Gapsets and numerical semigroups

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Introduction

• Let $n_g =$ the number of numerical semigroups S of genus g, *i.e.* with $|\mathbb{N} \setminus S| = g$ gaps.

• Values of n_g for $g = 0, 1, 2, \ldots$:

$$1, 1, 2, 4, 7, 12, 23, 39, 67, 118, 204, 343, 592, 1001, 1693, \ldots$$

Conjecture (Maria Bras-Amorós, 2008)

$$n_g \geq n_{g-1} + n_{g-2} \ \ \textit{for all} \ g \geq 2.$$

Widely open. A weaker version:

Conjecture

$$n_g \geq n_{g-1} \ \ \textit{for all} \ g \geq 1.$$

Zhai [2013]: asymptotically true. Still open for small g.

Displaying a numerical semigroup

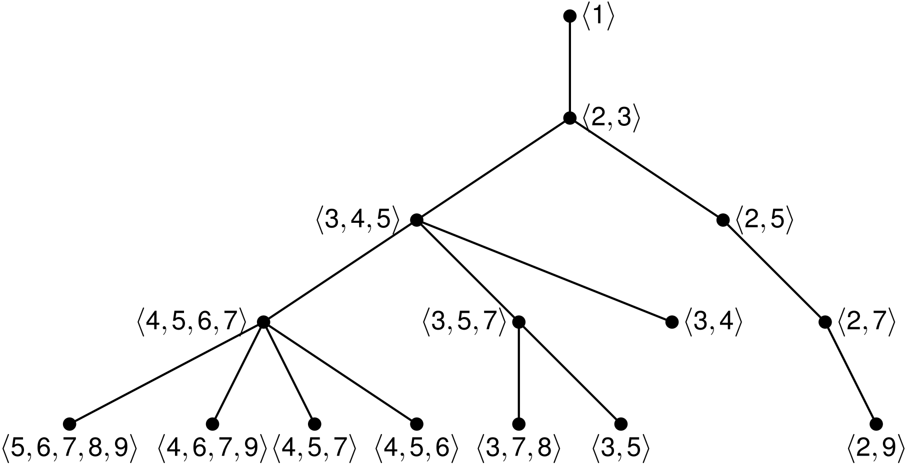
A numerical semigroup S may be displayed in quite different ways:

(1) As $S = \langle a_1, \dots, a_n \rangle$ with $\gcd(a_1, \dots, a_n) = 1$.

(2) As a cofinite submonoid of $\mathbb{N}$, e.g. by giving $\mathbb{N} \setminus S$.

- The transition $(1) \leftrightarrow (2)$ is costly. (E.g. the Frobenius problem.)
- Wilf's conjecture involves both $(1), (2)$. Maybe this is why it is hard?
- So, *whenever possible*, avoid mixing $(1), (2)$.
- The conjectures on n_g only concern representation (2).
- But...

The tree of numerical semigroups



Gapsets

Recall

• A prime ideal in a ring R is an ideal $P \subset R$ such that

- $xy \in P \implies x \in P$ or $y \in P$
- $1 \notin P$

Definition

A gapset is a finite subset $G \subset \mathbb{N}$ such that

- $x + y \in G \implies x \in G$ or $y \in G$
- $0 \notin G$

The first condition says that $\mathbb{N} \setminus G$ is stable under $+$:

$$x \notin G \text{ and } y \notin G \implies x + y \notin G.$$

Thus, a gapset G is just the **set of gaps** of a numerical semigroup $S \subseteq \mathbb{N}$, where $S = \mathbb{N} \setminus G$.

Terminology

Definition

Let G be a gapset. The multiplicity m, genus g, Frobenius number F and conductor c of G are those of $S = \mathbb{N} \setminus G$.

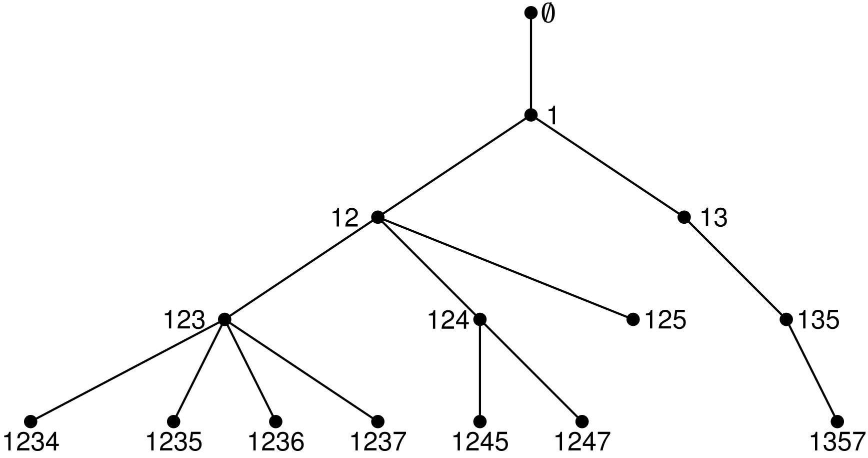
For instance, the gapset $G = \{1, 2, 4, 7\}$ has

$$m = 3,\ g = 4,\ F = 7 \text{ and } c = 8.$$

It corresponds to

$$S = \mathbb{N} \setminus G = \{0, \quad, \quad, 3, \quad, 5, 6, \quad, 8, 9, 10, \rightarrow\} = \langle 3, 5 \rangle.$$

The tree of gapsets



The depth

Definition

Let G be a gapset of multiplicity m and conductor c. The depth of G is the number $q = \lceil c/m \rceil$.

Thus,

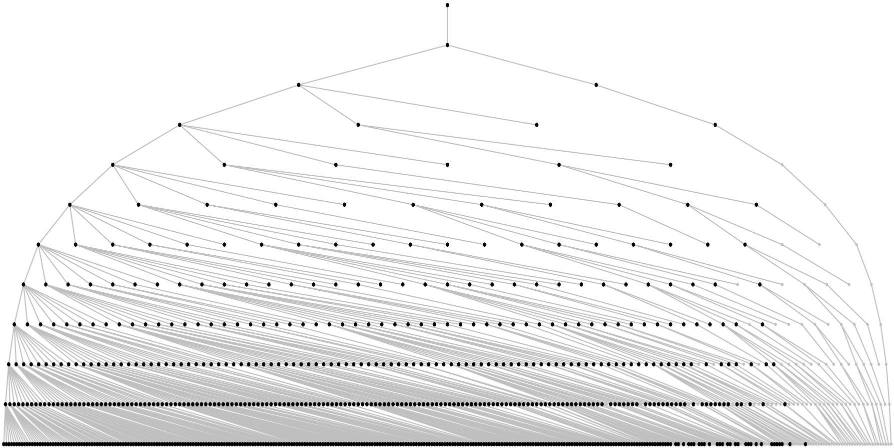
$$\begin{aligned} q = 1 &\iff m = c \iff S = \{0\} \cup [m, \infty[\\ q = 2 &\iff m < c \leq 2m \\ q = 3 &\iff 2m < c \leq 3m \\ &\dots \end{aligned}$$

Denote by

- $\Gamma = \{\text{all gapsets}\}$
- $\Gamma(g) = \{\text{gapsets of genus } g\}$
- $\Gamma'(g) = \{\text{gapsets of genus } g \text{ and depth } q \leq 3\}$

Zhai [2013]: $\boxed{|\Gamma'(g)|/|\Gamma(g)| \to 1 \text{ as } g \to \infty}$

Up to genus 11



- **Black dots**: $q \leq 3$
- Small gray dots: $q \geq 4$

Main result

Recall:

- $\Gamma = \{\text{all gapsets}\}$
- $\Gamma(g) = \{\text{gapsets of genus } g\}$
- $\Gamma'(g) = \{\text{gapsets of genus } g \text{ and depth } q \leq 3\}$

Thus, $|\Gamma(g)| = n_g$ for all g.

Maria's conjecture: $\boxed{|\Gamma(g)| \geq |\Gamma(g-1)| + |\Gamma(g-2)|}$

Our main result: $\boxed{|\Gamma'(g)| \geq |\Gamma'(g-1)| + |\Gamma'(g-2)|}$

The canonical partition

Lemma

Let G be a gapset of multiplicity m and depth q. Let $G_0 = [1, m-1]$ and

$$G_i = G \cap \underbrace{[im+1, (i+1)m-1]}_{m-1}.$$

for all $i \geq 0$. Then $G = G_0 \sqcup G_1 \sqcup \cdots \sqcup G_{q-1}$.

This is the canonical partition of G.

Proof.

Let $S = \mathbb{N} \setminus G$. Since $m = \min S^*$, all of $[1, m-1]$ are gaps of S, so $[1, m-1] \subseteq G$. Since $m\mathbb{N} \subset S$, we have $m\mathbb{N} \cap G = \emptyset$. Finally, since $qm \geq \text{conductor}$, we have $G \subseteq [1, qm-1]$. Hence $G_j = \emptyset$ for $j \geq q$. □

The associated filtration

Lemma

Let G be a gapset of multiplicity m and depth q. Let $G_i = G \cap [im+1, (i+1)m-1]$. Then $G_{i+1} \subseteq m + G_i$ for all $i \geq 0$.

Proof.

Let $x \in G_{i+1} \subseteq [(i+1)m+1, (i+2)m-1]$. Then

$$x - m \in [im+1, (i+1)m-1].$$

Now $x - m \in G$ since $x = m + (x-m)$ and $m \notin G$. So $x - m \in G_i$. □

Notation

Denote $F_i = -im + G_i$ for all $i \geq 0$. Then $F_i \subseteq [1, m-1]$ and

$$[1, m-1] = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_{q-1}.$$

We call $(F_0, F_1, \ldots, F_{q-1})$ the gapset filtration associated to G.

The 23 NS or gapsets of genus 6

- Minimal generators:

$\langle 2,13\rangle; \langle 3,7\rangle; \langle 3,8,13\rangle; \langle 3,10,11\rangle; \langle 4,5\rangle; \langle 4,6,9\rangle; \langle 4,6,11,13\rangle; \langle 4,7,9\rangle;$
$\langle 4,7,10,13\rangle; \langle 4,9,10,11\rangle; \langle 5,6,7\rangle; \langle 5,6,8\rangle; \langle 5,6,9,13\rangle; \langle 5,7,8,9\rangle;$
$\langle 5,7,8,11\rangle; \langle 5,7,9,11,13\rangle; \langle 5,8,9,11,12\rangle; \langle 6,7,8,9,10\rangle; \langle 6,7,8,9,11\rangle;$
$\langle 6,7,8,10,11\rangle; \langle 6,7,9,10,11\rangle; \langle 6,8,9,10,11,13\rangle; \langle 7,8,9,10,11,12,13\rangle.$

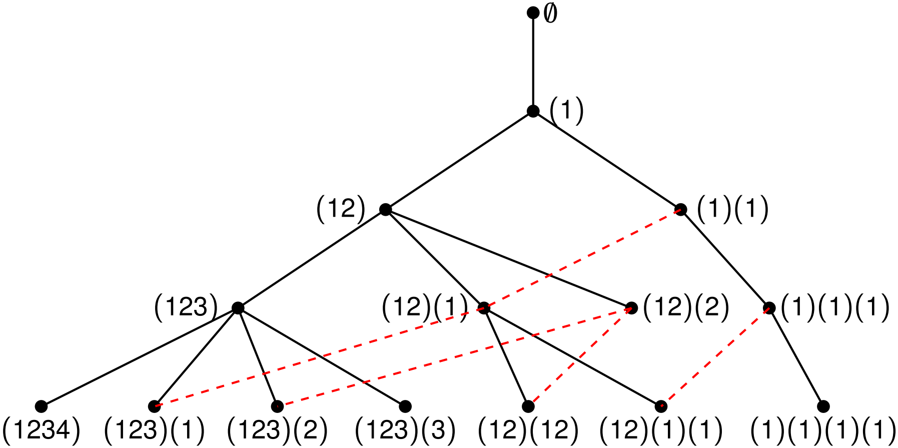
- Associated gapset filtration:

$(1)^6; (12)^2(2)^2; (12)^2(1)^2; (12)^3; (123)(23)(3); (123)(13)(3); (123)(13)(1);$
$(123)(12)(2); (123)(12)(1); (123)^2; (1234)(34); (1234)(24); (1234)(23);$
$(1234)(1)^2; (1234)(14); (1234)(13); (1234)(12); (12345)(5); (12345)(4);$
$(12345)(3); (12345)(2); (12345)(1); (123456).$

Remark

For a gapset filtration $(F_0, F_1, \ldots, F_{q-1})$, the genus is given by $g = |F_0| + |F_1| + \cdots + |F_{q-1}|$.

The graph of gapset filtrations



Main result

Theorem

Let $F = (F_0, F_1, F_2)$ be a gapset filtration of depth $q \leq 3$ and multiplicity $m \geq 2$, so that $F_0 = [1, m-1] \supseteq F_1 \supseteq F_2$. Let g be it genus. Let

$$\begin{aligned} F' &= (F_0 \sqcup \{m\}, F_1, F_2) \\ F'' &= (F_0 \sqcup \{m\}, F_1 \sqcup \{m\}, F_2). \end{aligned}$$

Then F', F'' are gapset filtrations, of genus $g+1, g+2$ respectively.

Proof.

Not one-line but straightforward, no brilliant idea needed. □

Corollary

For all $g \geq 2$, one has

$$|\Gamma'(g)| \geq |\Gamma'(g-1)| + |\Gamma'(g-2)|.$$

Proof.

We have just defined two injective maps

$$\begin{array}{rcl} \Gamma'(g-1) & \to & \Gamma'(g), \quad (A_0, A_1, A_2) \mapsto (A_0 \sqcup \{m_1\}, A_1, A_2) \\ \Gamma'(g-2) & \to & \Gamma'(g), \quad (B_0, B_1, B_2) \mapsto (B_0 \sqcup \{m_2\}, B_1 \sqcup \{m_2\}, B_2). \end{array}$$

Their images in $\Gamma'(g)$ are disjoint. □

Gapsets of multiplicity 3 and 4

Theorem (Case $m = 3$)

Let $a \geq 1, b \geq 0$. *Then*

- $(12)^a(1)^b$ *is a gapset filtration if and only if* $b \leq a+1$
- $(12)^a(2)^b$ *is a gapset filtration if and only if* $b \leq a$.

Theorem (Case $m = 4$)

Let $a \geq 1, b, c \geq 0$. *Then*

- $(123)^a(12)^b(1)^c$ *is a gapset filtration* $\iff b, c \leq a+1$
- $(123)^a(12)^b(2)^c$ *is a gapset filtration* $\iff b+c \leq a+1,\ c \leq a+b$
- $(123)^a(13)^b(1)^c$ *is a gapset filtration* $\iff c \leq a+1$
- $(123)^a(13)^b(3)^c$ *is a gapset filtration* $\iff c \leq a$
- $(123)^a(23)^b(2)^c$ *is a gapset filtration* $\iff b+c \leq a$
- $(123)^a(23)^b(3)^c$ *is a gapset filtration* $\iff b, c \leq a$.

Corollary

For all $g \geq 1$, there are explicit injections

$$\begin{array}{rcl} \Gamma(g-1, m=3) & \longrightarrow & \Gamma(g, m=3) \\ \Gamma(g-1, m=4) & \longrightarrow & \Gamma(g, m=4). \end{array}$$

In particular, $|\Gamma(g-1,m)| \leq |\Gamma(g,m)|$ *for* $m = 3,4$.

Result first obtained by [P. A. García-Sánchez, D. Marín-Aragón and A. M. Robles-Pérez] with computational methods on the Kunz polytope.

Theorem (PAGS, DMA & AMRP, +2018)

$|\Gamma(g-1,m)| \leq |\Gamma(g,m)|$ *for* $m = 3,4,5$.

The case $m \geq 6$ remains open.

Grazie mille per la sua attenzione :-)