Puiseux Monoids and Their Atomic Structure

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on Numerical Semigroups

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Outline

1. Basic Notions

2. Atomicity Conditions

3. Bounded Puiseux Monoids

4. Monotone Puiseux Monoids
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2 Atomicity Conditions

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4 Monotone Puiseux Monoids
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2. Atomicity Conditions
3. Bounded Puiseux Monoids
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2. Atomicity Conditions
3. Bounded Puiseux Monoids
4. Monotone Puiseux Monoids
**What is a Puiseux monoid?**

**Definition**

A Puiseux monoid is an additive submonoid of $\mathbb{Q}_{\geq 0}$.

**Remark:** Puiseux monoids are a generalization of numerical semigroups. However, the former are not necessarily
- finitely generated;
- atomic.

**Example:** For a prime $p$, consider the Puiseux monoid

$$M = \langle 1/p^n \mid n \in \mathbb{N} \rangle.$$ 

The set of atoms of $M$ is empty, i.e., $\mathcal{A}(M) = \emptyset$; hence $M$ is not atomic. In addition, $M$ fails to be finitely generated.
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Every numerical semigroup is finitely generated, while:

**Observation (1)**

A Puiseux monoid is finitely generated iff it is isomorphic to a numerical semigroup.

Numerical semigroups are atomic and minimally generated, while:

**Observation (2)**

A Puiseux monoid is atomic iff it is minimally generated.

Numerical semigroups have a *unique* minimal generating set, while:

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If a Puiseux monoid has a minimal generating set, then such a generating must be *unique*. 
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Let $P$ denote the set of primes.

**Example 1:** The Puiseux monoid $M = \langle 1/p \mid p \in P \rangle$ is atomic, and $A(M) = \{1/p \mid p \in P\}$. Therefore $|A(M)| = \infty$.

**Example 2:** Let $M$ be the Puiseux monoid generated by the set $S \cup T$, where $S = \{1/2^n \mid n \in \mathbb{N}\}$ and $T = \{1/p \mid n \in P\setminus\{2\}\}$. It follows that $M$ is not atomic; however, $A(M)$ is the infinite set $T$.

**Example 3** If $\{d_n\}$ is a sequence of natural numbers such that $d_n \mid d_{n+1}$ properly for every $n \in \mathbb{N}$, then $M = \langle 1/d_n \mid n \in \mathbb{N} \rangle$ is a Puiseux monoid satisfying $A(M) = \emptyset$; this is because

$$\frac{1}{d_n} = \frac{d_{n+1}}{d_n} \frac{1}{d_{n+1}}$$

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Let \( M \) be a Puiseux monoid. Then \( d(M\setminus\{0\}) \) is bounded iff \( M \) is atomic (indeed, isomorphic to a numerical semigroup).

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Let \( M \) be a Puiseux monoid. If 0 is not a limit point of \( M \), then \( M \) is atomic.
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As we have seen before, not every Puiseux monoid is atomic. However, every Puiseux monoid contains a nontrivial atomic submonoid.

**Theorem**

If $M$ is Puiseux monoid, then it satisfies **exactly one of the following conditions**:

- $M$ is isomorphic to a numerical semigroup;
- $M$ contains an atomic submonoid with infinitely many atoms.
Existence of Nontrivial Atomic Submonoids

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Realizability of $|\mathcal{A}(M)|$

**Theorem**

For every $m \in \mathbb{N}_0 \cup \{\infty\}$, there exists a Puiseux monoid $M$ such that $|\mathcal{A}(M)| = m$.

**Sketch of Proof:**

- For $m = 0$, we can take $M = \langle 1/p^n \mid n \in \mathbb{N} \rangle$, where $p$ is a prime.
- Let $m \in \mathbb{N}$. For distinct primes $p$ and $q$, define

$$M = \left\langle m, \ldots, 2m - 1, \frac{q}{p^{m+1}}, \frac{q}{p^{m+2}}, \ldots \right\rangle.$$ 

If $q > m$, then $\mathcal{A}(M) = \{m, \ldots, 2m - 1\}$ and so $|\mathcal{A}(M)| = m$.
- Finally, suppose $m = \infty$. Let $P$ denote the set of primes, and take $M = \langle 1/p \mid p \in P \rangle$. Then $\mathcal{A}(M) = \{1/p \mid p \in P\}$ and so $|\mathcal{A}(M)| = \infty$. 

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Bounded Puiseux Monoids

Definition

Let $M$ be a Puiseux monoid.

- We say that $M$ is *bounded* if it can be generated by a bounded subset of rationals.
- We say that $M$ is *strongly bounded* if it can be generated by a subset of rationals $R$ such that $\mathfrak{n}(R)$ is bounded.

Observations:

1. Every strongly bounded Puiseux monoid is bounded.
2. If $P$ denotes the set of primes, then $M = \langle \frac{p-1}{p} \mid p \in P \rangle$ is bounded but not strongly bounded.
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**Observations:**

1. Every strongly bounded Puiseux monoid is bounded.
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Definition
A Puiseux monoid $M$ is said to be *antimatter* if $\mathcal{A}(M)$ is empty.

Recall: If $\{d_n\} \subset \mathbb{N}$ such that $d_n \mid d_{n+1}$ properly, then $M = \langle 1/d_n \mid n \in \mathbb{N} \rangle$ satisfies that $\mathcal{A}(M) = \emptyset$, i.e., $M$ is antimatter. The next result is a generalization of this fact.

Definition: The *spectrum* of a sequence $\{a_n\}$ is the set of primes $p$ such that $p \mid a_n$ for every $n$ large enough.

Theorem
Let $\{r_n \mid n \in \mathbb{N}\}$ be a strongly bounded subset of rationals generating $M$. If $d(r_n)$ divides $d(r_{n+1})$, the sequence $\{d(r_n)\}$ is unbounded, and the spectrum of $\{n(r_n)\}$ is empty, then $M$ is antimatter.
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Antimatter Puiseux Monoids

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Definition

A Puiseux monoid $M$ is said to be \textit{antimatter} if $A(M)$ is empty.

Recall: If $\{d_n\} \subset \mathbb{N}$ such that $d_n | d_{n+1}$ properly, then $M = \langle 1/d_n \mid n \in \mathbb{N} \rangle$ satisfies that $A(M) = \emptyset$, i.e., $M$ is antimatter. The next result is a generalization of this fact.

Definition: The \textit{spectrum} of a sequence $\{a_n\}$ is the set of primes $p$ such that $p | a_n$ for every $n$ large enough.

Theorem

Let $\{r_n \mid n \in \mathbb{N}\}$ be a strongly bounded subset of rationals generating $M$. If $d(r_n)$ divides $d(r_{n+1})$, the sequence $\{d(r_n)\}$ is unbounded, and the spectrum of $\{n(r_n)\}$ is empty, then $M$ is antimatter.
A Puiseux monoid $M$ is said to be finite if there are only finitely many primes dividing elements of $d(M)$.

**Example:** If $P$ denotes the set of primes and $p \in P$, then $\langle 1/p^n \mid n \in \mathbb{N} \rangle$ is finite, but $\langle 1/q \mid q \in P \rangle$ is not.

**Theorem**

Let $M$ be a strongly bounded finite Puiseux monoid. Then $M$ is atomic iff $M$ is isomorphic to a numerical semigroup.
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We say that a subset of $\mathbb{R}$ is increasing (resp., decreasing) if we can list its elements increasingly (resp., decreasingly).

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A Puiseux monoid $M$ is said to be *increasing* (resp., *decreasing*) if it can be generated by an increasing (resp., decreasing) set of rationals. A Puiseux monoid is *monotone* if it is either increasing or decreasing.

**Observations:**

- Increasing Puiseux monoids are atomic.
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Prime Reciprocal Puiseux Monoid

Definition
A Puiseux monoid $M$ is prime reciprocal if there exists a subset of primes $P$ such that $M = \langle \frac{1}{p} \mid p \in P \rangle$.

Theorem (G-Gotti)
Every submonoid of a reciprocal Puiseux monoid is atomic.

Remark: In particular, a prime reciprocal Puiseux monoid is atomic. The next question suggests itself.

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For \( r \in \mathbb{Q}_{>0} \), we call \( \text{multiplicative } r\text{-cyclic} \) to the Puiseux monoid generated by the positive powers of \( r \), and we denote it by \( M_r \), that is \( M_r = \langle r^n \mid n \in \mathbb{N} \rangle \).

The next theorem describes the atomic structure of multiplicatively cyclic Puiseux monoids.

Theorem (G-Gotti)

For \( r \in \mathbb{Q}_{>0} \), let \( M_r \) be the multiplicative \( r\)-cyclic Puiseux monoid. Then the following statements hold.

- If \( d(r) = 1 \), then \( M_r \) is atomic with \( \mathcal{A}(M_r) = \{n(r)\} \).
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References

P. A. Garcia-Sanchez and J. C. Rosales. *Numerical Semigroups*.


THANK YOU FOR YOUR KIND ATTENTION!