Cyclotomic Numerical Semigroups II
–Polynomials playing pingpong–

Pieter Moree,

Max Planck Institute for Mathematics, Bonn

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Overview

1. The pingpong players: $P_S(x)$ and $\Phi_n(x)$

2. First match
   - Semigroup polynomial $P_{\langle p,q \rangle}(x)$
   - Binary cyclotomic polynomials
   - Exponent gaps
   - Gapblocks

3. Second match
   - General cyclotomic polynomials
   - Cyclotomic numerical semigroups
   - Symmetric non-cyclotomic numerical semigroups
   - Counting cyclotomic semigroups of given Frobenius number

4. Polynomially related numerical semigroups
   - An Application
Papers to be discussed

- Cyclotomic numerical semigroups. II, in preparation.
- Some other results from older papers by the speaker (and Y. Gallot)
Semigroup polynomials

We have \( H_S(x) = \sum_{s \in S} x^s = (1 - x)^{-1} - \sum_{s \not\in S} x^s \).
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P_S(x) := (1 - x)H_S(x) = 1 + (x - 1) \sum_{s \notin S} x^s.
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Observe that \( P_S(x) \) is a monic polynomial of degree \( F(S) + 1 \).
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Lemma

Write \( P_S(x) = a_0 + a_1x + \cdots + a_kx^k \).
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**Lemma**

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$$a_j = \begin{cases} 
1 & \text{if } j \in S \text{ and } j - 1 \notin S; \\
-1 & \text{if } j \notin S \text{ and } j - 1 \in S; \\
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**Corollary**

The nonzero coefficients of $P_S(x)$ alternate between 1 and $-1$. 
It follows that 
\[ P_{\langle 3,5 \rangle}(X) = 1 - X + X^3 - X^4 + X^5 - X^7 + X^8 \]

We have 
\[ \Phi_{15}(X) = 1 - X + X^3 - X^4 + X^5 - X^7 + X^8 \]

The equality is no coincidence!

Lemma (Folklore)
\[ P_{\langle p, q \rangle}(x) = \Phi_{pq}(x). \]

Corollary (Sylvester, 1884)
\[ F(\langle p, q \rangle) = \deg(\Phi_{pq}(X)) - 1 = (p - 1)(q - 1) - 1 = pq - p - q. \]

Corollary (Migotti, 1887)
Coefficients of \( \Phi_{pq}(x) \) are in \( \{-1, 0, 1\} \).
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| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | ... | ...
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*Coefficients of $\Phi_{pq}(x)$ are in $\{-1, 0, 1\}$.*
Write $1 + pq = \rho p + \sigma q$, $0 \leq \rho \leq q - 1$, $0 \leq \sigma \leq p - 1$. 

Note that $\rho p \equiv 1 \pmod{q}$ and $\sigma q \equiv 1 \pmod{p}$. Thus $\rho$ is the inverse of $p$ modulo $q$, $\sigma$ the inverse of $q$ modulo $p$. 

$\Phi_{pq}(X) = \phi(pq) \sum_{m=0}^\infty a_{pq}(m)x^m = \rho - 1 \sum_{i=0}^{\rho-1} X^{ip} \sigma - 1 \sum_{j=0}^{\sigma-1} X^{jq} - X^{pq}q - 1 \sum_{i=\rho}^{q-1} X^{ip} \sigma - 1 \sum_{j=\sigma}^{p-1} X^{jq}$

**Lemma** $a_{pq}(m) =$

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Binary cyclotomic polynomials

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**Corollary**

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$\rho = 3^{-1} \pmod{5} = 2$, $\sigma = 5^{-1} \pmod{3} = 2$,

$g(\langle p, q \rangle) = (p - 1)(q - 1)/2$
Sparse binary cyclotomic polynomials

Correspond to a NS having few (and hence large) gapblocks.
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Bzdęga (2012) showed: $c(\epsilon, \gamma)x^{1/2+\gamma-\epsilon} \leq H_{\gamma}(x) \leq C(\gamma)x^{1/2+\gamma}$.

Fouvry (2013): For $\gamma \in \left(\frac{1}{25}, \frac{1}{2}\right)$ we have $H_{\gamma}(x) \sim D(\gamma)x^{1/2+\gamma}\log x$, with $D(\gamma)$ an explicit constant.
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- Bounds for Kloosterman-Ramanujan sums over primes
- Bombieri-Vinogradov theorem
- Two-dimensional sieve
- Linnik’s famous theorem concerning the least prime in AP
Exponent gaps after Hong et al.

We describe some work of Hong-Lee-Lee-Park (2012).
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**Definition (Maximum gap)**

Given $f(x) = c_1 x^{e_1} + \cdots + c_t x^{e_t} \in \mathbb{Z}[x]$, with $c_i \neq 0$ and $e_1 < \cdots < e_t$, we define the **maximum gap** of $f$ as

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- Simple and exact formula for the minimum Miller loop length in the Ate pairing arising in elliptic curve cryptography.
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- Simple and exact formula for the minimum Miller loop length in the Ate\(_i\) pairing arising in elliptic curve cryptography.
- More manageable when turned into a problem involving the maximum gaps of inverse cyclotomic polynomials.
**Definition (Inverse cyclotomic polynomial)**

\[
\Psi_n(x) = \prod_{d \mid n, \ d < n} \Phi_d(x) = \frac{x^n - 1}{\Phi_n(x)} = \sum_{k=0}^{\infty} c_n(k)x^k.
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Theorem (Moree, JNTh, 2009)

We have \( B(pqr) \leq p - 1 \) and equality holds if and only if

\[ q \equiv r \equiv \pm 1 \pmod{p} \quad \text{and} \quad r < \frac{p - 1}{p - 2}(q - 1) \]
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In contrast: \( (2/3 - \epsilon)p \leq A(pqr) \leq 3p/4 \).
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\[ q \equiv r \equiv \pm 1 \pmod{p} \text{ and } r < \frac{p - 1}{p - 2}(q - 1) \]

In contrast: \( (2/3 - \epsilon)p \leq A(pqr) \leq 3p/4 \).
Conjecturally \( A(pqr) \leq 2p/3 \).
Exponent gaps

\[ g(\Phi_p) = 1, \quad g(\Psi_p) = 1, \quad g(\Phi_{pq}) = p - 1, \quad g(\Psi_{pq}) = q - p + 1 \]
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**Camburu, Ciolan, Luca, M., Shparlinski**

\[ R_3(x) = \frac{cx}{(\log x)^2} + O \left( \frac{x \log \log x}{(\log x)^3} \right), \quad c = (1 + \log 4) \log 4. \]
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Compare with the classical estimate (Gauss, Landau)

\[ Q_3(x) = (1 + o(1)) \frac{x(\log \log x)^2}{2 \log x}. \]
Lemma

Let $p < q$ be primes. Then $g(\Phi_{pq}) = p - 1$. 
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Proof.

Since \( S = \langle p, q \rangle \) is symmetric, there is a one to one correspondence between \( k \)-gapblocks and \( k \)-elementblocks. We have that \( g(\Phi_{pq}) \) equals the largest gap block in \( S \). Presence of \( \langle p \rangle \) in \( S = \langle p, q \rangle \) ensures that \( g(\Phi_{pq}) \leq p - 1 \). Since \( S = \{1, p, \ldots\} \), we have \( g(\Phi_{pq}) = p - 1 \). \( \square \)
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Theorem

(i) \( g(\Phi_{pq}) = p - 1 \) and the number of maximum gaps equals \( 2 \left\lfloor q/p \right\rfloor \);
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Theorem

(i) $g \left( \Phi_{pq} \right) = p - 1$ and the number of maximum gaps equals $2 \left\lfloor q/p \right\rfloor$;
(ii) $\Phi_{pq}$ contains the sequence of consecutive coefficients $\pm 1, \{0\}_m, \mp 1$ for all $m = 0, 1, \ldots, p - 2$ iff $q \equiv \pm 1 \pmod{p}$.

The notation $\{0\}_m$ indicates a string $0, \ldots, 0$ of $m$ consecutive zeros.
Gapblocks

Suppose $S = \langle a, b \rangle$ with $a$ and $b$ coprime.
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$$P_S(x) = \frac{(1 - x)(1 - x^{ab})}{(1 - x^a)(1 - x^b)} = \prod_{d \mid ab, \ d \nmid a, \ d \nmid b} \Phi_d(x)$$

is an inclusion-exclusion polynomial (Bachman, 2010).

**Theorem**

Let $2 \leq a < b$ be coprime positive integers. Then

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$$\prod_{d \mid ab, \ d \nmid a, \ d \nmid b} \Phi_d(x)$$

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\( \Phi_n(x) \) with more than two prime factors

\[ \Phi_n(x) \text{ with } n = 4849845 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \]
$\Phi_n(x)$ with more than two prime factors

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### Lemma (Value at 1)

\[ \Phi_n(1) = \begin{cases} 
  0 & \text{if } n = 1; \\
  p & \text{if } n = p^m; \\
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\end{cases} \]
Calculation of $\Phi_n(1)$

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$$\frac{x^n - 1}{x - 1} = \prod_{d|n, \ d>1} \Phi_d(x).$$
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Hence, $\Phi_{pq}(1) = 1 = P_{\langle p, q \rangle}(1)$. 

Pieter Moree

Cyclotomic Numerical Semigroups II

Levico Terme, July 7, 2016
Calculation of $\Phi_n(1)$

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Hence, by induction $\Phi_{p^f}(1) = p$. Next, note that

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Hence, $\Phi_{pq}(1) = 1 = P_{\langle p,q \rangle}(1)$. Now proceed with induction on the total number of prime factors.
Calculation of $\Phi_n(\pm1)$

For $n > 1$, we have $\log(\Phi_n(1)) = \Lambda(n)$, with $\Lambda$ the von Mangoldt function.
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$$\pi(x) := \sum_{p \leq x} 1 \sim \frac{x}{\log x},$$

or equivalently

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$$\Phi_n(-1) = \begin{cases} p & \text{if } n = 2p^m; \\ 1 & \text{otherwise.} \end{cases}$$
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Calculation of $\Phi_n(\zeta)$ with $\zeta$ a general root of unity.
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Calculation of $\Phi_n(\zeta)$ with $\zeta$ a general root of unity.

Not much known. Work in progress.
Consequences for cyclotomic ns

As we have seen, if a NS is cyclotomic, then

\[ P_S(x) = \prod_{d \in \mathcal{D}} \Phi_d(x)^{e_d}, \text{ with } e_d > 0 \text{ uniquely determined.} \]
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**Lemma (Cyclotomic restriction)**

*The set \( D \) does not contain 1 or prime powers.*
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**Lemma (Cyclotomic restriction)**

The set \( D \) does not contain 1 or prime powers.

**Proof.**

Since \( P_S(1) = 1 \) and \( \Phi_1(x) = x - 1 \) we infer that \( e_1 = 0 \).
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Restrictions on the set $D$?

Lemma (Cyclotomic restriction)

The set $D$ does not contain 1 or prime powers.

Proof.

Since $P_S(1) = 1$ and $\Phi_1(x) = x - 1$ we infer that $e_1 = 0$. Let $p^m$ be a prime power in $D$. Then by the value at 1 lemma we have $p | \Phi_{p^m}(1) | P_S(1)$. Contradiction.
Let $S \neq \mathbb{N}$ be a numerical semigroup. Then $P'_S(1) = g(S)$. 

Proof. There exist $2 \leq k_1 < \cdots < k_{2n+1}$ such that $P_S(x) = 1 - x + x^{k_1} - x^{k_2} + \cdots - x^{k_{2n}} + x^{k_{2n}+1}$. In fact, $k_1 = m(S) > 1$ and $k_{2n} + 1 = F(S) + 1$.

Gapblock correspondence: $N \setminus S = [1, k_1 - 1] \cup [k_2, k_3 - 1] \cup \cdots \cup [k_{2n}, k_{2n} + 1 - 1]$ (1)

$P'_S(x) = (1 + k_1 x^{k_1 - 1}) + \cdots + (1 - k_{2n} x^{k_{2n} - 1} + k_{2n+1} x^{k_{2n} + 1 - 1})$ (2)

The conclusion now follows on comparing (1) and (2).
Lemma (Connection with genus)

Let $S \neq \mathbb{N}$ be a numerical semigroup. Then $P'_S(1) = g(S)$.

Proof.

There exist $2 \leq k_1 < \cdots < k_{2n+1}$ such that

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In fact, $k_1 = m(S) > 1$ and $k_{2n+1} = F(S) + 1$. 
Lemma (Connection with genus)

Let $S \neq \mathbb{N}$ be a numerical semigroup. Then $P_S'(1) = g(S)$.

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In fact, $k_1 = m(S) > 1$ and $k_{2n+1} = F(S) + 1$. Gapblock correspondence:

$$\mathbb{N} \setminus S = [1, k_1 - 1] \cup [k_2, k_3 - 1] \cup \cdots \cup [k_{2n}, k_{2n+1} - 1] \quad (1)$$

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$$P_S'(1) = (k_1 - 1) + (k_3 - k_2) + \cdots + (k_{2n+1} - k_{2n}). \quad (2)$$
Semigroup Polynomials

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The conclusion now follows on comparing (1) and (2).
Lemma

Let $S$ be a cyclotomic numerical semigroup and $p > 2$ a prime. Then

\[ p \mid P_S(-1) \iff \Phi_{2^p}(x) \mid P_S(x) \]

for some $k \geq 1$. 

Example. Take $S = \langle 6, 9, 11 \rangle$. Then $P_S(-1) = 3$ and $P_S = \Phi_{18} \Phi_{33}$. 

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for some $k \geq 1$.

Proof.

“$\iff$”. The assumption $\Phi_{2p^k}(x) \mid P_S(x)$ implies that $\Phi_{2p^k}(-1) \mid P_S(-1)$. Now invoke the Lemma “Value at $-1$”.
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“$\Leftarrow$”. The assumption $\Phi_{2p^k}(x) \mid P_S(x)$ implies that $\Phi_{2p^k}(-1) \mid P_S(-1)$. Now invoke the Lemma “Value at $-1$”.

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Lemma

Let $S$ be a cyclotomic numerical semigroup and $p > 2$ a prime. Then

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Example. Take $S = \langle 6, 9, 11 \rangle$. Then $P_S(-1) = 3$ and $P_S = \Phi_{18}\Phi_{33}$. 

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Example. Take $S = \langle 6, 9, 11 \rangle$. Then $P_S(-1) = 3$ and $P_S = \Phi_{18}\Phi_{33}$. 
Even beats odd

We let $g(a, d) := \#\{g \notin S : g \geq 0, \ g \equiv a \ (\text{mod} \ d)\}$. 
We let $g(a, d) := \#\{g \not\in S : g \geq 0, \; g \equiv a \pmod{d}\}$. We have

$$P_S(-1) = 1 - 2 \sum_{s \not\in S} (-1)^s = 1 - 2(g(0, 2) - g(1, 2))$$

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**Lemma (Even beats odd)**

*If* \( g(1, 2) < g(0, 2) \), *then* \( S \) *is not cyclotomic.*
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This inequality is equivalent with \( P_S(-1) < 0 \). If \( S \) were cyclotomic, then by the value at \(-1\) lemma always \( \Phi_n(-1) \geq 0 \) and hence \( P_S(-1) \geq 0 \).
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Even beats odd in practice

Is the criterion actually of any practical use?
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- For $S = \langle 3, 5 \rangle$ we have $G = \{1, 2, 4, 7\}$ and so $g(0, 2) = g(1, 2) = 2$
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- $S = \langle 5, 6, 7, 8 \rangle$ is not cyclotomic. We have $g(0, 2) = 2$ and $g(1, 2) = 3$. Thus Lemma “Even beats odd” is not if and only if.
- We took all numerical semigroups $S$ that are symmetric and not complete intersection with $F(S) \leq k$ and determined how often on average Lemma “Even beats odd” applies. Our computations (with $k \leq 69$) indicate that likely an average exists and is in $[0.8, 0.85]$. 
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We have $P_S(\zeta) \in \mathbb{Z}[\zeta]$, the ring of integers of the cyclotomic field $\mathbb{Q}(\zeta) \simeq \mathbb{Q}[x]/(\Phi_m(x))$, which is of degree $\varphi(m)$.
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**Theorem**

If $Ps(-1) \equiv 1 \pmod{4}$ and $Ps(i)$ is not a real number, then $S$ is not cyclotomic.
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Work in progress...
Symmetric non-cyclotomic ns with $e(S) \geq 4$

**Theorem**

If $e(S) \leq 3$, then $S$ is cyclotomic iff $S$ is symmetric.
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For \( k \geq 5 \) put \( S_k = \{0, k, k + 1, \ldots, 2k - 2, 2k, \rightarrow\} \).
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**Example**

\( S = \langle 5, 6, 7, 8 \rangle \), with \( F(S) = 9 \) is the symmetric ns with the smallest Frobenius number that is not cyclotomic.
Symmetric non-cyclotomic ns with $e(S) \geq 4$

**Conjecture**

Put $P_{S_k}(x) = 1 - x + x^k - x^{2k-1} + x^{2k}$. For every $k \geq 5$ this polynomial has a root **not** on the unit circle.
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Put $P_{S_k}(x) = 1 - x + x^k - x^{2k-1} + x^{2k}$. For every $k \geq 5$ this polynomial has a root not on the unit circle.

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For every $k \geq 5$ the symmetric ns $S_k$ is non-cyclotomic and has embedding dimension $e(S_k) = k - 1 \geq 4$. 

Expect that the conjecture can be proved using the methods B. Gross, E. Hironaka and C. McMullen used in 2009 to study the cyclotomic factors of the Coxeter polynomial $E_n(x) = x^n - 2(x^3 - x - 1) + x^3 + x^2 - 1 x - 1$.

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Counting cyclotomic ns of given Frobenius number

Theorem (Upper bound)

Let \( k \geq 1 \) be odd and \( N(k) \) denote the number of cyclotomic numerical semigroups having Frobenius number \( k \).

\[
N(k) < e^{3.577 \sqrt{k}} \quad \text{for all } k \text{ large enough.}
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On the other hand:

Theorem (Backelin)

For all odd \( k \) large enough there are \( e^{(\log 2) \lfloor k/8 \rfloor} \) symmetric numerical semigroups having Frobenius number \( k \).

It follows that there are abundantly many symmetric numerical semigroups that are not cyclotomic.
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Counting cyclotomic ns of given Frobenius number

Sketch of proof of Theorem “Upper bound”.

Let $S$ be a cyclotomic ns with $F(S) = k$. Write $P_S(x) = \prod_{d \in D} \Phi_d(x)^{e_d}$, with $e_d \geq 1$. From this identity we obtain that $F(s) + 1 = k + 1 = \sum_{d \in D} e_d \varphi(d)$, which is a cyclotomic partition of $k + 1$. The number of cyclotomic partitions of $n$ we denote by $c(n)$. We infer that $N(k) \leq c(k + 1)$.

Theorem (Boyd and Montgomery, 1988) $c(n) \sim A e^{B \sqrt{n}/\sqrt{\log n}}$, $n \to \infty$. 

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Polynomially Related Numerical Semigroups

Definition
We say that the numerical semigroup $S$ is polynomially related to the numerical semigroup $T$, and denote this by $S \leq_P T$, if there exist $f(x) \in \mathbb{Z}[x]$ and an integer $w \geq 1$ such that

$$H_S(x^w)f(x) = H_T(x)$$

or equivalently,

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Example
a) $\langle p, q \rangle \leq_P \langle p^m, q^n \rangle$ if $1 \leq a \leq m$ and $1 \leq b \leq n$.

b) $\langle p, q \rangle \leq_P \langle p, q \rangle \langle n \rangle$ if $a, b \geq 1$ and $2 \leq a + b \leq n + 1$.

Problem
Find necessary and sufficient conditions such that $S \leq_P T$. 
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Example
a) $\langle p^a, q^b \rangle \leq_P \langle p^m, q^n \rangle$ if $1 \leq a \leq m$ and $1 \leq b \leq n$.
b) $\langle p^a, q^b \rangle \leq_P B_n(p, q)$ if $a, b \geq 1$ and $2 \leq a + b \leq n + 1$.

Problem
Find necessary and sufficient conditions such that $S \leq_P T$. 
Polynomially Related Numerical Semigroups

In proving the following, we make repeated use of the fact that $P_S(1) = 1$ and $P'_S(1) = g(S)$.

Lemma

Suppose that $H_S(x) f(x) = H_T(x)$ holds with $S$, $T$ numerical semigroups. Then

a) $f(0) = 1$.

b) $f(1) = w$.

c) $f'(1) = w (g(T) - wg(S) + (w - 1)/2)$.

d) $F(T) = wF(S) + \text{deg} f$.

e) If $w$ is even, then $f(-1) = 0$.

f) If $w$ is odd, then $f(-1) = P_T(-1)/P_S(-1)$.

g) If $T$ is cyclotomic, then so is $S$.

h) If $S$ is cyclotomic, then $T$ is cyclotomic iff $f$ is Kronecker.
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An Application

Theorem

Let \( p \neq q \) be primes and \( m, n \) positive integers. The quotient
\[
Q(x) = \frac{P_m(p, q^n)}{\Phi_p(mq^n)}(x)
\]
is in \( \mathbb{Z}[x] \), is monic and has constant coefficient 1. Its non-zero coefficients alternate between 1 and \(-1\).

In fact, a more general result holds.

Theorem

Suppose that \( S \) and \( T \) are numerical semigroups with
\[
H_S(x^w) = f(x)
\]
for some \( w \geq 1 \) and \( f \in \mathbb{N}[x] \). Put
\[
Q(x) = \frac{P_T(x)}{P_S(x^w)}(x).
\]
Then \( Q(0) = 1 \) and \( Q(x) \) is a monic polynomial having non-zero coefficients that alternate between 1 and \(-1\).
An Application

Theorem

Let $p \neq q$ be primes and $m, n$ positive integers. The quotient

$$Q(x) = P_{\langle p^m, q^n \rangle}(x)/\Phi_{p^m q^n}(x)$$

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Theorem

Let $p \neq q$ be primes and $m, n$ positive integers. The quotient

$$Q(x) = P_{\langle p^m, q^n \rangle}(x)/\Phi_{p^m q^n}(x)$$

is in $\mathbb{Z}[x]$, is monic and has constant coefficient 1. Its non-zero coefficients alternate between 1 and $-1$.

In fact, a more general result holds.

Theorem

Suppose that $S$ and $T$ are numerical semigroups with $H_S(x^w)f(x) = H_T(x)$ for some $w \geq 1$ and $f \in \mathbb{N}[x]$. Put $Q(x) = P_T(x)/P_S(x^w)$. Then $Q(0) = 1$ and $Q(x)$ is a monic polynomial having non-zero coefficients that alternate between 1 and $-1$. 
Thank you for attention!