Arithmetical properties of monomial curves obtained by gluing

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- The gluing of two numerical semigroups.
- Main results.
- Specific gluings.
- The proofs.
- Extensions.

The gluing of two numerical semigroups

- Let $S_1 = \langle m_1, \ldots, m_d \rangle$ and $S_2 = \langle n_1, \ldots, n_k \rangle$ be two numerical semigroups.

- Let $p \in S_1$ and $q \in S_2$ such that:

(1) $\gcd(p, q) = 1$, and

(2) $p \notin \{m_1, \ldots, m_d\}$ and $q \notin \{n_1, \ldots, n_k\}$.

Definition (C. Delorme 1976; J. C. Rosales, 1991)

The numerical semigroup $S = \langle qm_1, \ldots, qm_d, pn_1, \ldots, pn_k \rangle$ is called *a gluing of S_1 and S_2*.

Gluing is an operation particularly well behaved with respect to presentations.

And this allows to prove, for instance, the following (classical) results:

Proposition

A numerical semigroup other than $\mathbb{N}$ is a complete intersection if and only if it is the gluing of two complete intersection numerical semigroups.

Proposition

A gluing of two symmetric numerical semigroup is symmetric.

Proposition

A numerical semigroup S other than $\mathbb{N}$ is free if and only if S is a gluing of a free numerical semigroup with embedding dimension $e(S) - 1$ and $\mathbb{N}$. In particular, any free numerical semigroup is a complete intersection.

What about other properties, in particular the arithmetical properties of the tangent cone?

Example (Arslan-Mete-Şahin, 2008)

Let $S_1 = \langle 5, 12 \rangle$ and $S_2 = \langle 7, 8 \rangle$. Both have Cohen-Macaulay tangent cone. Then,

$$S = \langle 5 \cdot 21 = 105, 12 \cdot 21 = 252, 7 \cdot 17 = 119, 8 \cdot 17 = 136 \rangle$$

is a gluing of S_1 and S_2 but has not a Cohen-Macaulay tangent cone.

The following definition allows to give some positive answers:

Definition (Arslan-Mete-Şahin, 2008)

The numerical semigroup

$$S = \langle qm_1, \ldots, qm_d, pn_1, \ldots, pn_k \rangle$$

is called *a nice gluing of S_1 and S_2* if $q = an_1$ for some $1 < a < \mathrm{ord}_{S_1}(p)$.

Now, by using a combination of Gröbner basis techniques developed by F. Arslan (2000) in order to compute standard basis and the good behavior of gluing with respect to presentations, the following facts can be proved:

Proposition (Arslan-Mete-Şahin, 2008)

Assume that the numerical semigroup

$$S = \langle qm_1, \ldots, qm_d, pn_1, \ldots, pn_k \rangle$$

is a nice gluing of S_1 and S_2. Then:

(1) *If S_1 and S_2 have Cohen-Macaulay tangent cones, then S has a Cohen-Macaulay tangent cone.*

(2) *If S_1 has a non-decreasing Hilbert function and S_2 has a Cohen-Macaulay tangent cone, than S has a non-decreasing Hilbert function.*

Main results

We introduce a new kind of gluing, that we call specific gluing (to be defined later) that allows to complete and extend the previous results.

This new definition allows to use techniques based on Apéry sets, which maybe are more flexible and easier to handle in this context.

A nice gluing is not necessarily an specific gluing, but a nice gluing such that $G(S_2)$ is Cohen-Macaulay is always an specific gluing.

Concretely, we prove the following results about the Cohen-Macaulay property of the tangent cone:

Proposition (R. Jafari, S. Z, 2014)

Let S be an specific gluing of S_1 and S_2. Then, $G(S)$ is Cohen-Macaulay if and only if $G(S_1)$ is Cohen-Macaulay.

Corollary (R. Jafari, S. Z. , 2014)

Let S be a nice gluing of S_1 and S_2. If $G(S_2)$ is Cohen-Macaulay, then $G(S)$ is Cohen-Macaulay if and only if $G(S_1)$ is Cohen-Macaulay.

Next result concerns with the Gorenstein property. Namely:

Proposition (R. Jafari, S. Z., 2014)

Let $S = \langle qm_1, \ldots, qm_d, pn_1, \ldots, qn_k \rangle$ be an specific gluing of S_1 and S_2. Assume that S_2 is symmetric and M-pure with respect to q. Then, $G(S)$ is Gorenstein if and only if $G(S_1)$ is Gorenstein.

Corollary (R. Jafari, S. Z., 2014)

Let S be a nice gluing of S_1 and S_2. If $G(S_2)$ is Gorenstein, then $G(S)$ is Gorenstein if and only if $G(S_1)$ is Gorenstein.

And we may also prove the following general result about the behavior of the Hilbert function:

Proposition (R. Jafari, S. Z, 2014)

Let S be an specific gluing of S_1 and S_2. Assume that S_1 has a non-decreasing Hilbert function. Then, S has a non-decreasing Hilbert funtion.

Corollary (Arslan-Mete-Şahin, 2008)

Let S be a nice gluing of S_1 and S_2. If $G(S_2)$ is Cohen-Macaulay and S_1 has a non decreasing Hilbert function, then S has a non-decreasing Hilbert function.

Specific gluings

In order to explain the definition of specific gluing we need to recall some notions related with the study of the arithmetical properties of the tangent cone of a numerical semigroup ring.

• $S = \langle m_1, \ldots, m_d \rangle$ a numerical semigroup minimally generated by $m_1 < \cdots < m_d$; $M = S \setminus \{0\}$ is the maximal ideal of S.

• If k is a field, we denote by $k[[S]] = k[[t^{m_1}, \ldots, t^{m_d}]] \subseteq k[[t]]$ the numerical semigroup ring defined by S.

It is the (complete) local ring at the origin of the d-dimensional k-affine monomial curve given by $t \to (t^{m_1}, \ldots, t^{m_d}) \subset \mathbb{A}_k^d$.

- Let $\mathfrak{m} = (t^{m_1}, \ldots, t^{m_d})$ the maximal ideal of $k[[S]]$.
- And let

$$G(S) = \bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1}$$

the associated graded ring of $\mathfrak{m}$ or tangent cone of S.

It is the coordinate ring of the tangent cone at the origin of the corresponding monomial curve.

- We denote by $m(S)$ the multiplicity of S: $m(S) = m_1$.
- For a given $s \in S$, we set the order of s as

$$\mathrm{ord}_S(s) = \max\{n \mid s \in nM\}$$

Equivalently,

$$\mathrm{ord}_S(s) = \max\{n \mid t^s \in \mathfrak{m}^n\}$$

- Then, if $n = \mathrm{ord}_S(s)$,

$$0 \neq [t^s] \in \mathfrak{m}^n/\mathfrak{m}^{n+1} \hookrightarrow G(S)$$

and we denote this element by $(t^s)^*$, the initial form of t^s.

Let $e = m(S)$. Because $G(S)$ is a graded ring of dimension one and $(t^e)^*$ is a parameter of $G(S)$, it is Cohen-Macaulay if and only if $(t^e)^*$ is a non-zero divisor of $G(S)$.

- We set r the reduction number of S, that is

$$r = \min\{r \mid \mathfrak{m}^{r+1} = t^e\mathfrak{m}^r\} = \min\{r \mid (r+1)M = e + rM\}$$

- For any element s in S, we denote by $\mathrm{AP}(S, s)$ the Apéry set of S with respect to s.

Now, the Cohen-Macaulay property of the tangent cone can be detected in several ways. For instance:

Proposition

The following are equivalent:

(1) $G(S)$ *is Cohen-Macaulay.*

(2) $(t^e)^*$ *is a non-zero divisor of* $G(S)$.

(3) $(t^e)^*$ *is a non-zero divisor over the set of elements of the form* $(t^s)^* \in G(S)$.

(4) $\mathrm{ord}_S(s+e) = \mathrm{ord}_S(s) + 1$ *for all* $s \in S$.

(5) $\mathrm{ord}_S(s+e) = \mathrm{ord}_S(s) + 1$ *for all* $s \in S$ *with* $\mathrm{ord}_S(s) \leq r$.

(6) $\mathrm{ord}_S(w+ae) = \mathrm{ord}_S(w) + a$ *for all* $w \in AP(S,e)$ *and* $a \geq 0$.

Now, basically motivated by characterization (5) in the above proposition, we introduce the following number:

Definition

For any $x \in S$, let

$$l_x(S) := \max\{\mathrm{ord}_S(s+x) - \mathrm{ord}_S(x) - \mathrm{ord}_S(s); s \in S \mid \mathrm{ord}_S(s) \leq r\}$$

Observe that, in fact, we have that:

$G(S)$ is Cohen-Macaulay if and only if $l_x(S) = 0$ for any (some) $x = ae, a \geq 1$.

Definition

Let $S = \langle qm_1, \ldots, qm_d, pn_1, \ldots, pn_k \rangle$ be a gluing of S_1 and S_2. We call S a specific gluing of S_1 and S_2 if

$$\mathrm{ord}_{S_2}(q) + l_q(S_2) \leq \mathrm{ord}_{S_1}(p).$$

• Assume that S is a nice gluing of S_1 and S_2 and $G(S_2)$ is Cohen-Macaulay.

Then, $q = an_1$ and so $l_q(S_2) = 0$. Therefore, the condition of being a nice gluing:

$$q = an_1 \text{ for some } 1 < a \leq \mathrm{ord}_{S_1}(p)$$

just tell us that S is an specific gluing of S_1 and S_2.

But observe that in our definition of specific, even being $G(S_2)$ Cohen-Macaulay we do not need the condition $q = an_1$.

In fact, it is clear by the definition that for a given q, all possible gluings of S_1 and S_2 with q are specific, except finitely many of them.

Observe that the example of Arslan-Mete-Şahin:

$$S = \langle 5 \cdot 21 = 105, 12 \cdot 21 = 252, 7 \cdot 17 = 119, 8 \cdot 17 = 136 \rangle$$

is not an specific gluing (although in this case both have Cohen-Macaulay tangent cone) because

$$\mathrm{ord}_{S_2}(21) = 3 > \mathrm{ord}_{S_1}(17) = 2$$

Here is another example:

$$S_1 = \langle 2, 3 \rangle, S_2 = \langle 5, 6, 13 \rangle$$

We have that $G(S_2)$ is not Cohen-Macaulay: $(t^5)^*(t^{13})^* = 0$.

We also have that 11 is an element of order 2 with $l_{11}(S_2) = 2$ (computations with the numerical semigroups package of GAP).

Now take $p = 8$ and $q = 11$. Then

$$\mathrm{ord}_{S_2}(11) + l_{11}(S_2) = 2 + 2 = 4 = \mathrm{ord}_{S_1}(8)$$

Hence $S = \langle 22, 33, 40, 48, 104 \rangle$ is an specific gluing of S_1 and S_2. In particular, we have that $G(S)$ is Cohen-Macaulay.

The proofs

In order to prove our results, we study carefully a particular presentation of the elements of an specific gluing.

In fact, we show that there exists a somehow unique way to present such elements in terms of some concrete Apéry sets.

The following is the crucial result in our approach:

Proposition

Let

$$S = \langle qm_1, \ldots, qm_d, pn_1, \ldots, pn_k \rangle$$

be a specific gluing of S_1 *and* S_2*. If* $u \in S$*, then*

(1) *There exist* $z_1 \in S_1$ *and* $z_2 \in \mathrm{AP}(S_2, q)$ *such that* $u = qz_1 + pz_2$ *and* $\mathrm{ord}_S(u) = \mathrm{ord}_{S_1}(z_1) + \mathrm{ord}_{S_2}(z_2)$*.*

(2) *Let* $u = qz_1 + pz_2$ *be a representation of* u *as in part (1). If* $u = qs_1 + ps_2$ *for some* $s_1 \in S_1$ *and* $s_2 \in \mathrm{AP}(S_2, q)$*, then* $s_1 = z_1$*,* $s_2 = z_2$*.*

Essentially, the above fact is all what we need in order to prove the results concerning the Cohen-Macaulay property and the non-decreasing of the Hilbert function.

(For instance, as a first step one can prove that $m(S) = qm_1$.)

But for the Gorenstein property we need some extra considerations.

In fact, we need to extend the characterization given by L. Bryant (2010) of the Gorensteiness of the tangent cone of a numerical semigroup ring.

Remember that given a numerical semigroup S we have two different partial orderings: for all elements $x, y \in S$

(1) $x \preceq y$ if there exists $z \in S$ such that $y = x + z$;

(2) $x \preceq_M y$ if there exists $z \in S$ such that $y = x + z$ and $\mathrm{ord}_S(y) = \mathrm{ord}_S(x) + \mathrm{ord}_S(z)$.

Now, given an Apéry set $\mathrm{AP}(S, x)$ with respect to some $x \in S$, we denote respectively by $\mathrm{Max\ AP}(S, x)$ and $\mathrm{Max}_M\mathrm{AP}(S, x)$ the corresponding maximal elements.

Definition

S is called pure (resp. M-pure) with respect to $x \in S$ if all elements in $\mathrm{Max\ AP}(S, x)$ (resp. $\mathrm{Max}_M\mathrm{AP}(S, x)$) have the same order.

It's easy to see that S is M-pure with respect to x if and only if S is pure with respect to x and $\mathrm{Max\ AP}(S, x) = \mathrm{Max}_M\mathrm{AP}(S, x)$.

In particular, S is symmetric and M-pure with respect to x if and only if $\mathrm{Max\ AP}_M(S, x)$ has only one element.

Next result extends the one obtained by Bryant for $x = e$.

It is the extra ingredient we need to prove our result on the Gorenstein property of the tangent cone of an specific gluing:

Proposition

Assume that $G(S)$ is Cohen-Macaulay. Then, $G(S)$ is Gorenstein if and only if S is symmetric and any of the following conditions hold:

(1) *S is M-pure (with respect to e);*

(2) *S is M-pure with respect to $x = ke$ for all $k > 0$;*

(3) *S is M pure with respect to $x = ke$ for some $k > 0$.*

Extensions

Remember that a semigroup

$$S = \langle qm_1, , \ldots , qm_d, p \rangle$$

obtained by gluing S_1 and $\mathbb{N}$ is called an extension of S_1.

If moreover it is a nice gluing ($q \leq \mathrm{ord}_{S_1}(p)$) we will call it a nice extension (in this case this is the same as being specific).

Now, we have by the previous results that if $G(S_1)$ is Cohen-Macaulay (resp. Gorenstein) then any nice extension of S_1 has Cohen-Macaulay (resp. Gorenstein) tangent cone.

We observe that the nice condition cannot be removed.

Let:

$$S_1 = \langle 2, 5 \rangle, p = 2 + 5 = 7, q = 3$$

Then,

$$S = \langle 6, 15, 7 \rangle$$

is an extension of S_1 which is not nice.

And $G(S)$ is not Cohen-Macaulay because $(t^6)^*(t^{15})^* = 0$.

Next result provides a way to get extensions whose tangent cone are always Cohen-Macaulay independently from the tangent cone of S_1:

Proposition (R. Jafari, S. Z, (2014))

Let $S = \langle qm_1, \ldots, qm_d, p \rangle$ be an extension of S_1. If $p < q$, then $G(S)$ is Cohen-Macaulay and so the Hilbert function of S is non-decreasing.

We also observe that condition $p < q$ cannot be removed. Let:

$$S_1 = \langle 2, 7 \rangle, p = 5, q = 4$$

Then,

$$S = \langle 5, 8, 28 \rangle$$

is an extension of S_1.

And $G(S)$ is not Cohen-Macaulay because $(t^{20})^*(t^{28})^* = 0$.

We finish by summarizing the above results with the following statement:

Proposition (R. Jafari, S. Z., 2014)

Let $S = \langle qm_1, \ldots, qm_d, p \rangle$ be an extension of S_1. Assume that S_1 has a non-decreasing Hilbert function. If the Hilbert function of S is decreasing, then $\mathrm{ord}_{S_1}(p) < q < p$.

Hence all extensions of S_1 by q except finitely many of them have non-decreasing Hilbert functions.