On symmetric oversemigroups of a numerical semigroup

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All the new results in this talk are obtained with Lance Bryant.

Look at one of the (apparently) simplest commutative monoid: a numerical semigroup S, i.e. an additive submonoid of $\mathbb{N}$ with zero and finite complement in $\mathbb{N}$.

f_S is the *Frobenius number* of S, that is the greatest integer which does not belong to S.

A *relative ideal* of S is a nonempty subset I of $\mathbb{Z}$ (which is the quotient group of S) such that $I + S \subseteq I$ and $I + s \subseteq S$, for some $s \in S$. A relative ideal $I \subset S$ is an *integral ideal*.

If I, J are (relative) ideals of S, then the following is a relative ideals too:

$$I - J = \{z \in \mathbb{Z} \mid z + J \subseteq I\}$$

In particular, if $I = S$ and $J = T$ is an oversemigroup of S then

$$S - T$$

is the biggest integral ideal which S and T share.

For each relative ideal I of S we have

$$I \subset \bigcap_{I \subset z+S} z+S = S-(S-I)$$

and I is *bidual* whenever $I = S-(S-I)$

Define

$$I \sim J$$

if $J = z + I$, for some $z \in \mathbb{Z}$.

- If $I \sim J$ then I is bidual if and only if J is bidual.
- In each equivalence class of ideals of S we can choose an ideal I, $S \subset I \subset \mathbb{N}$

The *blowup* of an ideal I of S is

$$B(I) = \bigcup_{n \in \mathbb{N}} (nI - nI) = (nI - nI), \text{ for } n >> 0$$

If $I \sim J$, then $B(I) = B(J)$

If $S \subset I \subset \mathbb{N}$, then

$(nI - nI) = ((n+1)I - (n+1)I) \Leftrightarrow nI = (n+1)I \Leftrightarrow nI$ is a semigroup

Thus, if $S \subset I \subset \mathbb{N}$, then

$$I \subset 2I \subset \cdots \subset nI = B(I)$$

and $B(I)$ is the smallest semigroup containing I.

In terms of generators: If $a_1 < a_2 < \cdots < a_m$ and

$$I = (a_1 + S) \cup (a_2 + S) \cup \cdots \cup (a_m + S)$$

is an ideal of S, then the blowup of I is the semigroup

$$B(I) = \langle S, a_2 - a_1, \ldots, a_m - a_1 \rangle$$

In particular, if $M = S \setminus \{0\}$ is the maximal ideal of

$$S = \langle a_1 = e, a_2, \ldots, a_m \rangle$$

then

$$B(M) = \langle e, a_2 - e, \ldots, a_m - e \rangle$$

A particular relative ideal of S plays a special role. It is the *canonical ideal*

$$\Omega = \{f_S - x \mid x \in \mathbb{Z} \setminus S\}$$

So

$$x \notin S \Rightarrow f_S - x \in \Omega$$

$$x \in S \Rightarrow f_S - x \notin \Omega$$

It is

$$S \subseteq \Omega \subseteq \mathbb{N}$$

and the blowup $B(\Omega)$ is the smallest semigroup containing Ω

Example

$$S = \{0, 4, 7, 8, 11, 12, \rightarrow\}$$

Frobenius number $f_S = 10$

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$$

x is *symmetric* to $f_S - x = 10 - x$
0 is symmetric to 10
1 is symmetric to 9
2 is symmetric to 8 ...

$$\Omega = \{\text{black}\} \cup \{\text{red}\}$$

$B(\Omega) = \mathbb{N}$

Facts:

- For each ideal I of S:

$$(\Omega - I) = \{f_S - x \mid x \in \mathbb{Z} \setminus I\}$$

- $\Omega - (\Omega - I) = I$, for each ideal I of S.

- Let Ω' be an ideal of S. Then

$$\Omega' - (\Omega' - I) = I$$

  for each ideal I of S if and only if $\Omega' = z + \Omega$, for some $z \in \mathbb{Z}$.

S is called *symmetric* if one of the equivalent conditions holds:

1. $f_S - x \in S$ for each $x \in \mathbb{Z} \setminus S$
2. Each ideal I of S is bidual, i.e. $S - (S - I) = I$
3. $\Omega = S$

Facts:

- Every two-generated semigroup $S = \langle a, b \rangle$ is symmetric
- Every semigroup of values of a plane algebroid branch is symmetric.
  Example. $S = \langle 4, 6, 13 \rangle$ is the semigroup of values of $\mathbb{C}[[t^4, t^6 + t^7]]$
- Symmetric semigroups correspond to one-dimensional Gorenstein rings

Thus it seems relevant to see how many bidual ideals there are in a semigroup, to see how far it is from being so good as a symmetric one.

Lemma

Let S be a semigroup, I an ideal of S and $z \in \mathbb{Z}$. Then

$$I \not\subset S - z \quad \Leftrightarrow \quad f_S - z \in I + \Omega$$

Proof. Suppose that $I \not\subset S - z$. Then there exists an $i \in I$ such that $i + z = x \notin S$. So $f_S - x \in \Omega$. Hence

$$i + (f_S - x) = i + (f_S - (i + z)) = f_S - z \in I + \Omega$$

Conversely, let $f_S - z \in I + \Omega$. Thus, $f_S - z = i + b$, where $i \in I$ and $b \in \Omega$, i.e. $f_S - (z + i) = b \in \Omega$. Thus $z + i \notin S$, i.e. $i \notin S - z$ and so $I \not\subset S - z$. $\square$

By the Lemma above, we have a constructive way to get $I^{**} = S - (S - I)$, from a given ideal I:

- construct $I + \Omega$ and take $x \in \mathbb{Z} \setminus (I + \Omega)$.
- for each $x \in \mathbb{Z} \setminus I + \Omega$, take the symmetric $z = f_S - x$.
- intersect all the principal ideals $S - z$

If the ideal I is chosen such that $S \subset I \subset \mathbb{N}$, then $I + \Omega \subset \mathbb{N}$ and also $I^{**} \subset \mathbb{N}$. Among the principal ideals $S - z$ above, those with $z \geq f_S + 1$ give components which contain $\mathbb{N}$. Thus I^{**} can be described as a finite intersection

$$\mathbb{N} \cap \{(S - z); I \subset S - z, 0 \leq z \leq f_S\}$$

Proposition

$$I^{**} \subset I + B(\Omega)$$

for all ideals I of S.

Proof. If $x \in I^{**}$, then x is in all the principal ideals containing I. Since $x \notin S - (f_S - x)$ (otherwise $f_S \in S$), we have

$$I \not\subset S - (f_S - x)$$

By the Lemma $f_S - (f_S - x) = x \in I + \Omega \subset I + B(\Omega)$. $\square$

Theorem (A)

Let $S \subset T$ be semigroups. Then the following are equivalent:
(1) $\Omega \subset T$
(2) *Every ideal of T is a bidual S-ideal.*

Proof. (1) $\Rightarrow$ (2): Let I be an ideal of T. Since $\Omega \subset T$, I is also an ideal of $B(\Omega)$. Thus by previous Proposition

$$I \subset I^{**} \subset I + B(\Omega) = I$$

and $I = I^{**}$.

Theorem (A)

Let $S \subset T$ be semigroups. Then the following are equivalent:
(1) $\Omega \subset T$
(2) *Every ideal of T is a bidual S-ideal.*

(2) $\Rightarrow$ (1): Suppose that $\Omega \not\subset T$. Then there exists $x \notin T$ such that $f_S - x \notin S$. We have that $U = T \cup \{x+1, \rightarrow\}$ is a semigroup containing T with Frobenius number $f_T = x$, and we claim that U is not a bidual S-ideal. It is enough to show that each principal ideal of S containing U also contains x. Let $z + S$ be a principal relative ideal of S containing U.

$$z + f_S \notin z + S \Rightarrow z + f_S \notin U \Rightarrow z + f_S \leq x$$

If $z + f_S = x$, then, since $f_S - x \notin S$, we have $0 = x - x = z + f_S - x \notin z + S$. This is a contradiction since $0 \in U \subset z + S$. Therefore, $z + f_S < x$ and $x \in z + S$. We conclude that U is a semigroup containing T that is not a bidual S-ideal. □

For a semigroup S, with maximal ideal $M = S \setminus \{0\}$ we have in general the following inclusions:

$$S \quad \subset \quad \begin{array}{c} M - M \quad \subset \quad B(M) \\ \\ \Omega \quad \subset \quad B(\Omega) \end{array}$$

S is *almost symmetric* if $\Omega \subset M - M$. The following conditions are equivalent:

- S is almost symmetric but not symmetric
- $B(\Omega) = M - M$
- $B(\Omega) = \Omega \cup \{f_S\}$

So for S almost symmetric but not symmetric we have

$$S \subset \Omega \subset M - M = B(\Omega) = \Omega \cup \{f_S\}$$

Example

$$S = \langle 5, 8, 9, 12 \rangle = \{0, 5, 8, 9, 10, 12, \rightarrow\}$$

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \rightarrow$$

is almost symmetric because

$$\Omega = \{0, 4, 5, 7, 8, 9, 10, 12, \rightarrow\} = \langle 0, 4, 7 \rangle$$

$$4, 7 \in M - M = B(\Omega) = \{0, 4, 5, 7, \rightarrow\} = \Omega \cup \{f_S = 11\}$$

We can now characterize almost symmetric semigroups in two ways:

Proposition

The following are equivalent:

1. *S is almost symmetric*
2. *Every ideal of $M - M$ is a bidual S-ideal*
3. *Every non-principal bidual S-ideal survives as ideal in $B(\Omega)$ (i.e. $I + B(\Omega) = I$)*

Proof. $1 \Leftrightarrow 2$: By Theorem A, because $\Omega \subset T = M - M$.
$1 \Rightarrow 3$: It needs a little work ...
$3 \Rightarrow 1$: If $S = \mathbb{N}$, then S is almost symmetric. If $S \neq \mathbb{N}$, then M is a non-principal bidual S-ideal. By 3), $M + B(\Omega) \subset M$ and so $\Omega \subset B(\Omega) \subset M - M$. $\square$

The multiplicity *e* of S is the smallest positive integer in S. Recall that every relative ideal I of a semigroup S of multiplicity e needs at most e generators and I is called *maximally generated* if I needs e generators.

Corollary

If S is an almost symmetric semigroup, then every maximally generated ideal of S is bidual.

Proof. It is known (V. B. - K. Pettersson , 1995):

$$I \text{ is maximally generated} \Leftrightarrow I + B(M) = I$$

$M - M \subset B(M)$. Thus if I is maximally generated, i.e. an ideal in $B(M)$, it is also an ideal of $M - M$, i.e. by the previous Proposition is a bidual S-ideal. □

Corollary

If S is an almost symmetric semigroup of maximal embedding dimension (i.e. M is maximally generated), then every maximally generated ideal of S is bidual and, conversely, every non principal bidual ideal is maximally generated.

Proof. $M - M = B(M)$. □

The following is a result which received attention by several people and has been formulated in ring theory context, for any Cohen Macaulay ring ....

Theorem (V. B. - R. Fröberg, 1997)

S is an almost symmetric semigroup of maximal embedding dimension if and only if the oversemigroup $T = M - M$ is symmetric.

In the situation of the theorem we have three facts:

- T is symmetric
- $\Omega \subset T$ (because S is almost symmetric)
- $S - T = M = T + e = T + (f_S - f_T)$ (because S is maximal embedding dimension)

Theorem

Let $S \subset T$ be semigroups and suppose that one of the following condition holds:

- *T is symmetric*
- $\Omega \subset T$
- $S - T = T + \delta$, *where* $\delta = f_S - f_T$

Then the remaining two conditions are equivalent.

An ideal I of a semigroup S is *stable* if it is principal in $I - I$.
Easy fact: Let $S \subset T$ be semigroups and let T be a bidual S-ideal. Then

$$S - T = T + \delta \Leftrightarrow S - T \text{ is a stable ideal}$$

When an oversemigroup T of S is symmetric?

Corollary

Let $S \subset T$ be semigroups.

1. *If $\Omega \subset T$, then T is symmetric if and only if $S - T$ is a stable ideal.*
2. *If S is almost symmetric and T is a bidual S-ideal, then T is symmetric if and only if $S - T$ is a stable ideal.*
3. *If S is symmetric, then T is symmetric if and only if $S - T$ is a stable ideal.*

Can we say something more precise, in case the oversemigroup T is the blowup of the maximal ideal?

Example. A symmetric semigroup may have a blowup of the maximal ideal which is not even almost symmetric:

$S = \langle 8, 11, 14, 18, 21 \rangle$ (symmetric)

$B(M) = \langle 3, 8, 10 \rangle$ (not almost symmetric)

Recall that

$$S = \langle a_1, \ldots, a_\nu \rangle \ \Rightarrow \ B(M) = \langle a_1, a_2 - a_1, \ldots, a_\nu - a_1 \rangle$$

However in special cases, we have a positive answer:

If $a \in S$, set $\mathrm{d}(a, S) :=$ number of S-factorizations of a.

Proposition

Let $S = \langle a_1, \dots, a_\nu \rangle$ and consider the following conditions:

1. *S is 2-generated, i.e. $\nu = 2$*
2. *S is symmetric and for each i, $i = 1 \dots, \nu$ and for each $w \in \mathrm{Ap}(S, a_i)$, $\mathrm{d}(w, S) = 1$*
3. *S is symmetric and for each $w \in \mathrm{Ap}(S, a_1)$, $\mathrm{d}(w, S) = 1$*
4. *S is symmetric, M-pure and additive*
5. *$B(M)$ is symmetric*

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$

Remark. All the implications in the Proposition are strict, e.g. (3) $\not\Rightarrow$ (2): A semigroup satisfies condition (2) of Proposition if and only if there exist paiswise relatively prime numbers $\ell_1, \ell_2, \ldots, \ell_\nu$ such that

$$a_i = (\prod_{j-1}^{\nu} \ell_j)/\ell_i$$

Thus we have $\prod_{i=1}^{\nu} a_i = (\prod_{i=1}^{\nu} \ell_i)^{\nu-1}$, i.e. the $(\nu - 1)$-the root of the product of the generators is an integer.
The semigroup of an algebroid plane branch, as $\mathbb{C}[[t^4, t^6 + t^7]]$, which is $S = \langle 4, 6, 13 \rangle$ satisfies condition (3) (wellknown by Zariski) , but $4 \cdot 6 \cdot 13 = 312$ is not a square, so S does not satisfy condition (2).