linear inequalities for the Hilbert depth of graded modules over polynomial rings

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Introduction
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- Invariants of Hilbert series $\rightarrow$ numerical semigroups
- New interpretation of some characterization already explained in Vila-Real and Cortona
- This talk is based on a series of common works with
  * Lukas Katthän, Goethe-Universität Frankfurt am Main
  * Jan Uliczka, Universität Osnabrück

All available on the arXiv.
The setting
Let $K$ be a field.

Let $R := K[X_1, \ldots, X_n]$ be a polynomial ring endowed with a grading, typically

- standard-$\mathbb{Z}$-grading, i.e., $\deg X_i = 1$
- nonstandard-$\mathbb{Z}$-grading
- $(\mathbb{Z}^r$-grading)

Let $0 \neq M = \bigoplus_{\ell} M_{\ell}$ be a finitely generated graded $R$-module, with Hilbert series

$$H_M(t) = \sum_{\ell \in \mathbb{Z}^r} (\dim_K M_{\ell}) t^\ell \in \mathbb{Z}[t][t^{-1}]$$

Series without negative coefficients: nonnegative series.
Previous results
For the moment, let us restrict ourselves to $\mathbb{Z}$-gradings.

Set $d_i := \deg X_i \in \mathbb{N}$ for all $i = 1, \ldots, n$.

**Definition [Hilbert depth]**

$$\text{Hdep}(M) := \max\{\text{depth}N \mid N \text{ a f.g. gr. module with } H_N = H_M\}.$$ 

This is a well-defined but opaque quantity!

Characterizations?
Theorem [—, Uliczka 13]

A formal Laurent series $H$ with denominator $\prod_i (1 - t^{d_i})$ is the Hilbert series of a f.g. graded $R$-module $M$ if and only if

$$H(t) = \sum_{I \subseteq \{1, \ldots, n\}} \frac{Q_I(t)}{\prod_{j \in I} (1 - t^{d_j})}$$

with nonnegative $Q_I(t)$.

Definition [Decomposition Hilbert depth]

$$\text{decHdep}(M) := \max \left\{ r \in \mathbb{N} \mid H_M \text{ admits a decompos. as above with } Q_I = 0 \ \forall \ I \text{ such th. } |I| < r \right\}.$$
Case of two variables

Let \( R = K[X, Y] \) be with \( \alpha := \deg X, \beta := \deg Y \) coprime.

Set \( \Gamma := \langle \alpha, \beta \rangle \) the numerical semigroup generated by \( \alpha \) and \( \beta \).

Theorem [---, Uliczka 13]

Let \( M \) be a finitely generated graded \( R \)-module. Then

\[ \text{Hdep}(M) > 0 \quad \text{if and only if} \quad H_M(t) = \sum_n h_n t^n \]

satisfies the condition

\[ \sum_{i \in I} h_{i+n} \leq \sum_{j \in J} h_{j+n} \]

for all \( n \in \mathbb{Z} \) and all “fundamental couples” \([I, J] \).
(I) What is a “fundamental couple” \([I, J]\)?

Let \(L\) be the set of gaps of \(\langle \alpha, \beta \rangle\).

An \((\alpha, \beta)-\text{fundamental couple}\) \([I, J]\) consists of two integer sequences \(I = (i_k)_{k=0}^{m}\) and \(J = (j_k)_{k=0}^{m}\), such that

1. \(i_0 = 0\).
2. \(i_1, \ldots, i_m, j_1, \ldots, j_{m-1} \in L\) and \(j_0, j_m \leq \alpha \beta\).

\[(2)\]
\[
\begin{align*}
  i_k &\equiv j_k \pmod{\alpha} \quad \text{and} \quad i_k < j_k \quad \text{for } k = 0, \ldots, m; \\
  j_k &\equiv i_{k+1} \pmod{\beta} \quad \text{and} \quad j_k > i_{k+1} \quad \text{for } k = 0, \ldots, m-1; \\
  j_m &\equiv i_0 \pmod{\beta} \quad \text{and} \quad j_m \geq i_0.
\end{align*}
\]

3. \(|i_k - i_\ell| \in L\) for \(1 \leq k < \ell \leq m\).
What is a “fundamental couple” $[I, J]$?

- $I$ consists of minimal generators of “relative ideals” = “semimodules” $\Delta$ of $\Gamma$.
- $J$ contains “small shifts” of $I$-sets which turn out to generate a sort of syzygy $\text{Syz}_\Delta$.

Syzygy in the sense that any element in $\text{Syz}_\Delta$ admits more than one presentation in the form $i + x$ with $i \in I$ and $x \in \Gamma$. 
In the special case $\Gamma = \langle 3, 5 \rangle$ the criterion is given by the inequalities

\[
\begin{align*}
    h_{n+0} & \leq h_{n+15}, \\
    h_{n+0} + h_{n+1} & \leq h_{n+6} + h_{n+10}, \\
    h_{n+0} + h_{n+2} & \leq h_{n+12} + h_{n+5}, \\
    h_{n+0} + h_{n+4} & \leq h_{n+9} + h_{n+10}, \\
    h_{n+0} + h_{n+7} & \leq h_{n+12} + h_{n+10}, \\
    h_{n+0} + h_{n+1} + h_{n+2} & \leq h_{n+5} + h_{n+6} + h_{n+7}, \\
    h_{n+0} + h_{n+2} + h_{n+4} & \leq h_{n+5} + h_{n+7} + h_{n+9}.
\end{align*}
\]
Lattice paths for $\Gamma = \langle 5, 7 \rangle$

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Lattice paths for \( \Gamma = \langle 5, 7 \rangle \)

\[ I = [0, 8, 6, 9] \text{ and } J = [15, 13, 16, 14]. \]
New results
A deep algebraic meaning of the inequalities
\[ \sum_{i \in I} h_{i+n} \leq \sum_{j \in J} h_{j+n} \] remained rather hidden.

New insights appeared when considering the \( \mathbb{Z}^r \)-grading.

The starting question arose by looking at the decomposition theorem (already mentioned):

A formal Laurent series \( H \) with denominator \( \prod_i (1 - t^{d_i}) \) is the Hilbert series of a f.g. graded \( R \)-module \( M \) iff

\[
H(t) = \sum_{I \subseteq \{1, \ldots, n\}} \frac{Q_I(t)}{\prod_{j \in I} (1 - t^{d_j})} \quad \text{with nonnegative } Q_I.
\]

**Question**: Is the condition of the Thm satisfied by every rational function with the given denominator and nonnegative coefficients?
Question: Which formal Laurent series arise as Hilbert series of $R$-modules (in a certain class)?

Conditions: The series must...

- ... have nonnegative coefficients.
- ... be rational function with denominator $\prod_i (1 - t^\deg X_i)$.
- ...

Related work:

- Macaulay, 1927: cyclic modules, standard $\mathbb{Z}$-grading.
- Boij & Smith, 2015: modules generated in degree 0, standard $\mathbb{Z}$-grading + technical details
Theorem [Katthän, —, Uliczka 2016]

Let $H \in \mathbb{Z}[t][t^{-1}]$ be a formal Laurent series, which is the Hilbert series of some finitely generated graded $R$-module $M$. Let further $S := R/(X^\beta - Y^\alpha)$.

Then the following statements are equivalent:

(a) $H_{\text{dep}}(M) > 0$

(b) For any finitely generated torsionfree $S$-module $N$, it holds that

$$\frac{H \cdot H_N}{H_R} \geq 0.$$

(c) Condition (b) holds for any finitely generated torsionfree $S$-module of rank 1.

(d) For all $n \in \mathbb{Z}$, $[I, J]$ fundamental couple, $H = \sum_i h_i t^i$ satisfies

$$\sum_{i \in I} h_{i+n} \leq \sum_{j \in J} h_{j+n} \quad (\star)$$
We need the following result about the structure of fundamental couples.

**Lemma**

Let \([l = (i_k), J = (j_k)]\) be a fundamental couple of length \(m\). Then there exist two integer sequences

\[
\beta > a_0 > a_1 > \cdots > a_m = 0, \quad \text{and} \quad 0 = b_0 < b_1 < \cdots < b_m < \alpha
\]

such that

\[
i_k = \alpha \beta - a_{k-1} \alpha - b_k \beta \quad \text{for } 1 \leq k \leq m, \quad \text{and} \quad j_k = \alpha \beta - a_k \alpha - b_k \beta \quad \text{for } 0 \leq k \leq m
\]
(c) ⇒ (d): Let \([I, J]\) be a fundamental couple.

Recall that \(S = K[t^\alpha, t^\beta]\) is the monoid algebra of \(\Gamma\). Let 
\(N \subseteq K[t]\) be the \(S\)-module generated by \(t^{\alpha \beta - j_0}, \ldots, t^{\alpha \beta - j_m}\).

This module is torsionfree, hence \(\frac{H_M H_N}{H_R} \geq 0\) by assumption.

To see that this inequality implies \((\star)\), we need to compute \(H_N\).

Let \((a_k)_{k=0}^m, (b_k)_{k=0}^m\) be the sequences as in Lemma and let 
\[\tilde{N} := (X^{a_0} Y^{b_0}, \ldots, X^{a_m} Y^{b_m}).\]

It is easy to see that \(\tilde{N}\) is the preimage of \(N\) under the projection 
\(R \to S\).

In particular, note that \(X^\beta - Y^\alpha \in \tilde{N}\), because \(X^{a_0}, Y^{b_m} \in \tilde{N}\).

Hence \(N \cong \tilde{N}/(X^\beta - Y^\alpha)\) and thus \(H_N = H_{\tilde{N}} - t^{\alpha \beta} H_R\).
By considering the minimal free resolution of $\tilde{N}$, one sees that its syzygies are generated in the degrees $a_{k-1}\alpha + b_k\beta$ for $1 \leq k \leq m$.

Therefore

$$\frac{H_N}{H_R} = \frac{H_{\tilde{N}} - t^{\alpha\beta}H_R}{H_R} = \sum_{k=0}^{m} t^{a_k\alpha+b_k\beta} - \sum_{k=1}^{m} t^{a_{k-1}\alpha+b_k\beta} - t^{\alpha\beta}$$

$$= \sum_{k=0}^{m} t^{\alpha\beta-j_k} - \sum_{k=1}^{m} t^{\alpha\beta-i_k} - t^{\alpha\beta-i_0} = t^{\alpha\beta} \left( \sum_{j \in J} t^{-j} - \sum_{i \in I} t^{-i} \right)$$

Then we obtain

$$0 \leq \frac{H \cdot H_N}{H_R} = (\sum_{n \in \mathbb{Z}} h_n t^n) t^{\alpha\beta} \left( \sum_{j \in J} t^{-j} - \sum_{i \in I} t^{-i} \right)$$

$$= t^{\alpha\beta} \sum_{n \in \mathbb{Z}} t^n \left( \sum_{j \in J} h_{n+j} - \sum_{i \in I} h_{n+i} \right),$$

and $(\star)$ is satisfied for $[I, J]$. 