The Milnor number of plane irreducible singularities in positive characteristic

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In this talk we present some results of

First definitions: intersection multiplicity

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For any power series \( f, h \in K[[x, y]] \) we define the **intersection multiplicity** \( i_0(f, h) \) by putting

\[
i_0(f, h) = \dim_K K[[x, y]]/(f, h),
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where \( (f, h) \) is the ideal of \( K[[x, y]] \) generated by \( f \) and \( h \).
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**Property**

Let $f, h$ be non-zero power series without constant term. Then $i_0(f, h) < +\infty$ if and only if $\{ f = 0 \}$ and $\{ h = 0 \}$ have no common branch.
First definitions: semigroup of a branch

Properties

- $i_0(f, h_1 h_2) = i_0(f, h_1) + i_0(f, h_2)$.
- $i_0(f, 1) = 0$. 

For any irreducible power series $f \in K[[x, y]]$, where $K$ is an algebraically closed field of characteristic $p \geq 0$, we put $\Gamma(f) = \{i_0(f, h) : h \text{ runs over all power series such that } h \not\equiv 0 \text{ (mod } f)\}$. $\Gamma(f)$ is a semigroup called the semigroup associated with the branch $\{f = 0\}$. 
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$\Gamma(f)$ is a semigroup called the **semigroup associated with the branch** $\{f = 0\}$. 
Properties of the semigroup

Lemma

- $\Gamma(f)$ is a numerical semigroup (i.e. $\gcd(\Gamma(f)) = 1$).
- There exists a unique sequence $v_0, \ldots, v_g$ such that
  - $v_0 = \min(\Gamma(f) \setminus \{0\}) = \text{ord } f$,
  - $v_k = \min(\Gamma(f) \setminus Nv_0 + \cdots + Nv_{k-1})$ for $k \in \{1, \ldots, g\}$,
  - $\Gamma(f) = Nv_0 + \cdots + Nv_g$. 
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The sequence $v_0, \ldots, v_g$ is called the minimal sequence of generators of $\Gamma(f)$. 
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**Definition**

\( \Gamma(f) \) is a **tame semigroup** if \( p \) does not divide \( v_k \) for all \( k \in \{0, 1, \ldots, g\} \).
Properties of the semigroup

Let $e_k := \gcd(v_0, \ldots, v_k)$ for $k \in \{1, \ldots, g\}$. Then
- $e_0 > e_1 > \cdots e_{g-1} > e_g = 1$ and
- $e_{k-1}v_k < e_kv_{k+1}$ for $k \in \{1, \ldots, g - 1\}$.

Let $n_k := e_{k-1}/e_k$ for $k \in \{1, \ldots, g\}$. Then
- $n_k > 1$ for $k \in \{1, \ldots, g\}$ and
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Properties

- $\Gamma(f)$ is a strongly increasing semigroup.
- $\Gamma(f)$ has **conductor**

$$c(f) = \sum_{k=1}^{g} (n_k - 1) v_k - v_0 + 1.$$
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Milnor number

The **Milnor number** of $f$ is the intersection multiplicity

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In **characteristic zero** we have

$$\mu(f) = c(f),$$

for any irreducible power series $f \in K[[x, y]]$, and consequently $\mu(f)$ **is determined by** $\Gamma(f)$. 
But in positive characteristic is not, in general, true:

**Example (Boubakri-Greuel-Markwig)**

\[ f = x^p + y^{p-1} \text{ and } g = (1 + x)f, \text{ where } p > 2. \]

Then \( \Gamma(f) = \Gamma(g) \), \( c(f) = c(g) = (p - 1)(p - 2) \) but \( \mu(f) = +\infty \) and \( \mu(g) = p(p - 2) \).
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In positive characteristic it is well-known that \( \mu(f) \geq c(f). \)
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We give necessary and sufficient conditions for the equality \( \mu(f) = c(f) \) in terms of the semigroup associated with \( f \), provided that \( p > v_0 = \text{ord } f = \text{multiplicity of } \Gamma(f) \).
Main result

**Theorem (GB-P, May 2015)**

Let $f \in K[[x, y]]$ be an irreducible singularity and let $v_0, \ldots, v_g$ be the minimal system of generators of $\Gamma(f)$. Suppose that $p = \text{char } K > v_0$. Then the following two conditions are equivalent:

- $\mu(f) = c(f)$
- $\Gamma(f)$ is a tame semigroup ($v_k \not\equiv 0 \pmod{p}$ for $k = 1, \ldots, g$).
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**Example**

Let $f(x, y) = (y^2 + x^3)^2 + x^5y$. Then $f$ is irreducible and $\Gamma(f) = 4N + 6N + 13N$, so the conductor is $c(f) = 16$. Let $p = \text{char } K > v_0 = 4$. If $p \neq 13$ then $\mu(f) = c(f)$ by Theorem. If $p = 13$ then a direct calculation shows that $\mu(f) = 17$. 
Ingredients of the proof

Let \( f \in K[[x, y]] \) be an irreducible singularity with

\[
\Gamma(f) = N v_0 + \cdots + N v_g.
\]

Since \( f \) is unitangent \( i_0(f, x) = \text{ord} f = v_0 \) or \( i_0(f, y) = \text{ord} f = v_0 \).

We assume that \( i_0(f, x) = \text{ord} f = v_0 \).
Ingredients of the proof

We need a sharpened version of Merle’s factorization theorem on polar curves:

**Theorem (Factorization of the polar curve)**

Suppose that \( v_0 = \text{ord } f \not\equiv 0 \pmod{p} \). Then \( \frac{\partial f}{\partial y} = \psi_1 \cdots \psi_g \) in \( K[[x, y]] \), where

(i) \( \text{ord } \psi_k = \frac{v_0}{e_k} - \frac{v_0}{e_{k-1}} \) for \( k \in \{1, \ldots, g\} \).

(ii) If \( \phi \in K[[x, y]] \) is an irreducible factor of \( \psi_k \), \( k \in \{1, \ldots, g\} \), then

\[
\frac{i_0(f, \phi)}{\text{ord } \phi} = \frac{e_{k-1} v_k}{v_0},
\]

and

(iii) \( \text{ord } \phi \equiv 0 \pmod{\frac{v_0}{e_{k-1}}} \).
Ingredients of the proof

Lemma

Suppose that \( v_0 = \text{ord } f \not\equiv 0 \pmod{p} \). Then

\[
i_0 \left( f, \frac{\partial f}{\partial y} \right) = c(f) + \text{ord } f - 1.
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**Lemma**

Suppose that $p > \text{ord } f$. Then $i_0 \left( f, \frac{\partial f}{\partial y} \right) \leq \mu(f) + \text{ord } f - 1$ with equality if and only if $v_k \not\equiv 0 \pmod{p}$ for $k \in \{1, \ldots, g\}$. 
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**Proof**: we use the factorization of the polar curve.
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Proof: we use the factorization of the polar curve.

Proof of main theorem: it is a consequence of Lemmas.
What happens if \( p = \text{char } K \leq \nu_0 = \text{ord } f \)?

What happens if we do not suppose \( p = \text{char } K > \nu_0 = \text{ord } f \)?
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Proposición (Case $g = 1$)

If $\Gamma(f) = N\nu_0 + N\nu_1$ (so $c(f) = (\nu_0 - 1)(\nu_1 - 1)$) then

$$\mu(f) \geq (\nu_0 - 1)(\nu_1 - 1)$$

with equality if and only if $\nu_0 \not\equiv 0 \pmod{p}$ and $\nu_1 \not\equiv 0 \pmod{p}$. 
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Conjecture

Let $f \in \mathbb{K}[[x, y]]$ be an irreducible singularity with semigroup $\Gamma(f) = Nv_0 + \cdots + Nv_g$. Suppose that $p = \text{char} \mathbb{K} > \text{ord } f$. Then the following two conditions are equivalent:

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Conjecture

Let $f \in K[[x, y]]$ be an irreducible singularity with semigroup $\Gamma(f) = N v_0 + \cdots + N v_g$. Suppose that $p = \text{char } K > \text{ord } f$. Then the following two conditions are equivalent:

1. $\mu(f) = c(f)$
2. $\Gamma(f)$ is a tame semigroup ($v_k \not\equiv 0 \pmod{p}$ for $k = 1, \ldots, g$).

Second lemma fails if we remove the hypothesis $p > \text{ord } f$. 

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Let $f \in K[[x, y]]$ be an irreducible singularity with semigroup $\Gamma(f) = Nv_0 + \cdots + Nv_g$. Suppose that $\rho = \text{char } K > \text{ord } f$. Then the following two conditions are equivalent:

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If $\Gamma(f)$ is a tame semigroup then $\mu(f) = c(f)$.
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The other implication is still open.