Duality and syzygies for semimodules over numerical semigroups.

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Reference

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Outline

Our motivation was to gain a better understanding of certain semimodules over numerical semigroups with 2 generators appearing in previous investigations concerning *Hilbert depth*.

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1. Lattice paths and $\langle\alpha,\beta\rangle$-lean sets
2. Syzygies of $\langle\alpha,\beta\rangle$-semimodules
3. Dual semimodules

Γ-lean sets and Γ-semimodules

Definition

Let Γ be a numerical semigroup. A set $\{x_0 = 0, x_1, \ldots, x_n\} \subseteq \mathbb{N}$ is called Γ-lean if $|x_i - x_j| \notin \Gamma$ for $0 \leq i < j \leq n$.

A key notion will be that of a *module* over a numerical semigroup Γ:

Definition

A Γ-semimodule Δ is a non-empty subset of $\mathbb{N}$ such that $\Delta + \Gamma \subseteq \Delta$.

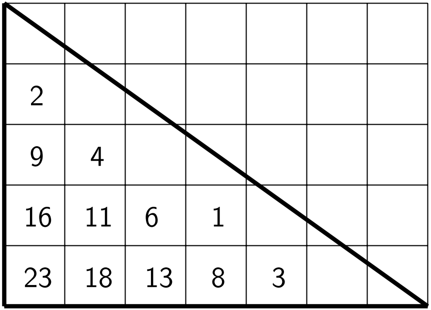
Every Γ-semimodule Δ has a unique minimal system of generators.

The minimal system of generators of a normalized Γ-semimodule is Γ-lean, and conversely, every Γ-lean subset of $\mathbb{N}$ generates minimally a normalized Γ-semimodule.

Gaps of $\langle \alpha, \beta \rangle$ and lattice points

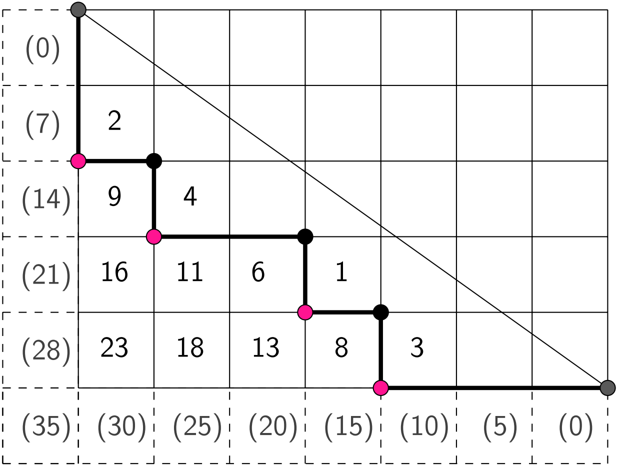
From now on we only consider semigroups $\Gamma = \langle \alpha, \beta \rangle$ with $\alpha < \beta$.

There is a map $G \to \mathbb{N}^2$, $\alpha\beta - a\alpha - b\beta \mapsto (a, b)$ which identifies a gap with a lattice point. Since $\alpha\beta - a\alpha - b\beta > 0$, the point lies inside the triangle with corners $(0, 0), (\beta, 0), (0, \alpha)$.



Gaps of $\langle 5, 7 \rangle$

$\langle \alpha, \beta \rangle$-lean sets and lattice paths



$I = [0, 8, 6, 9]$ and $J = [15, 13, 16, 14]$.

Gaps and ordering

For $\Gamma = \langle \alpha, \beta \rangle$ it holds

$$\ell \in \mathbb{N} \setminus \Gamma \quad \Longleftrightarrow \quad \exists\ a, b \in \mathbb{N}_{>0} \text{ with } \ell = \alpha\beta - a\alpha - b\beta.$$

This means that, for gaps $i_k = \alpha\beta - a_k\alpha - b_k\beta$, $k = 1, 2$, we have that

$$|i_1 - i_2| \in \mathbb{N} \setminus \Gamma \quad \Longleftrightarrow \quad (a_2 - a_1)(b_2 - b_1) < 0.$$

This allows us to introduce a partial ordering for the gaps:

$$i_1 \prec i_2 \quad :\Longleftrightarrow \quad a_1 > a_2 \ \wedge\ b_1 < b_2.$$

Syzygies of $\langle\alpha,\beta\rangle$-semimodules

Next we explain the meaning of J in terms of $\langle\alpha,\beta\rangle$-semimodules: Every $\langle\alpha,\beta\rangle$-semimodule Δ yields another $\langle\alpha,\beta\rangle$-semimodule $\mathrm{Syz}(\Delta)$.

Definition

Let I be an $\langle\alpha,\beta\rangle$-lean set, and let Δ be the $\langle\alpha,\beta\rangle$-semimodule generated by I. The syzygy of Δ is the $\langle\alpha,\beta\rangle$-semimodule

$$\mathrm{Syz}(\Delta) := \bigcup_{\substack{i,i'\in I\\ i\neq i'}} \left((i+\langle\alpha,\beta\rangle)\cap(i'+\langle\alpha,\beta\rangle)\right).$$

The semimodule $\mathrm{Syz}(\Delta)$ consists of those elements in Δ which admit more than one presentation of the form $i+x$ with $i\in I, x\in\langle\alpha,\beta\rangle$.

Fundamental couples and syzygies

$\mathrm{Syz}(\Delta)$ can be also recognized in the lattice path corresponding to Δ:

Theorem

Let I, J sets of turning points as in the example. Let Δ be the $\langle\alpha,\beta\rangle$-semimodule generated by the elements of I. Then

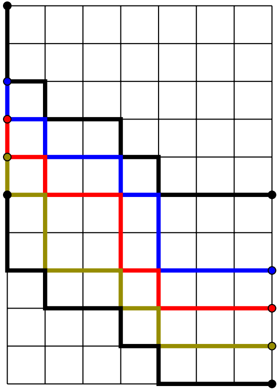
$$\mathrm{Syz}(\Delta) = \bigcup_{0\leq k<m\leq n} \Big(i_k + \langle\alpha,\beta\rangle \cap i_m + \langle\alpha,\beta\rangle\Big) = \bigcup_{k=0}^{n} (j_k + \langle\alpha,\beta\rangle).$$

i. e. , $\mathrm{Syz}(\Delta)$ is generated by the elements of J.

Iterated syzygies and their orbits

The procedure of building a syzygy can be iterated; we set

$$\mathrm{Syz}^{\ell}(\Delta) := \mathrm{Syz}(\mathrm{Syz}^{\ell-1}(\Delta)), \quad \ell \geq 2.$$

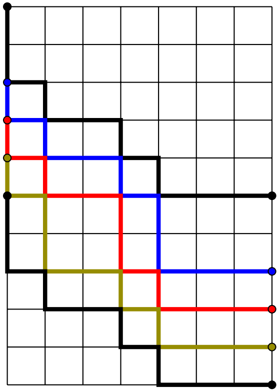


Since all semimodules $\mathrm{Syz}^{\ell}(\Delta)$ share the same number of generators, it is clear that this sequence must be periodic up to isomorphism.

The set of isomorphism classes appearing in such a sequence of syzygies will be called an orbit.

Syzygies and the matrix description

It is easily seen that taking the syzygy cyclically permutates the top row of the matrix by one position to the left:



$$\Delta \mapsto \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}$$

$$\mathrm{Syz}(\Delta) \mapsto \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 3 \end{pmatrix}$$

$$\mathrm{Syz}^2(\Delta) \mapsto \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}$$

$$\mathrm{Syz}^3(\Delta) \mapsto \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}$$

Dual semimodules

For any Γ-semimodule Δ we set the *dual* of Δ

$$\Delta^* := \mathrm{Hom}_\Gamma(\Delta, \Gamma) \cong \{c \in \mathbb{Z} \mid c + \Delta \subseteq \Gamma\} =: \Gamma - \Delta.$$

Dual semimodules behave as expected:

Let Δ, Δ' be Γ-semimodules, and let $d \in \mathbb{Z}$. Then

(a) $(\Delta + d)^* = \Delta^* - d$.

(b) $(\Delta \cup \Delta')^* = \Delta^* \cap (\Delta')^*$.

(c) $\Gamma^* = \Gamma$.

We found a describing formula:

Theorem

Let $I = \{0, i_1, \ldots, i_n\}$ be a Γ-lean set with gaps

$$i_k = \alpha\beta - a_k\alpha - b_k\beta$$

which are ordered increasingly with respect to $\prec$, and let $\Delta_I = \bigcup_{i \in I} (\Gamma + i)$, then

$$\Delta_I^* = (\Gamma + a_1\alpha) \cup \bigcup_{k=1}^{n-1} (\Gamma + a_{k+1}\alpha + b_k\beta) \cup (\Gamma + b_n\beta).$$

Corollary

$$(\Delta_I^*)^* = \Delta_I.$$

Let $\mathbb{F}$ be a field. Consider $\mathbb{F}[\Gamma]$, which may be identified with $R = \mathbb{F}[t^\alpha, t^\beta]$.

The counterparts of Γ-semimodules are the graded R-submodules of $\mathbb{F}[t]$.

Let $I = \{0, i_1, \ldots, i_n\}$ be a Γ-lean set with $i_k > 0$, and let $M_I = \sum_{i \in I} Rt^i$.

Consider the first syzygy of M_I, the kernel of the map

$$\begin{array}{ccc} \bigoplus_{i \in I} R(-i) & \xrightarrow{\varphi_1} & M_I \\ (f_0, \ldots, f_n) & \longmapsto & \sum_{k=0}^{n} f_k t^{i_k}. \end{array}$$

By a result of Piontkowski this kernel is generated by *bivectors*

$$(0,\ldots,0,t^{\gamma_k},0,\ldots,0,-t^{\gamma_m},0,\ldots,0) \text{ with } i_k+\gamma_k=i_m+\gamma_m.$$

In fact $n+1$ special bivectors are sufficient, namely

$$\begin{aligned}
f_0 &= (t^{(\beta-a_1)\alpha},-t^{b_1\beta},0,\ldots,0)\\
f_k &= (0,\ldots,0,t^{(a_k-a_{k+1})\alpha},-t^{(b_{k+1}-b_k)\beta},0,\ldots,0) \quad \text{for } k=1,\ldots,n-1\\
f_n &= (-t^{(\alpha-b_n)\beta},0,\ldots,0,t^{a_n\alpha}).
\end{aligned}$$

The degrees $\deg f_k = j_k$ are exactly the elements of the set J.

Hence, the support of the syzygy $\ker\varphi_1$ agrees with the object we called the syzygy of Δ_I.

The second step of the free resolution of M_I is the map

$$\bigoplus_{j\in J} R(-j) \xrightarrow{\varphi_2} \ker \varphi_1$$

$$(g_0,\dots,g_n) \longmapsto \sum_{k=0}^{n} g_k f_k.$$

The condition $\varphi_2(g_0,\dots,g_n)=0$ yields the following system of equations:

$$\begin{array}{lcllll} g_0 t^{(\beta-a_1)\alpha} & - & g_n t^{(\alpha-b_n)\beta} & = & 0 & \\ g_1 t^{(a_1-a_2)\alpha} & - & g_0 t^{b_1\beta} & = & 0 & \\ g_k t^{(a_k-a_{k+1})\alpha} & - & g_{k-1} t^{(b_k-b_{k-1})\beta} & = & 0 & \text{for } k=2,\dots,n-1 \\ g_n t^{a_n\alpha} & - & g_{n-1} t^{(b_n-b_{n-1})\beta} & = & 0 & \end{array}$$

We can solve for g_0 and get

$$\begin{aligned} g_k &= g_0 t^{b_k\beta-(a_1-a_{k+1})\alpha} \quad \text{for } k=1,\ldots,n-1 \\ g_n &= g_0 t^{b_n\beta-a_1\alpha}, \end{aligned}$$

as one easily checks by induction on k.

Hence $g=(g_0,\ldots,g_n)$ is an element of $\ker\varphi_2$ if and only if it can be written in the form

$$g = g_0\left(1, t^{b_1\beta-(a_1-a_2)\alpha}, \ldots, t^{b_{n-1}\beta-(a_1-a_n)\alpha}, t^{b_n\beta-a_1\alpha}\right)$$

with some $g_0 \in R$ such that all the entries are in R as well.

In the language of Γ-semimodules this means that we are looking for the dual of the semimodule

$$\widehat{\Delta_I} := \Gamma \cup \bigcup_{k=1}^{n-1} (\Gamma + (b_k\beta - (a_1 - a_{k+1})\alpha) \cup (\Gamma + b_n\beta - a_1\alpha).$$

The Theorem above implies

$$\widehat{\Delta_I}^* = a_1\alpha + \Delta_I,$$

hence $\ker \varphi_2$ equals

$$\left\{ g_0 \left(1, t^{b_1\beta-(a_1-a_2)\alpha}, \ldots, t^{b_{n-1}\beta-(a_1-a_n)\alpha}, t^{b_n\beta-a_1\alpha}\right) \mid g_0 \in M_I \cdot t^{a_1\alpha} \right\},$$
$$\cong M_I.$$

Therefore we have shown:

Theorem

Let $\Gamma = \langle \alpha, \beta \rangle$ be a numerical semigroup. Let I be a Γ-lean set, and let $M_I = \sum_{i \in I} Rt^i$ with $R = \mathbb{F}[t^\alpha, t^\beta]$. Then the minimal graded free resolution of M_I is —up to a shift— periodic of period 2.

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So we recover part of a result of Eisenbud (TAMS 1980):

Theorem

Let A be a regular local ring, $x \in A$, and let $B = A/x$. If $\mathbf{F} : \cdots \to F_1 \to F_0$ is the minimal B-free resolution of a finitely generated B-module M, then:

(i) $\mathbf{F}$ becomes periodic of period 2 after $\dim A + 1$ steps;

(ii) $\mathbf{F}$ is periodic (necessarily of period 2) iff M is a maximal CM B-module with no free summand.