Studying the catenary and the tame degrees in 4-generated symmetric non complete intersection numerical semigroups

Caterina Viola

International meeting on numerical semigroups with applications
Levico Terme - July 2016
Every numerical semigroup is finitely generated and admits a unique minimal system of generators.

Example

\[ S = \langle 5, 8, 11, 14, 17 \rangle \]

\[ e(S) = 5 \]
Numerical Semigroups

Every numerical semigroup is finitely generated and admits a unique minimal system of generators. The cardinality of this minimal system of generators is called embedding dimension of $S$ and will be denoted by $e(S)$. 

Example

$S = \langle 5, 8, 11, 14, 17 \rangle$

$e(S) = 5$
Every numerical semigroup is finitely generated and admits a unique minimal system of generators. The cardinality of this minimal system of generators is called embedding dimension of \( S \) and will be denoted by \( e(S) \).

Example

\[
S = \langle 5, 8, 11, 14, 17 \rangle \\
\text{e}(S) = 5
\]
Setup

Let $S = \langle n_1, \ldots, n_p \rangle$ be a $p$-generated numerical semigroup.
Let $S = \langle n_1, \ldots, n_p \rangle$ be a $p$-generated numerical semigroup.

$$
\varphi: \mathbb{N}^p \to S, \varphi(x_1, \ldots, x_p) = x_1 n_1 + \cdots + x_p n_p
$$

$$
\ker \varphi = \{(x, y) \in \mathbb{N}^p \times \mathbb{N}^p \mid \varphi(x) = \varphi(y)\}
$$
Let $S = \langle n_1, \ldots, n_p \rangle$ be a $p$-generated numerical semigroup.

$$\varphi: \mathbb{N}^p \rightarrow S, \quad \varphi(x_1, \ldots, x_p) = x_1 n_1 + \cdots + x_p n_p$$

$$\ker \varphi = \{(x, y) \in \mathbb{N}^p \times \mathbb{N}^p | \varphi(x) = \varphi(y)\}$$

The factorization set of $s \in S$ is the set of the solutions to

$$x_1 n_1 + \cdots + x_p n_p = s,$$

$$Z(s) = \{x \in \mathbb{N}^e | \varphi(x) = s\} = \varphi^{-1}(s).$$
Setup

Let $S = \langle n_1, \ldots, n_p \rangle$ be a $p$-generated numerical semigroup.

$$\varphi : \mathbb{N}^p \to S, \varphi(x_1, \ldots, x_p) = x_1 n_1 + \cdots + x_p n_p$$

$\ker \varphi = \{(x, y) \in \mathbb{N}^p \times \mathbb{N}^p | \varphi(x) = \varphi(y)\}$

- The factorization set of $s \in S$ is the set of the solutions to $x_1 n_1 + \cdots + x_p n_p = s$, $Z(s) = \{x \in \mathbb{N}^e | \varphi(x) = s\} = \varphi^{-1}(s)$.
- The length of $x \in Z(s)$ is $|x| = x_1 + \cdots + x_p$. 
Let $S = \langle n_1, \ldots, n_p \rangle$ be a $p$-generated numerical semigroup.

$$\varphi : \mathbb{N}^p \to S, \quad \varphi(x_1, \ldots, x_p) = x_1 n_1 + \cdots + x_p n_p$$

$$\ker \varphi = \{(x, y) \in \mathbb{N}^p \times \mathbb{N}^p \mid \varphi(x) = \varphi(y)\}$$

- The **factorization set** of $s \in S$ is the set of the solutions to $x_1 n_1 + \cdots + x_p n_p = s$, $Z(s) = \{x \in \mathbb{N}^e \mid \varphi(x) = s\} = \varphi^{-1}(s)$.
- The **length** of $x \in Z(s)$ is $|x| = x_1 + \cdots + x_p$.
- Given another factorization $y = (y_1, \ldots, y_p)$, the **distance** between $x$ and $y$ is
  $$d(x, y) = \max\{|x - \gcd(x, y)|, |y - \gcd(x, y)|\},$$
  where $\gcd(x, y) = (\min\{x_1, y_1\}, \ldots, \min\{x_p, y_p\})$. 
Setup

Let $S = \langle n_1, \ldots, n_p \rangle$ be a $p$-generated numerical semigroup.

$$\varphi : \mathbb{N}^p \to S, \quad \varphi(x_1, \ldots, x_p) = x_1 n_1 + \cdots + x_p n_p$$

$$\ker \varphi = \{(x, y) \in \mathbb{N}^p \times \mathbb{N}^p \mid \varphi(x) = \varphi(y)\}$$

- The **factorization set** of $s \in S$ is the set of the solutions to $x_1 n_1 + \cdots + x_p n_p = s$, $Z(s) = \{x \in \mathbb{N}^e \mid \varphi(x) = s\} = \varphi^{-1}(s)$.
- The **length** of $x \in Z(s)$ is $|x| = x_1 + \cdots + x_p$.
- Given another factorization $y = (y_1, \ldots, y_p)$, the **distance** between $x$ and $y$ is
  $$d(x, y) = \max\{|x - \gcd(x, y)|, |y - \gcd(x, y)|\},$$
  where $\gcd(x, y) = (\min\{x_1, y_1\}, \ldots, \min\{x_p, y_p\})$.
- A **presentation** of $S$ is a congruence $\sigma$ on $\mathbb{N}^p$ contained in $\ker \varphi$. 
The graph $G_n$

Let $S = \langle n_1, \ldots, n_p \rangle$ be a $p$-generated numerical semigroup, $n \in S$ we define the graph $G_n = (V_n, E_n)$ such that, for any $1 \leq i, j \leq p$, $i \neq j$:

- $n_i \in V_n \iff n - n_i \in S$;
- $(n_i, n_j) \in E_n \iff n - (n_i + n_j) \in S$. 

Example: $S = \langle 3, 5, 7 \rangle$

$G_{13}$

$G_6$

$G_{10}$
The graph $G_n$

Let $S = \langle n_1, \ldots, n_p \rangle$ be a $p$-generated numerical semigroup, $n \in S$ we define the graph $G_n = (V_n, E_n)$ such that, for any $1 \leq i, j \leq p$, $i \neq j$:

- $n_i \in V_n \iff n - n_i \in S$;
- $(n_i, n_j) \in E_n \iff n - (n_i + n_j) \in S$.

Example: $S = \langle 3, 5, 7 \rangle$

![Graphs $G_{13}$, $G_6$, $G_{10}$]
The graph $G_n$

Let $S = \langle n_1, \ldots, n_p \rangle$ be a $p$-generated numerical semigroup, $n \in S$ we define the graph $G_n = (V_n, E_n)$ such that, for any $1 \leq i, j \leq p$, $i \neq j$:

- $n_i \in V_n \iff n - n_i \in S$;
- $(n_i, n_j) \in E_n \iff n - (n_i + n_j) \in S$.

Example: $S = \langle 3, 5, 7 \rangle$

We define

$$\text{Betti}(S) = \{ n \in S \mid G_n \text{ is not connected} \}.$$
**Minimal presentations**

A minimal presentation is a presentation that is minimal with respect to set inclusion (in this setting it is also minimal with respect to cardinality).
Minimal presentations

A minimal presentation is a presentation that is minimal with respect to set inclusion (in this setting it is also minimal with respect to cardinality).

A numerical semigroup is uniquely presented if for every two of its minimal presentations $\sigma$ and $\tau$ and every $(a, b) \in \sigma$, either $(a, b) \in \tau$ or $(b, a) \in \tau$.
Minimal presentations

A **minimal presentation** is a presentation that is minimal with respect to set inclusion (in this setting it is also minimal with respect to cardinality).

A numerical semigroup is **uniquely presented** if for every two of its minimal presentations $\sigma$ and $\tau$ and every $(a, b) \in \sigma$, either $(a, b) \in \tau$ or $(b, a) \in \tau$.

For each $n \in S$ let $C_1, \ldots, C_t$ be the connected components of $G_n$ ($\mathcal{R}$-classes)

- pick $\alpha_i \in C_i$;
- set $\sigma_n = \{(\alpha_1, \alpha_2), (\alpha_1, \alpha_3), \ldots, (\alpha_1, \alpha_t)\}$.

$$\sigma = \bigcup_{n \in S} \sigma_n$$

is a (minimal) presentation of $S$. 
Minimal presentations

A **minimal presentation** is a presentation that is minimal with respect to set inclusion (in this setting it is also minimal with respect to cardinality).

A numerical semigroup is **uniquely presented** if for every two of its minimal presentations $\sigma$ and $\tau$ and every $(a, b) \in \sigma$, either $(a, b) \in \tau$ or $(b, a) \in \tau$.

For each $n \in S$ let $C_1, \ldots, C_t$ be the connected components of $G_n$ ($R$-classes)
- pick $\alpha_i \in C_i$;
- set $\sigma_n = \{(\alpha_1, \alpha_2), (\alpha_1, \alpha_3), \ldots, (\alpha_1, \alpha_t)\}$.

$$\sigma = \bigcup_{n \in S} \sigma_n$$

is a (minimal) presentation of $S$.

Actually,

$$\sigma = \bigcup_{b \in \text{Betti}(S)} \sigma_b.$$
\( \rho \) minimal presentation for \( S \), then \( |\rho| \geq e(S) - 1 \).

A numerical semigroup \( S \) is a complete intersection, (CI), if the cardinality of any of its minimal presentations is equal to \( e(S) - 1 \).

A numerical semigroup \( S \) is irreducible if it cannot be expressed as the intersection of two numerical semigroups properly containing it.

A numerical semigroup \( S \) is symmetric if it is irreducible and the Frobenius number, \( F(S) \), is odd.

Proposition

\( S \) is a complete intersection \( \Rightarrow \) \( S \) symmetric.

If \( e(S) \leq 3 \), \( S \) is a complete intersection \( \iff \) \( S \) symmetric (Herzog).
\( \rho \) minimal presentation for \( S \), then \( |\rho| \geq e(S) - 1 \).

A numerical semigroup \( S \) is a complete intersection, (CI), if the cardinality of any of its minimal presentations is equal to \( e(S) - 1 \).

A numerical semigroup is irreducible if it cannot be expressed as the intersection of two numerical semigroups properly containing it.

Proposition: \( S \) is a complete intersection \( \implies \) \( S \) symmetric.

If \( e(S) \leq 3 \), \( S \) is a complete intersection \( \iff \) \( S \) symmetric (Herzog).
\( \rho \) minimal presentation for \( S \), then \( |\rho| \geq e(S) - 1 \).

A numerical semigroup \( S \) is a complete intersection, (CI), if the cardinality of any of its minimal presentations is equal to \( e(S) - 1 \).

A numerical semigroup is irreducible if it cannot be expressed as the intersection of two numerical semigroups properly containing it.

A numerical semigroup \( S \) is symmetric if it is irreducible and the Frobenius number, \( F(S) \), is odd.

\[ \text{Proposition} \quad S \text{ is a complete intersection} \implies S \text{ symmetric}. \]

If \( e(S) \leq 3 \), \( S \) is a complete intersection \iff \( S \) symmetric (Herzog).
ρ minimal presentation for S, then $|\rho| \geq e(S) - 1$.

A numerical semigroup S is a complete intersection, (CI), if the cardinality of any of its minimal presentations is equal to $e(S) - 1$.

A numerical semigroup is irreducible if it cannot be expressed as the intersection of two numerical semigroups properly containing it.

A numerical semigroup S is symmetric if it is irreducible and the Frobenius number, $F(S)$, is odd.

**Proposition**

$S$ is a complete intersection $\Rightarrow S$ symmetric.
minimal presentation for $S$, then $|\rho| \geq e(S) - 1$.

A numerical semigroup $S$ is a complete intersection, (CI), if the cardinality of any of its minimal presentations is equal to $e(S) - 1$.

A numerical semigroup is irreducible if it cannot be expressed as the intersection of two numerical semigroups properly containing it.

A numerical semigroup $S$ is symmetric if it is irreducible and the Frobenius number, $F(S)$, is odd.

**Proposition**

$S$ is a complete intersection $\Rightarrow$ $S$ symmetric.

If $e(S) \leq 3$, $S$ is a complete intersection $\iff$ $S$ symmetric (Herzog).
The catenary degree

The catenary degree of $s \in S$, $c(s)$, is the minimum nonnegative integer $N$ such that for any two factorizations $x$ and $y$ of $s$, there exists a sequence of factorizations $x_1, \ldots, x_t$ of $s$ such that

- $x_1 = x$, $x_t = y$,
- for all $i \in \{1, \ldots, t - 1\}$, $d(x_i, x_{i+1}) \leq N$.

The catenary degree of $S$, $c(S)$, is the supremum (maximum) of the catenary degrees of the elements of $S$. 
Example: $66 \in S = \langle 6, 9, 11 \rangle$, $c(66) = 4$

The factorizations of $66 \in \langle 6, 9, 11 \rangle$ are

$$Z(66) = \{(0, 0, 6), (1, 3, 3), (2, 6, 0), (4, 1, 3), (5, 4, 0), (8, 2, 0), (11, 0, 0)\}$$

The distance between $(11, 0, 0)$ and $(0, 0, 6)$ is 11.
Example: $66 \in S = \langle 6, 9, 11 \rangle$, $c(66) = 4$

The factorizations of $66 \in \langle 6, 9, 11 \rangle$ are

$$Z(66) = \{(0, 0, 6), (1, 3, 3), (2, 6, 0), (4, 1, 3), (5, 4, 0), (8, 2, 0), (11, 0, 0)\}$$

The distance between $(11, 0, 0)$ and $(0, 0, 6)$ is 11.
Example: $66 \in S = \langle 6, 9, 11 \rangle$, $c(66) = 4$

The factorizations of $66 \in \langle 6, 9, 11 \rangle$ are

$$Z(66) = \{(0, 0, 6), (1, 3, 3), (2, 6, 0), (4, 1, 3), (5, 4, 0), (8, 2, 0), (11, 0, 0)\}$$

The distance between $(11, 0, 0)$ and $(0, 0, 6)$ is 11.

- $(11, 0, 0)$
- $(8, 2, 0)$
- $(0, 0, 6)$
- $(3, 0, 0)$
- $(0, 2, 0)$
- $(8, 2, 0)$
- $(0, 0, 6)$
Example: $66 \in S = \langle 6, 9, 11 \rangle$, $c(66) = 4$

The factorizations of $66 \in \langle 6, 9, 11 \rangle$ are

$$Z(66) = \{(0, 0, 6), (1, 3, 3), (2, 6, 0), (4, 1, 3), (5, 4, 0), (8, 2, 0), (11, 0, 0)\}$$

The distance between $(11, 0, 0)$ and $(0, 0, 6)$ is $11$. 

![Diagram showing the factorizations and distances]
Example: $66 \in S = \langle 6, 9, 11 \rangle$, $c(66) = 4$

The factorizations of $66 \in \langle 6, 9, 11 \rangle$ are

$$Z(66) = \{(0, 0, 6), (1, 3, 3), (2, 6, 0), (4, 1, 3), (5, 4, 0), (8, 2, 0), (11, 0, 0)\}$$

The distance between $(11, 0, 0)$ and $(0, 0, 6)$ is 11.
Example: $66 \in S = \langle 6, 9, 11 \rangle$, $c(66) = 4$

The factorizations of $66 \in \langle 6, 9, 11 \rangle$ are

$Z(66) = \{ (0, 0, 6), (1, 3, 3), (2, 6, 0), (4, 1, 3), (5, 4, 0), (8, 2, 0), (11, 0, 0) \}$

The distance between $(11, 0, 0)$ and $(0, 0, 6)$ is 11.
The tame degree

The **tame degree** of $S$, $t(S)$, is defined as the minimum $N$ such that for any $s \in S$ and any factorization $x$ of $s$, if $s - n_i \in S$ for some $i \in \{1, \ldots, p\}$, then there exists another factorization $y$ of $s$ such that $d(x, y) \leq N$ and the $i$th coordinate of $y$ is nonzero ($n_i$ “occurs” in this factorization).
Example: $66 \in S = \langle 6, 9, 11 \rangle$, $t(66) = 7$

The factorizations of $66 \in \langle 6, 9, 11 \rangle$ are

$$Z(66) = \{(0, 0, 6), (1, 3, 3), (2, 6, 0), (4, 1, 3), (5, 4, 0), (8, 2, 0), (11, 0, 0)\}$$

Besides, 9 divides 66

$$(11, 0, 0)$$
Example: $66 \in S = \langle 6, 9, 11 \rangle$, $t(66) = 7$

The factorizations of $66 \in \langle 6, 9, 11 \rangle$ are

$Z(66) = \{(0, 0, 6), (1, 3, 3), (2, 6, 0), (4, 1, 3), (5, 4, 0), (8, 2, 0), (11, 0, 0)\}$

and 11 also divides 66

$$
\begin{align*}
3 & \mid (8, 2, 0) \\
3 & \mid (11, 0, 0)
\end{align*}
$$
Example: $66 \in S = \langle 6, 9, 11 \rangle$, $t(66) = 7$

The factorizations of $66 \in \langle 6, 9, 11 \rangle$ are

$$Z(66) = \{(0, 0, 6), (1, 3, 3), (2, 6, 0), (4, 1, 3), (5, 4, 0), (8, 2, 0), (11, 0, 0)\}$$

$$\begin{array}{c}
(8, 2, 0) \\
3 \mid \\
(11, 0, 0) \\
7 \mid \\
(4, 1, 3)
\end{array}$$
The catenary degree of $S$ is less than or equal to the tame degree of $S$.

$$c(S) \leq t(S)$$
The catenary degree of $S$ is less than or equal to the tame degree of $S$.

\[ c(S) \leq t(S) \]

**Goal:** Say if the inequality is strict or not for numerical semigroups $S$ with $e(S) = 4$ that are symmetric but not complete intersection.
Bresinsky’s theorem

The numerical semigroup \( S \) is 4-generated symmetric, not complete intersection, if and only if there are integers \( \alpha_i, 1 \leq i \leq 4 \), \( \alpha_{ij}, i, j \in \{21, 31, 32, 42, 13, 43, 14, 24\} \), s.t.:

\[
\begin{align*}
0 < \alpha_{ij} < \alpha_i, & \quad \text{for all } i, j, \\
\alpha_1 &= \alpha_{21} + \alpha_{31}, \\
\alpha_2 &= \alpha_{32} + \alpha_{42}, \\
\alpha_3 &= \alpha_{13} + \alpha_{43}, \\
\alpha_4 &= \alpha_{14} + \alpha_{24}, \\
n_1 &= \alpha_2 \alpha_3 + \alpha_{14} \alpha_{43} + \alpha_{24} \alpha_{31}, \\
n_2 &= \alpha_3 \alpha_4 + \alpha_{21} \alpha_{43} + \alpha_{31} \alpha_{42}, \\
n_3 &= \alpha_1 \alpha_4 + \alpha_{24} \alpha_{31} + \alpha_{13} \alpha_{42}, \\
n_4 &= \alpha_1 \alpha_2 + \alpha_{42} \alpha_{31} + \alpha_{14} \alpha_{23}. 
\end{align*}
\]

Then

\[
\begin{align*}
b_1 &= \alpha_1 n_1 = \alpha_3 n_3 + \alpha_4 n_4, \\
b_2 &= \alpha_2 n_2 = \alpha_1 n_1 + \alpha_4 n_4, \\
b_3 &= \alpha_3 n_3 = \alpha_2 n_2 + \alpha_1 n_1, \\
b_4 &= \alpha_4 n_4 = \alpha_3 n_3 + \alpha_2 n_2, \\
b_5 &= \alpha_2 n_1 + \alpha_4 n_3 = \alpha_3 n_2 + \alpha_1 n_4. 
\end{align*}
\]
Bresinsky’s theorem

The numerical semigroup $S$ is 4-generated symmetric, not complete intersection, if and only if there are integers $\alpha_i$, $1 \leq i \leq 4$, $\alpha_{ij}$, $i, j \in \{21, 31, 32, 42, 13, 43, 14, 24\}$, s.t.:

- $0 < \alpha_{ij} < \alpha_i$, for all $i, j$, 

Then $Betti(S) =$

\[
\begin{align*}
&b_1 = \alpha_1 n_1 = \alpha_{13} n_3 + \alpha_{14} n_4 \\
&b_2 = \alpha_2 n_2 = \alpha_{21} n_1 + \alpha_{24} n_4 \\
&b_3 = \alpha_3 n_3 = \alpha_{31} n_1 + \alpha_{32} n_2 \\
&b_4 = \alpha_4 n_4 = \alpha_{42} n_2 + \alpha_{43} n_3 \\
&b_5 = \alpha_{21} n_1 + \alpha_{43} n_3 = \alpha_{32} n_2 + \alpha_{14} n_4 
\end{align*}
\]
Bresinsky’s theorem

The numerical semigroup $S$ is 4-generated symmetric, not complete intersection, if and only if there are integers $\alpha_i$, $1 \leq i \leq 4$, $\alpha_{ij}$, $i, j \in \{21, 31, 32, 42, 13, 43, 14, 24\}$, s.t.:

- $0 < \alpha_{ij} < \alpha_i$, for all $i, j$,
- $\alpha_1 = \alpha_{21} + \alpha_{31}$, $\alpha_2 = \alpha_{32} + \alpha_{42}$, $\alpha_3 = \alpha_{13} + \alpha_{43}$,
- $\alpha_4 = \alpha_{14} + \alpha_{24}$, and
Bresinsky’s theorem

The numerical semigroup $S$ is 4-generated symmetric, not complete intersection, if and only if there are integers $\alpha_i$, $1 \leq i \leq 4$, $\alpha_{ij}$, $i, j \in \{21, 31, 32, 42, 13, 43, 14, 24\}$, s.t.:

- $0 < \alpha_{ij} < \alpha_i$, for all $i, j$,
- $\alpha_1 = \alpha_{21} + \alpha_{31}$, $\alpha_2 = \alpha_{32} + \alpha_{42}$, $\alpha_3 = \alpha_{13} + \alpha_{43}$, $\alpha_4 = \alpha_{14} + \alpha_{24}$, and
- $n_1 = \alpha_2 \alpha_3 \alpha_{14} + \alpha_{32} \alpha_{13} \alpha_{24}$, $n_2 = \alpha_3 \alpha_4 \alpha_{21} + \alpha_{31} \alpha_{43} \alpha_{24}$, $n_3 = \alpha_1 \alpha_4 \alpha_{32} + \alpha_{14} \alpha_{42} \alpha_{31}$, $n_4 = \alpha_1 \alpha_2 \alpha_{43} + \alpha_{42} \alpha_{21} \alpha_{13}$. 
Bresinsky’s theorem

The numerical semigroup $S$ is 4-generated symmetric, not complete intersection, if and only if there are integers $\alpha_i$, $1 \leq i \leq 4$, $\alpha_{ij}$, $i,j \in \{21, 31, 32, 42, 13, 43, 14, 24\}$, s.t.:

- $0 < \alpha_{ij} < \alpha_{i}$, for all $i, j$,
- $\alpha_1 = \alpha_{21} + \alpha_{31}$, $\alpha_2 = \alpha_{32} + \alpha_{42}$, $\alpha_3 = \alpha_{13} + \alpha_{43}$, $\alpha_4 = \alpha_{14} + \alpha_{24}$, and
- $n_1 = \alpha_2 \alpha_3 \alpha_{14} + \alpha_{32} \alpha_{13} \alpha_{24}$, $n_2 = \alpha_3 \alpha_4 \alpha_{21} + \alpha_{31} \alpha_{43} \alpha_{24}$, $n_3 = \alpha_1 \alpha_4 \alpha_{32} + \alpha_{14} \alpha_{42} \alpha_{31}$, $n_4 = \alpha_1 \alpha_2 \alpha_{43} + \alpha_{42} \alpha_{21} \alpha_{13}$.

Then

$$
\text{Betti}(S) = \begin{cases} 
    b_1 = \alpha_1 n_1 = \alpha_{13} n_3 + \alpha_{14} n_4 \\
    b_2 = \alpha_2 n_2 = \alpha_{21} n_1 + \alpha_{24} n_4 \\
    b_3 = \alpha_3 n_3 = \alpha_{31} n_1 + \alpha_{32} n_2 \\
    b_4 = \alpha_4 n_4 = \alpha_{42} n_2 + \alpha_{43} n_3 \\
    b_5 = \alpha_{21} n_1 + \alpha_{43} n_3 = \alpha_{32} n_2 + \alpha_{14} n_4 
\end{cases}
$$
Observations on the catenary degree of $S$

- The catenary degree is reached in one of the Betti elements, $c(S) = \max\{c(b) \mid b \in \text{Betti}(S)\}$;
Observations on the catenary degree of $S$

- The catenary degree is reached in one of the Betti elements, $c(S) = \max\{c(b) | b \in \text{Betti}(S)\}$;
- 4-generated symmetric and non complete intersection numerical semigroups are uniquely presented (Katsabekis & Ojeda) and therefore each Betti element has exactly two factorizations having $\gcd = (0, 0, 0, 0)$ (García-Sánchez & Ojeda);
Observations on the catenary degree of $S$

- The catenary degree is reached in one of the Betti elements, $c(S) = \max\{c(b) \mid b \in \text{Betti}(S)\}$;
- 4-generated symmetric and non complete intersection numerical semigroups are uniquely presented (Katsabekis & Ojeda) and therefore each Betti element has exactly two factorizations having $\gcd = (0, 0, 0, 0)$ (García-Sánchez & Ojeda);
- for each one of the Betti elements the catenary degree is the distance between its two factorizations, i.e., since their $\gcd$ is zero, $c(b) = \max\{|z| \mid z \in Z(b)\}$.
Observations on the catenary degree of $S$

- The catenary degree is reached in one of the Betti elements, $c(S) = \max\{c(b) \mid b \in \text{Betti}(S)\}$;

- 4-generated symmetric and non complete intersection numerical semigroups are uniquely presented (Katsabekis & Ojeda) and therefore each Betti element has exactly two factorizations having $\gcd = (0, 0, 0, 0)$ (García-Sánchez & Ojeda);

- for each one of the Betti elements the catenary degree is the distance between its two factorizations, i.e., since their $\gcd$ is zero, $c(b) = \max\{|z| \mid z \in Z(b)\}$.

Then,

$$c(S) = \max\{\alpha_1, \alpha_{13} + \alpha_{14}, \alpha_2, \alpha_{21} + \alpha_{24}, \alpha_3, \alpha_{31} + \alpha_{32}, \alpha_4, \alpha_{42} + \alpha_{43}, \alpha_{21} + \alpha_{43}, \alpha_{32} + \alpha_{14}\}.$$
Conjecture: For 4-generated symmetric non complete intersection numerical semigroups $c(S) < t(S)$. 

Known: $t(S) = \max \{ t(n) \mid n \in \text{Prim}(S) \cap \text{NC}(S) \}$, where $\text{Prim}(S) = \{ n \in S \mid \exists x, y \in \mathbb{Z} \text{ that are minimal positive solutions to} x_1 n_1 + x_2 n_2 + x_3 n_3 + x_4 n_4 - y_1 n_1 - y_2 n_2 - y_3 n_3 - y_4 n_4 = 0 \}$ and $\text{NC}(S) = \{ n \in S \mid G_n \text{is not complete} \}$. 

But since each Betti element $b_i$ has just two factorizations with gcd $= (0, 0, 0, 0)$, $t(b_i) = c(b_i)$. 

Idea: find an element $n$ in $(\text{Prim}(S) \cap \text{NC}(S)) \setminus \text{Betti}(S)$ s.t. $t(n) > c(S)$. 

Conjecture: For 4-generated symmetric non complete intersection numerical semigroups \( c(S) < t(S) \).

How to prove it?
Conjecture: For 4-generated symmetric non complete intersection numerical semigroups $c(S) < t(S)$.

How to prove it?

Known: $t(S) = \max\{t(n) \mid n \in \text{Prim}(S) \cap \text{NC}(S)\}$,
Conjecture: For 4-generated symmetric non complete intersection numerical semigroups $c(S) < t(S)$.

How to prove it?

Known: $t(S) = \max\{t(n) \mid n \in \text{Prim}(S) \cap \text{NC}(S)\}$, where

$\text{Prim}(S) = \{n \in S \mid \exists x, y \in Z(n) \text{ that are minimal positive solutions to} \ x_1n_1 + x_2n_2 + x_3n_3 + x_4n_4 - y_1n_1 - y_2n_2 - y_3n_3 - y_4n_4 = 0 \ \text{and} \ x \neq y\}$
Conjecture: For 4-generated symmetric non complete intersection numerical semigroups $c(S) < t(S)$.

How to prove it?

Known: $t(S) \equiv \max\{t(n) \mid n \in \text{Prim}(S) \cap \text{NC}(S)\}$, where

$\text{Prim}(S) = \{ n \in S \mid \exists x, y \in Z(n) \text{ that are minimal positive solutions to } x_1n_1 + x_2n_2 + x_3n_3 + x_4n_4 - y_1n_1 - y_2n_2 - y_3n_3 - y_4n_4 = 0 \text{ and } x \neq y \}$

$\text{NC}(S) = \{ n \in S \mid G_n \text{ is not complete} \}$
Conjecture: For 4-generated symmetric non complete intersection numerical semigroups $c(S) < t(S)$.

How to prove it?

Known: $t(S) = \max\{t(n) \mid n \in \text{Prim}(S) \cap \text{NC}(S)\}$, where

$\text{Prim}(S) = \{n \in S \mid \exists x, y \in Z(n) \text{ that are minimal positive solutions to} x_1 n_1 + x_2 n_2 + x_3 n_3 + x_4 n_4 - y_1 n_1 - y_2 n_2 - y_3 n_3 - y_4 n_4 = 0$ and $x \neq y\}$

$\text{NC}(S) = \{n \in S \mid G_n \text{ is not complete}\}$

$\text{Betti}(S) \subseteq \text{Prim}(S) \cap \text{NC}(S)$. 
Conjecture: For 4-generated symmetric non complete intersection numerical semigroups $c(S) < t(S)$.

How to prove it?

Known: $t(S) = \max\{t(n) \mid n \in \text{Prim}(S) \cap \text{NC}(S)\}$, where

$$\text{Prim}(S) = \{n \in S \mid \exists x, y \in Z(n) \text{ that are minimal positive solutions to } x_1n_1 + x_2n_2 + x_3n_3 + x_4n_4 - y_1n_1 - y_2n_2 - y_3n_3 - y_4n_4 = 0$$
$$\text{and } x \neq y\}$$

$$\text{NC}(S) = \{n \in S \mid G_n \text{ is not complete}\}$$

$\text{Betti}(S) \subseteq \text{Prim}(S) \cap \text{NC}(S)$. But since each Betti element $b_i$ has just two factorizations with gcd = $(0, 0, 0, 0)$, $t(b_i) = c(b_i)$

Idea: find an element $n$ in $(\text{Prim}(S) \cap \text{NC}(S)) \setminus \text{Betti}(S)$ s.t. $t(n) > c(S)$.
The case $c(S) = \alpha_i, i \in \{1, 2, 3, 4\}$

Take $k = \min\{h \in \mathbb{N} \mid hn_i - n_j \in S, j \equiv i + 1, \pmod{4}\}$. $(k > \alpha_i)$
The case $c(S) = \alpha_i$, $i \in \{1, 2, 3, 4\}$

Take $k = \min\{h \in \mathbb{N} \mid hn_i - n_j \in S, j \equiv i + 1, \pmod{4}\}$. 
($k > \alpha_i$)

Take $kn_i$. 
The case $c(S) = \alpha_i, i \in \{1, 2, 3, 4\}$

Take $k = \min\{h \in \mathbb{N} \mid hn_i - n_j \in S, j \equiv i + 1, \quad (\text{mod} \ 4)\}$. 
($k > \alpha_i$)

Take $kn_i$.

$Z(kn_i) \ni z$ such that $|z| = k$. 
The case \( c(S) = \alpha_i, \; i \in \{1, 2, 3, 4\} \)

Take \( k = \min\{h \in \mathbb{N} \mid hn_i - n_j \in S, j \equiv i + 1, \; (\text{mod} \; 4)\} \).
\( (k > \alpha_i) \)

Take \( kn_i \).

\( Z(kn_i) \ni z \) such that \( |z| = k \).

\( \exists z' \in Z(kn_i) \) in which \( n_j \) occurs and \( n_i \) does not occur.

\[ \text{Note:} \; kn_i \in \text{Prim}(S) \cap \text{NC}(S). \]

\[ t(kn_i) \geq d(z, z') \geq k > \alpha_i = c(S) \]
The case $c(S) = \alpha_i, i \in \{1, 2, 3, 4\}$

Take $k = \min\{h \in \mathbb{N} \mid hn_i - n_j \in S, j \equiv i + 1, \mod 4\}$. ($k > \alpha_i$)

Take $kn_i$.

$Z(kn_i) \ni z$ such that $|z| = k$.

$\exists z' \in Z(kn_i)$ in which $n_j$ occurs and $n_i$ does not occur.

Note: $kn_i \in \text{Prim}(S) \cap \text{NC}(S)$.

$$t(kn_i) \geq d(z, z') \geq k > \alpha_i = c(S)$$
The case \( c(S) = \alpha_i, \ i \in \{1, 2, 3, 4\} \)

Take \( k = \min\{h \in \mathbb{N} \mid hn_i - n_j \in S, j \equiv i + 1, \ (\mod 4)\} \).
\( (k > \alpha_i) \)

Take \( kn_i \).
\( Z(kn_i) \ni z \) such that \( |z| = k \).
\( \exists z' \in Z(kn_i) \) in which \( n_j \) occurs and \( n_i \) does not occur.

Note: \( kn_i \in \text{Prim}(S) \cap \text{NC}(S) \).

\[
t(kn_i) \geq d(z, z') \geq k > \alpha_i = c(S)
\]

\( \Downarrow \quad \Downarrow \)

\[
t(S) \geq t(kn_i) > c(S)
\]
Thank you