1}{g}, \ldots, \frac{n_{d-1}}{g} \rangle$.
- Let

$$L := n_{d-1} n_{d-2} \left(\frac{gc + dn_{d-1}}{n_1} + d \right) - m_d$$

where c is the conductor of T.

Proposition (4)

Let $j > L$ and $s \in S + j$. If $\mathbf{a}, \mathbf{a'}$ are two factorizations of s with $|\mathbf{a}| > |\mathbf{a'}|$, then there exists a factorization $\mathbf{b}$ of s such that $|\mathbf{a}| = |\mathbf{b}|$ and $b_1 \neq 0$.

- Let $n_i := m_d - m_{d-i}$, for all $1 \leq i \leq d-1$.
- Let $g := gcd(n_1, \dots, n_{d-1})$ and $T := \langle \frac{n_1}{g}, \dots, \frac{n_{d-1}}{g} \rangle$.
- Let

$$L := n_{d-1} n_{d-2} \left(\frac{gc + dn_{d-1}}{n_1} + d \right) - m_d$$

where c is the conductor of T.

Proposition (4)

*Let $j > L$ and $s \in S + j$. If **a**, **a'** are two factorizations of s with $|\mathbf{a}| > |\mathbf{a'}|$, then there exists a factorization **b** of s such that $|\mathbf{a}| = |\mathbf{b}|$ and $b_1 \neq 0$.*

- Let $n_i := m_d - m_{d-i}$, for all $1 \leq i \leq d-1$.
- Let $g := gcd(n_1, \ldots, n_{d-1})$ and $T := \langle \frac{n_1}{g}, \ldots, \frac{n_{d-1}}{g} \rangle$.
- Let

$$L := n_{d-1} n_{d-2} \left(\frac{gc + dn_{d-1}}{n_1} + d \right) - m_d$$

where c is the conductor of T.

Proposition (4)

Let $j > L$ and $s \in S + j$. If $\mathbf{a}, \mathbf{a'}$ are two factorizations of s with $|\mathbf{a}| > |\mathbf{a'}|$, then there exists a factorization $\mathbf{b}$ of s such that $|\mathbf{a}| = |\mathbf{b}|$ and $b_1 \neq 0$.

- Let $n_i := m_d - m_{d-i}$, for all $1 \leq i \leq d-1$.
- Let $g := gcd(n_1, \ldots, n_{d-1})$ and $T := \langle \frac{n_1}{g}, \ldots, \frac{n_{d-1}}{g} \rangle$.

- Let
$$L := n_{d-1} n_{d-2} \left(\frac{gc + dn_{d-1}}{n_1} + d \right) - m_d$$
where c is the conductor of T.

Proposition (4)

*Let $j > L$ and $s \in S + j$. If **a**, **a'** are two factorizations of s with $|\mathbf{a}| > |\mathbf{a'}|$, then there exists a factorization **b** of s such that $|\mathbf{a}| = |\mathbf{b}|$ and $b_1 \neq 0$.*

Corollary (5)

For any $j > L$, the j-th shifted numerical semigroup $S + j$ is homogeneous and $G(S + j)$ is Cohen-Macaulay. In particular, $S + j$ is of homogeneous type.

Proof:

Take E any system of binomials generators of $I(S + j)$. By the previous Proposition 4, for any binomial $x^{\mathbf{a}} - x^{\mathbf{a'}} \in E$ such that $|\mathbf{a}| > |\mathbf{a'}|$, there exists a binomial $x^{\mathbf{a}} - x^{\mathbf{b}}$ such that $|\mathbf{a}| = |\mathbf{b}| > |\mathbf{a'}|$ and $b_1 \neq 0$. Then, substituting $x^{\mathbf{a}} - x^{\mathbf{a'}}$ by $x^{\mathbf{a}} - x^{\mathbf{b}}$ and $x^{\mathbf{b}} - x^{\mathbf{a'}}$ and then refining to a minimal system of generators, we get that $S + j$ fulfills condition (2) in Proposition 1 and so we are done.

Corollary (5)

For any $j > L$, the j-th shifted numerical semigroup $S + j$ is homogeneous and $G(S + j)$ is Cohen-Macaulay. In particular, $S + j$ is of homogeneous type.

Proof:

Take E any system of binomials generators of $I(S + j)$. By the previous Proposition 4, for any binomial $x^{\mathbf{a}} - x^{\mathbf{a'}} \in E$ such that $|\mathbf{a}| > |\mathbf{a'}|$, there exists a binomial $x^{\mathbf{a}} - x^{\mathbf{b}}$ such that $|\mathbf{a}| = |\mathbf{b}| > |\mathbf{a'}|$ and $b_1 \neq 0$. Then, substituting $x^{\mathbf{a}} - x^{\mathbf{a'}}$ by $x^{\mathbf{a}} - x^{\mathbf{b}}$ and $x^{\mathbf{b}} - x^{\mathbf{a'}}$ and then refining to a minimal system of generators, we get that $S + j$ fulfills condition (2) in Proposition 1 and so we are done.

Definition

We say that a family of positive integers $0 < n_1 \leq \cdots \leq n_{d-1}$ is the shifting type of a family of integers $\underline{\mathbf{m}} = 0 < m_1 < \cdots < m_d$ if $n_i = m_d - m_{d-i}$ for all $1 \leq \cdots \leq i \leq \cdots \leq d-1$.

Note that in this case $m_{d-i} = n_{d-1} - n_i + m_1$ for all $i = 1, \ldots, d$ (assuming $n_0 = 0$).

- Hence the family of elements $\underline{\mathbf{m}} = 0 < m_1 < \cdots < m_d$ is uniquely determined by its shifting type and m_1.

Definition

We say that a family of positive integers $0 < n_1 \leq \cdots \leq n_{d-1}$ is the shifting type of a family of integers $\underline{\mathbf{m}} = 0 < m_1 < \cdots < m_d$ if $n_i = m_d - m_{d-i}$ for all $1 \leq \cdots \leq i \leq \cdots \leq d - 1$.

Note that in this case $m_{d-i} = n_{d-1} - n_i + m_1$ for all $i = 1, \ldots, d$ (assuming $n_0 = 0$).

- Hence the family of elements $\underline{\mathbf{m}} = 0 < m_1 < \cdots < m_d$ is uniquely determined by its shifting type and m_1.

Definition

We say that a family of positive integers $0 < n_1 \leq \cdots \leq n_{d-1}$ is the shifting type of a family of integers $\underline{\mathbf{m}} = 0 < m_1 < \cdots < m_d$ if $n_i = m_d - m_{d-i}$ for all $1 \leq \cdots \leq i \leq \cdots \leq d - 1$.

Note that in this case $m_{d-i} = n_{d-1} - n_i + m_1$ for all $i = 1, \ldots, d$ (assuming $n_0 = 0$).

- Hence the family of elements $\underline{\mathbf{m}} = 0 < m_1 < \cdots < m_d$ is uniquely determined by its shifting type and m_1.

- The shifting type is invariant under shifting.

- Two families of d positive integers are shifted one from the other if and only if they have the same shifting type.

Note that the family

$$1, n_{d-1} - n_{d-2} + 1, \ldots, n_{d-1} - n_2 + 1, n_{d-1} - n_1 + 1, n_{d-1} + 1$$

is the one with smallest possible m_1 and with shifting type $n_1, \ldots, n_{d-1}$.

- We then also say that the semigroup $S = \langle m_1, \ldots, m_d \rangle$ is of shifting type $0 < n_1 \leq \cdots \leq n_{d-1}$.

- The shifting type is invariant under shifting.

- Two families of d positive integers are shifted one from the other if and only if they have the same shifting type.

Note that the family

$$1, n_{d-1} - n_{d-2} + 1, \ldots, n_{d-1} - n_2 + 1, n_{d-1} - n_1 + 1, n_{d-1} + 1$$

is the one with smallest possible m_1 and with shifting type $n_1, \ldots, n_{d-1}$.

- We then also say that the semigroup $S = \langle m_1, \ldots, m_d \rangle$ is of shifting type $0 < n_1 \leq \cdots \leq n_{d-1}$.

- The shifting type is invariant under shifting.

- Two families of d positive integers are shifted one from the other if and only if they have the same shifting type.

Note that the family

$$1, n_{d-1} - n_{d-2} + 1, \ldots, n_{d-1} - n_2 + 1, n_{d-1} - n_1 + 1, n_{d-1} + 1$$

is the one with smallest possible m_1 and with shifting type $n_1, \ldots, n_{d-1}$.

- We then also say that the semigroup $S = \langle m_1, \ldots, m_d \rangle$ is of shifting type $0 < n_1 \leq \cdots \leq n_{d-1}$.

Let:

- $g = gcd(n_1, \ldots, n_{d-1})$;
- $T = \langle \frac{n_1}{g}, \ldots, \frac{n_{d-1}}{g} \rangle$ and c the conductor of T;
- $L = n_{d-1} n_{d-2} (\frac{gc + dn_{d-1}}{n_1} + d) - n_{d-1}$.

Proposition

For any $e > L$ all the semigroups S of shifting type $0 < n_1 \leq \cdots \leq n_{d-1}$ with $m_1 = e$ are homogeneous and $G(S)$ is Cohen-Macaulay. In particular their Hilbert function is non decreasing.

Let:

- $g = gcd(n_1, \ldots, n_{d-1})$;
- $T = \langle \frac{n_1}{g}, \ldots, \frac{n_{d-1}}{g} \rangle$ and c the conductor of T;
- $L = n_{d-1} n_{d-2} (\frac{gc + dn_{d-1}}{n_1} + d) - n_{d-1}$.

Proposition

For any $e > L$ all the semigroups S of shifting type $0 < n_1 \leq \cdots \leq n_{d-1}$ with $m_1 = e$ are homogeneous and $G(S)$ is Cohen-Macaulay. In particular their Hilbert function is non decreasing.

SOME REFERENCES

A. V. Jayanthan and Hema Srinivasan, *Periodic occurrence of complete intersection monomial curves*, Proc. Amer. Math. Soc. **141** (2013), 4199-4208.

P. Gimenez, I. Sengupta, and Hema Srinivasan, *Minimal graded free resolutions for monomial curves defined by arithmetical sequences*, J. Algebra **388** (2013), 249–310.

Thanh Vu, *Periodicity of Betti numbers of monomial curves*, J. Algebra **418** (2014), 66–90.

J, Herzog and D. Stamate, *On the defining equations of the tangent cone of a numerical semigroup ring*, J. Algebra **418** (2014), 8–28.

MUITO OBRIGADO!

Thank you very much for your attention!