Seeds of numerical semigroups

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\[ \Lambda = \{\lambda_0 = 0, \lambda_1, \lambda_2, \ldots \}, \text{ with } \lambda_i < \lambda_{i+1} \text{ for all } i \in \mathbb{N}_0, \text{ is a numerical semigroup.} \]
Notation

\[ \Lambda = \{ \lambda_0 = 0, \lambda_1, \lambda_2, \ldots \} \], with \( \lambda_i < \lambda_{i+1} \) for all \( i \in \mathbb{N}_0 \), is a numerical semigroup.

\[ g = g(\Lambda) = |\mathbb{N}_0 \setminus \Lambda| \] is the genus of \( \Lambda \).
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Example (A). \( \Lambda = \{ \lambda_0 = 0, 8, 10, 11, 14, 15, 16, 17, 18, \ldots \} \)
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\( g = 10 \)
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Example (A).

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\[ g = 10 \quad k = 4 \]
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Example (A).  
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A generator of \( \Lambda \) is a non-gap \( \lambda_t \neq 0 \) such that, equivalently,
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\[g = 10 \quad k = 4\]

A generator of \(\Lambda\) is a non-gap \(\lambda_t \neq 0\) such that, equivalently,

- \(\Lambda \setminus \{\lambda_t\}\) is a numerical semigroup,
- \(\lambda_t \neq \lambda_j + \lambda_{j'}\) for all \(0 < j, j' < t\).
The tree of numerical semigroups

Semigroups are represented by the non-zero non-gaps up to the conductor
Seeds of a numerical semigroup

For any index $i \geq 0$, define

$$\Lambda_i := \Lambda \setminus \{\lambda_1, \ldots, \lambda_i\}.$$
Seeds of a numerical semigroup

For any index \( i \geq 0 \), define

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\Lambda_i := \Lambda \setminus \{\lambda_1, \ldots, \lambda_i\}.
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It is a semigroup of genus \( g + i \).
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**Definition.** Given $0 \leq i < k$, a non-gap $\lambda_t$ with $\lambda_t \geq c$ is an order-$i$ seed of $\Lambda$ if, equivalently,
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**Definition.** Given \( 0 \leq i < k \), a non-gap \( \lambda_t \) with \( \lambda_t \geq c \) is an order-\( i \) seed of \( \Lambda \) if, equivalently,

- \( \lambda_t + \lambda_i \) is a generator of \( \Lambda_i \),
- \( \Lambda_i \setminus \{\lambda_t + \lambda_i\} \) is a numerical semigroup,
- \( \lambda_t + \lambda_i \neq \lambda_j + \lambda_{j'} \) for all \( i < j, j' < t \).
What makes $\lambda_t \geq c(\Lambda)$ an order-zero seed?
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\( \lambda_t + 0 = \lambda_t \) should be a generator of \( \Lambda_0 = \Lambda \).
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That is, the order-zero seeds of $\Lambda$ are its generators $\geq c(\Lambda)$.
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That is, the order-zero seeds of $\Lambda$ are its generators $\geq c(\Lambda)$.

They are in bijection with its immediate descendants in the semigroup tree.
Seeds of a numerical semigroup
Seeds of a numerical semigroup

Example (A).

\[ \Lambda = \{ \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \} = \{ 0, 8, 10, 11, 14, 15, 16, 17, 18, \ldots \} \]
Seeds of a numerical semigroup

Example (A).

\[ \Lambda = \{ 0, 8, 10, 11, 14, 15, 16, 17, 18, \ldots \} \]

\[ \lambda_4 = 14 \text{ is not an order-one seed because} \]

\[ \lambda_4 + \lambda_1 = 22 = 11 + 11 = \lambda_3 + \lambda_3 \]
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\[ \Lambda = \{ 0, 8, 10, 11, 14, 15, 16, 17, 18, \ldots \} \]

- \( \lambda_4 = 14 \) is not an order-one seed because
  \[ \lambda_4 + \lambda_1 = 22 = 11 + 11 = \lambda_3 + \lambda_3 \]

- \( \lambda_5 = 15 \) is an order-one seed because
  \[ \lambda_5 + \lambda_1 = 23 \notin \{ \lambda_2, \lambda_3, \lambda_4 \} + \{ \lambda_2, \lambda_3, \lambda_4 \} = \{ 20, 21, 22, 24, 25, 28 \} \]
The table of seeds of a semigroup

**Lemma.** Any order-$i$ seed of $\Lambda$ is at most $c + \lambda_{i+1} - \lambda_i - 1$. 
The table of seeds of a semigroup

Lemma. Any order-$i$ seed of $\Lambda$ is at most $c + \lambda_{i+1} - \lambda_i - 1$.

Example (B). Table of seeds of $\{0, 5, 8, 9, 10, \ldots \}$

$$c = \lambda_2$$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>1</th>
<th>0</th>
<th>1</th>
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</tbody>
</table>
The table of seeds of a semigroup

**Lemma.** Any order-\(i\) seed of \(\Lambda\) is at most \(c + \lambda_{i+1} - \lambda_i - 1\).

**Example (B).** Table of seeds of \(\{0, 5, 8, 9, 10, \ldots\}\) with \(c = \lambda_2\)

\[
\begin{array}{cccccc}
1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

Its rows are indexed by the possible seed orders, \(0 \leq i \leq k - 1\).
The table of seeds of a semigroup

Lemma. Any order-$i$ seed of $\Lambda$ is at most $c + \lambda_{i+1} - \lambda_i - 1$.

Example (B). Table of seeds of $\{0, 5, 8, 9, 10, \ldots \}$

$\begin{array}{c|cccc}
\leftarrow \lambda_{i+1} - \lambda_i \rightarrow \\
1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}$

Its rows are indexed by the possible seed orders, $0 \leq i \leq k - 1$.

The $i$-th row has $\lambda_{i+1} - \lambda_i$ entries.
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The \(j\)-th entry in the \(i\)-th row is \(\begin{cases} 1 & \text{if } c + j \text{ is an order-}i \text{ seed}, \\ 0 & \text{otherwise}. \end{cases}\)
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Lemma. Any order-\(i\) seed of \(\Lambda\) is at most \(c + \lambda_{i+1} - \lambda_i - 1\).

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\begin{cases}
1 & \text{if } c + j \text{ is an order-}i \text{ seed,} \\
0 & \text{otherwise.}
\end{cases}
\]

The total number of entries in the table is \(c\).
The table of seeds of a semigroup

Example (A).

\[ \Lambda = \{ \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \ldots \} \]
The table of seeds of a semigroup

Example (A).

\[ \Lambda = \{ \lambda_0, 8, 10, 11, 14, 15, 16, 17, 18, \ldots \} \]

<table>
<thead>
<tr>
<th>order</th>
<th>0</th>
<th>1</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>\lambda_1 - \lambda_0 = 8</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1</td>
<td></td>
<td></td>
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<td></td>
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<td>\lambda_2 - \lambda_1 = 2</td>
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<tr>
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<td>1</td>
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<td></td>
<td>\lambda_3 - \lambda_2 = 1</td>
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<tr>
<td>order</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
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<td></td>
<td>\lambda_4 - \lambda_3 = 3</td>
</tr>
</tbody>
</table>
The table of seeds of a semigroup

Example (A).

\[ \Lambda = \{ \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \} \]

<table>
<thead>
<tr>
<th>Order</th>
<th>Seed</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 8 10 11 14 15 16 17 18  (c)</td>
<td>(\lambda_1 - \lambda_0 = 8)</td>
</tr>
<tr>
<td>1</td>
<td>0 1</td>
<td>(\lambda_2 - \lambda_1 = 2)</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>(\lambda_3 - \lambda_2 = 1)</td>
</tr>
<tr>
<td>3</td>
<td>1 1 1</td>
<td>(\lambda_4 - \lambda_3 = 3)</td>
</tr>
</tbody>
</table>

\(\lambda_4 = c + 0\) is not an order-one seed
The table of seeds of a semigroup

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\[ \Lambda = \{ \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \} \]

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<tbody>
<tr>
<td>( \lambda_0 = 0 )</td>
<td>1</td>
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<td>( \lambda_1 = 8 )</td>
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<td>1</td>
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<td>( \lambda_2 = 10 )</td>
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<td>( \lambda_3 = 11 )</td>
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<td>( \lambda_4 = 14 )</td>
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<td>( \lambda_5 = 15 )</td>
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<td>( \lambda_6 = 16 )</td>
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<td>( \lambda_7 = 17 )</td>
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<td>( \lambda_8 = 18 )</td>
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</tbody>
</table>

\( \lambda_4 = c + 0 \) is not an order-one seed

\( \lambda_5 = c + 1 \) is an order-one seed
Behavior of seeds along the semigroup tree

Suppose $\lambda_s$ is an order-zero seed of $\Lambda$ ($s \geq k$).
Behavior of seeds along the semigroup tree

Suppose $\lambda_s$ is an order-zero seed of $\Lambda$ ($s \geq k$).

$\tilde{\Lambda} := \Lambda \setminus \{\lambda_s\}$ is a semigroup of genus $g + 1$. 
Behavior of seeds along the semigroup tree

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$\tilde{\Lambda} := \Lambda \setminus \{\lambda_s\}$ is a semigroup of genus $g + 1$.

**Example (B).** Descendants of

$\Lambda = \{0, 5, \underbrace{8}_c, 9, 10, \underbrace{11}_\lambda, \underbrace{12}_\lambda, 13, \ldots\}$
Behavior of seeds along the semigroup tree

Suppose $\lambda_s$ is an order-zero seed of $\Lambda (s \geq k)$.

$\tilde{\Lambda} := \Lambda \setminus \{\lambda_s\}$ is a semigroup of genus $g + 1$.

**Example (B).** Descendants of

\[
\Lambda = \{0, 5, \underbrace{8}_c, 9, 10, \underbrace{11}_d, 12, 13, \ldots \}
\]

$s = 2$ \quad $s = 3$ \quad $s = 5$ \quad $s = 6$

$\tilde{\Lambda} = \{0, 5, 9, 10, \ldots \} \quad \tilde{\Lambda} = \{0, 5, 8, 10, \ldots \} \quad \tilde{\Lambda} = \{0, 5, 8, 9, 10, 12, \ldots \} \quad \tilde{\Lambda} = \{0, 5, 8, 9, 10, 11, 13, \ldots \}$
Behavior of seeds along the semigroup tree

Suppose $\lambda_s$ is an order-zero seed of $\Lambda$ ($s \geq k$).

$\tilde{\Lambda} := \Lambda \setminus \{\lambda_s\}$ is a semigroup of genus $g + 1$.

**Example (B).** Descendants of

$$\Lambda = \{0, 5, 8, 9, 10, 11, 12, 13, \ldots\}$$

<table>
<thead>
<tr>
<th>$s$</th>
<th>$\tilde{\Lambda}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>${0, 5, 9, 10, \ldots}$</td>
</tr>
<tr>
<td>3</td>
<td>${0, 5, 8, 10, \ldots}$</td>
</tr>
<tr>
<td>5</td>
<td>${0, 5, 8, 9, 10, 12, \ldots}$</td>
</tr>
<tr>
<td>6</td>
<td>${0, 5, 8, 9, 10, 11, 13, \ldots}$</td>
</tr>
</tbody>
</table>

**Goal.** Obtain the seeds of $\tilde{\Lambda}$ from those of $\Lambda$. 
Suppose $i < k$. 

Old-order recycled seeds
Old-order recycled seeds

Suppose $i < k$.

Any order-$i$ seed $\lambda_t$ of $\Lambda$ with $t > s$ is also an order-$i$ seed of $\tilde{\Lambda}$. 
Old-order recycled seeds

Suppose $i < k$.

Any order-$i$ seed $\lambda_t$ of $\Lambda$ with $t > s$ is also an order-$i$ seed of $\tilde{\Lambda}$.

Example (B). Descendants of

\[
\Lambda = \{0, 5, \underbrace{8}_c, 9, 10, \underbrace{11}_c, \underbrace{12}_c, 13, \ldots\}
\]
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Old-order recycled seeds

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**Example (B).** Descendants of

$\Lambda = \{0, 5, \underbrace{8, 9, 10, 11, 12, 13, \ldots}_{c}\}$
Suppose $i < k$. 

Old-order new seeds
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**Theorem 1.** $\lambda_t > \lambda_s$ is an order-$i$ seed of $\tilde{\Lambda}$ if and only if either

1. $\lambda_t$ is an order-$i$ seed of $\Lambda$


Old-order new seeds

Suppose \( i < k \).

**Theorem 1.** \( \lambda_t > \lambda_s \) is an order-\( i \) seed of \( \tilde{\Lambda} \) if and only if either

1. \( \lambda_t \) is an order-\( i \) seed of \( \Lambda \)
2. \( i < k - 1, \lambda_t = \lambda_s + \lambda_{i+1} - \lambda_i \) and \( \lambda_s \) is an order-(\( i + 1 \)) seed of \( \Lambda \)
Old-order new seeds

Suppose $i < k$.

**Theorem 1.** $\lambda_t > \lambda_s$ is an order-$i$ seed of $\tilde{\Lambda}$ if and only if either

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2. $i < k - 1$, $\lambda_t = \lambda_s + \lambda_{i+1} - \lambda_i$ and $\lambda_s$ is an order-$(i + 1)$ seed of $\Lambda$
3. $i = k - 1$, $\lambda_s = c$, and either

$$\begin{cases} 
\lambda_t = \lambda_s + \lambda_k - \lambda_{k-1} \\
\lambda_t = \lambda_s + \lambda_k - \lambda_{k-1} + 1
\end{cases}$$


Old-order new seeds

Suppose $i < k$.

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   \[
   \begin{align*}
   \lambda_t &= \lambda_s + \lambda_k - \lambda_{k-1} \\
   \lambda_t &= \lambda_s + \lambda_k - \lambda_{k-1} + 1
   \end{align*}
   \]
4. $i = k - 1$, $\lambda_s = c + 1$, and $\lambda_t = \lambda_s + \lambda_k - \lambda_{k-1}$
Old-order new seeds

Suppose $i < k$.

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   - $\lambda_t = \lambda_s + \lambda_k - \lambda_{k-1}$
   - $\lambda_t = \lambda_s + \lambda_k - \lambda_{k-1} + 1$

4. $i = k - 1$, $\lambda_s = c + 1$, and $\lambda_t = \lambda_s + \lambda_k - \lambda_{k-1}$

![Diagram](image)
Suppose $i < k$.

**Theorem 1.** $\lambda_t > \lambda_s$ is an order-$i$ seed of $\tilde{\Lambda}$ if and only if either

1. $\lambda_t$ is an order-$i$ seed of $\Lambda$
2. $i < k - 1$, $\lambda_t = \lambda_s + \lambda_{i+1} - \lambda_i$ and $\lambda_s$ is an order-$(i + 1)$ seed of $\Lambda$
3. $i = k - 1$, $\lambda_s = c$, and either
   \[
   \begin{cases}
   \lambda_t = \lambda_s + \lambda_k - \lambda_{k-1} \\
   \lambda_t = \lambda_s + \lambda_k - \lambda_{k-1} + 1
   \end{cases}
   \]
4. $i = k - 1$, $\lambda_s = c + 1$, and $\lambda_t = \lambda_s + \lambda_k - \lambda_{k-1}$
Suppose $i \geq k$. 
New-order seeds

Suppose $i \geq k$.

Theorem 2.

- If $i < s - 2$, then $\tilde{\Lambda}$ has no order-$i$ seeds.
Suppose $i \geq k$.

**Theorem 2.**

- If $i < s - 2$, then $\tilde{\Lambda}$ has no order-$i$ seeds.
- If $i = s - 2$, then the only order-$i$ seed of $\tilde{\Lambda}$ is $\lambda_s + 1$. 

Suppose $i \geq k$.

**Theorem 2.**

- If $i < s - 2$, then $\tilde{\Lambda}$ has no order-$i$ seeds.
- If $i = s - 2$, then the only order-$i$ seed of $\tilde{\Lambda}$ is $\lambda_s + 1$.
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<table>
<thead>
<tr>
<th>$s = 2$</th>
<th>$s = 3$</th>
<th>$s = 5$</th>
<th>$s = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="" /></td>
<td><img src="image2.png" alt="" /></td>
<td><img src="image3.png" alt="" /></td>
<td><img src="image4.png" alt="" /></td>
</tr>
</tbody>
</table>

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Example (A). $\Lambda = \{0, 8, 10, 11, \underbrace{14}_c, 15, 16, \underbrace{17}_c, 18, \ldots \}$
Example (A). \( \Lambda = \{ 0, 8, 10, 11, 14, 15, 16, 17, 18, \ldots \} \)

\[
\begin{array}{cccccccc}
\lambda_4 & \lambda_5 & \lambda_7 \\
(14) & (15) & (16) \\
\end{array}
\]

\( s = 4 \)

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\( s = 5 \)

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\( s = 7 \)

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]
Example (A). \( \Lambda = \{0, 8, 10, 11, \begin{array}{c} 14 \\ \hline \end{array}, 15, 16, 17, 18, \ldots \} \)
Example (A). $\Lambda = \{0, 8, 10, 11, 14, 15, 16, 17, 18, \ldots\}$

$$s = 4$$

$$s = 5$$

$$s = 7$$
Example (A). $\Lambda = \{0, 8, 10, 11, \lambda_4, \lambda_5, 15, 16, \lambda_7, 18, \ldots\}$

$s = 4$

\[
\begin{array}{ccccccc}
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & & & & & \\
1 & 1 & & & & & \\
\end{array}
\]

$s = 5$

\[
\begin{array}{ccccccc}
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & & & & & \\
1 & & & & & & \\
1 & 1 & & & & & \\
\end{array}
\]

$s = 7$

\[
\begin{array}{ccccccccc}
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & & & & & & & \\
1 & & & & & & & & \\
1 & 1 & & & & & & & \\
\end{array}
\]
The string $G = G(\Lambda)$ stores the gaps of $\Lambda$. 
Strings $G, S$

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The string $S = S(\Lambda)$ stores all the rows of the table of seeds of $\Lambda$ merged in a single string.
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**Example (A).** For $\Lambda = \{0, 8, 10, 11, 14, 15, 16, \ldots \}$,

\[
c = \lambda_4
\]
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**Example (A).** For $\Lambda = \{0, 8, 10, 11, 14, 15, 16, \ldots \}$,

with table of seeds

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 \\
1 \\
1 & 1 & 1 \\
\end{array}
\]
Strings $G, S$

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**Example (A).** For $\Lambda = \{0, 8, 10, 11, 14, 15, 16, \ldots \}$, with table of seeds

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 \\
0 & 1 \\
1 \\
1 & 1 & 1 \\
\end{array}
\]

the strings $G, S$ are

\[
\begin{array}{cccccccccccc}
G \rightarrow & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
S \rightarrow & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]
$G, S$ in the tree of semigroups
Descending algorithm for $G, S$

Let $\tilde{\Lambda} = \Lambda \setminus \{\lambda_s\}$.
Descending algorithm for $G, S$

Let $\tilde{\Lambda} = \Lambda \setminus \{\lambda_s\}$. Set $\Delta = s - k$. 

Update of $G$: $G(\tilde{\Lambda})$ is obtained from $G(\Lambda)$ by replacing the 0 bit $G^{c+\Delta-1}$ with 1.

Update of $S$: Let $\tilde{S} = \tilde{S}_0 \tilde{S}_1 \cdots \tilde{S}_\ell \cdots$ with $\tilde{S}_\ell := \begin{cases} 0 & \text{if } \ell = \lambda_i + j \text{ with } 1 \leq i < k, 0 \leq j < \Delta, \\ S_\ell & \text{otherwise} \end{cases}$.

Then, $S(\tilde{\Lambda}) = \tilde{S}_{\Delta+1} \tilde{S}_{\Delta+2} \cdots \tilde{S}_{c-1} 2 \Delta 0 \cdots 0 1 1 1$. 

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Descending algorithm for $G, S$

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$$
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S_\ell & \text{otherwise.}
\end{cases}
$$

Then,

$$S(\tilde{\Lambda}) = \tilde{S}_{\Delta+1} \tilde{S}_{\Delta+2} \cdots \tilde{S}_{c-1} 0 \cdots 0 1 1 1.$$
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It is useful to manipulate $G(\Lambda)$ and $S(\Lambda)$ as integers in binary form.
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Bitwise operations on binary strings:
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Bitwise operations on binary strings:

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Bitwise operations on binary strings:

- & \textit{and},
- | \textit{inclusive or},
- $\gg$ \textit{right shift} by a non-negative integer $x$ (i.e., multiplying by $2^x$),
- $\ll$ \textit{left shift} by a non-negative integer $x$.

Then,

$$G(\tilde{\Lambda}) = G \mid (1 \gg \tilde{c} - 2)$$
Descending algorithm for $G, S$

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Then,

\[
G(\tilde{\Lambda}) = G \mid (1 \gg \tilde{c} - 2)
\]
\[
S(\tilde{\Lambda}) = (\tilde{S} \ll \Delta + 1) \mid (1 1 1 \gg c + \Delta - 2)
\]
Descending algorithm for $G, S$

Input: $c := c(\Lambda)$, $G := G(\Lambda)$, $S := S(\Lambda)$, $\Delta$

Output: $c(\tilde{\Lambda})$, $G(\tilde{\Lambda})$, $S(\tilde{\Lambda})$

1. $\tilde{S} := S$
2. rake := $G$
3. from 1 to $\Delta$ do
4. rake := rake $\gg$ 1
5. $\tilde{S} := \tilde{S} \&$ rake
6. return $\tilde{c} := c + \Delta + 1$, $G | (1 \gg \tilde{c} - 2)$, $(\tilde{S} \ll \Delta + 1) | (111 \gg \tilde{c} - 3)$
## Comparing algorithms

Time in seconds to compute $n_g$:

<table>
<thead>
<tr>
<th>Method</th>
<th>30</th>
<th>31</th>
<th>32</th>
<th>33</th>
<th>34</th>
<th>35</th>
<th>36</th>
<th>37</th>
<th>38</th>
<th>39</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Apéry - DFS</td>
<td>13</td>
<td>24</td>
<td>39</td>
<td>67</td>
<td>114</td>
<td>193</td>
<td>327</td>
<td>554</td>
<td>933</td>
<td>1577</td>
<td>2657</td>
</tr>
<tr>
<td>Apéry - recursive</td>
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<td>16</td>
<td>28</td>
<td>47</td>
<td>81</td>
<td>136</td>
<td>232</td>
<td>393</td>
<td>634</td>
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<td>1805</td>
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<tr>
<td>decomposition - DFS</td>
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<td>16</td>
<td>27</td>
<td>46</td>
<td>79</td>
<td>131</td>
<td>222</td>
<td>373</td>
<td>626</td>
<td>1050</td>
<td>1762</td>
</tr>
<tr>
<td>single check - DFS</td>
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<td>14</td>
<td>23</td>
<td>39</td>
<td>65</td>
<td>110</td>
<td>185</td>
<td>310</td>
<td>518</td>
<td>868</td>
<td>1448</td>
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<tr>
<td>decomposition - recursive</td>
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<td>165</td>
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<td>1297</td>
</tr>
<tr>
<td>single check - recursive</td>
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<td>4</td>
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<td>87</td>
<td>145</td>
<td>241</td>
<td>400</td>
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<tr>
<td>seeds - DFS</td>
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<td>4</td>
<td>8</td>
<td>12</td>
<td>21</td>
<td>36</td>
<td>58</td>
<td>96</td>
<td>161</td>
<td>269</td>
</tr>
<tr>
<td>seeds - recursive</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>9</td>
<td>15</td>
<td>26</td>
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