When the catenary degree meets the tame degree in embedding dimension three numerical semigroups

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based on

P. A. García-Sánchez, C. Viola,
When the catenary degree meets the tame degree in embedding dimension three numerical semigroups, to appear in Involve.

S. T. Chapman, P. A. García-Sánchez, Z. Tripp, C. Viola,
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M. Delgado, P. A. García-Sánchez, J. J. Morais,
GAP package numericalsgps
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- A factorization of $s \in S$ is an element $x = (x_1, \ldots, x_p) \in \mathbb{N}^p$ such that $x_1 n_1 + \cdots + x_p n_p = s$. 

- The length of $x$ is $|x| = x_1 + \cdots + x_p$.

- Given another factorization $y = (y_1, \ldots, y_p)$, the distance $d(x, y)$ is $\max\{|x - \gcd(x, y)|, |y - \gcd(x, y)|\}$, where $\gcd(x, y) = (\min\{x_1, y_1\}, \ldots, \min\{x_p, y_p\})$. 

Setup

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- The length of $x$ is $|x| = x_1 + \cdots + x_p$.
- Given another factorization $y = (y_1, \ldots, y_p)$, the distance between $x$ and $y$ is
  \[ d(x, y) = \max\{ |x - \gcd(x, y)|, |y - \gcd(x, y)| \} , \]
  where $\gcd(x, y) = (\min\{x_1, y_1\}, \ldots, \min\{x_p, y_p\})$. 


66 \in S = \langle 6, 9, 11 \rangle, \ c(S) = 4

The factorizations of 66 \in \langle 6, 9, 11 \rangle are

F(66) = \{(0, 0, 6), (1, 3, 3), (2, 6, 0), (4, 1, 3), (5, 4, 0), (8, 2, 0), (11, 0, 0)\}

The distance between (11, 0, 0) and (0, 0, 6) is 11.
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The catenary degree

The catenary degree of $s \in S$, $c(s)$, is the minimum nonnegative integer $N$ such that for any two factorizations $x$ and $y$ of $s$, there exists a sequence of factorizations $x_1, \ldots, x_t$ of $s$ such that

- $x_1 = x$, $x_t = y$,
- for all $i \in \{1, \ldots, t - 1\}$, $d(x_i, x_{i+1}) \leq N$.

The catenary degree of $S$, $c(S)$, is the supremum (maximum) of the catenary degrees of the elements of $S$. 
The catenary degree of $77 \in \langle 10, 11, 23, 35 \rangle$
The catenary degree

\[(1, 4, 1, 0) \quad 3 \quad (2, 2, 0, 1)\]

\[(0, 7, 0, 0) \quad 3 \quad (2, 1, 2, 0)\]
The catenary degree

(1, 4, 1, 0)  3  (2, 2, 0, 1)

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\[(1, 4, 1, 0) \quad (2, 2, 0, 1)\]

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Besides, $9$ divides $66$
$66 \in S = \langle 6, 9, 11 \rangle$, $t(S) = 7$

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and 11 also divides 66

$$(8, 2, 0)$$

$$3 \mid (11, 0, 0)$$
$66 \in S = \langle 6, 9, 11 \rangle$, $t(S) = 7$

The factorizations of $66 \in \langle 6, 9, 11 \rangle$ are

$F(66) = \{(0, 0, 6), (1, 3, 3), (2, 6, 0), (4, 1, 3), (5, 4, 0), (8, 2, 0), (11, 0, 0)\}$

\[
\begin{align*}
(8, 2, 0) \\
3 & \mid \\
(11, 0, 0) \\
7 & \mid \\
(4, 1, 3)
\end{align*}
\]
The tame degree of $S$, $t(S)$, is defined as the minimum $N$ such that for any $s \in S$ and any factorization $x$ of $s$, if $s - n_i \in S$ for some $i \in \{1, \ldots, p\}$, then there exists another factorization $y$ of $s$ such that $d(x, y) \leq N$ and the $i$th coordinate of $y$ is nonzero ($n_i$ “occurs” in this factorization).
The catenary degree of $S$ is less than or equal to the tame degree of $S$.

$$c(S) \leq t(S)$$
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It is known that in some cases both coincide (for instance for monoids with a generic presentation).
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It is known that in some cases both coincide (for instance for monoids with a generic presentation).

We want to characterize when the equality holds if the embedding dimension of $S$ is three.
Embedding dimension three numerical semigroups

Let $S = \langle n_1 < n_2 < n_3 \rangle$ be a numerical semigroup with embedding dimension 3.

Define

$$c_i = \min \{k \in \mathbb{N} \setminus \{0\} | kn_i \in \langle n_j, n_k \rangle, \{i, j, k\} = \{1, 2, 3\}\}.$$ 

Then, for all $\{i, j, k\} = \{1, 2, 3\}$, there exist some $r_{ij}, r_{ik} \in \mathbb{N}$ such that

$$c_i n_i = r_{ij} n_j + r_{ik} n_k.$$
We know that

$$\text{Betti}(S) = \{c_1 n_1, c_2 n_2, c_3 n_3\}.$$ 

Hence $1 \leq \# \text{Betti}(S) \leq 3$. 
Embedding dimension three numerical semigroups

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Hence \( 1 \leq \# \text{Betti}(S) \leq 3. \)

Herzog proved that \( S \) is symmetric if and only if \( r_{ij} = 0 \) for some \( i, j \in \{1, 2, 3\} \), or equivalently, \( \# \text{Betti}(S) \in \{1, 2\} \).
We know that

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Hence $1 \leq \# \text{Betti}(S) \leq 3$.

Herzog proved that $S$ is symmetric if and only if $r_{ij} = 0$ for some $i, j \in \{1, 2, 3\}$, or equivalently, $\# \text{Betti}(S) \in \{1, 2\}$. Therefore, $S$ is nonsymmetric if and only if $\# \text{Betti}(S) = 3$. 
The nonsymmetric case

Let $S$ be a numerical semigroup minimally generated by \{\(n_1, n_2, n_3\)\} with $n_1 < n_2 < n_3$. 

V. Blanco, P. A. García-Sánchez, A. Geroldinger proved that $c(S) = t(S)$ for $S$ a nonsymmetric embedding dimension three numerical semigroup. For this reason we focus henceforth in the case $S$ is symmetric, and thus $\#\text{Betti}(S) \in \{1, 2\}$. 
The nonsymmetric case

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For this reason we focus henceforth in the case $S$ is symmetric, and thus $\# \text{Betti}(S) \in \{1, 2\}$.
When $S$ has two Betti elements, we distinguish the three subcases:

- $c_1 n_1 = c_2 n_2 \neq c_3 n_3$;
- $c_1 n_1 = c_3 n_3 \neq c_2 n_2$;
- $c_1 n_1 \neq c_2 n_2 = c_3 n_3$;
The case $c_1 n_1 = c_2 n_2 \neq c_3 n_3$

Proposition

Let $S = \langle n_1, n_2, n_3 \rangle$ with $n_1 < n_2 < n_3$ and $c_1 n_1 = c_2 n_2 \neq c_3 n_3$. Then $c(S) < t(S)$.

Example

$S = \langle 4, 6, 7 \rangle$ $c(S) = 3 < t(S) = 5$
The case $c_1 n_1 = c_3 n_3 \neq c_2 n_2$

**Proposition**

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**Example**

$S = \langle 4, 5, 6 \rangle$ $c(S) = 3 < t(S) = 4$
The case $c_1 n_1 \neq c_2 n_2 = c_3 n_3$

**Proposition**

Let $S = \langle n_1, n_2, n_3 \rangle$ with $n_1 < n_2 < n_3$ and $c_1 n_1 \neq c_2 n_2 = c_3 n_3$. If $c_2 n_2 \nmid c_1 n_1$, then $c(S) < t(S)$.

**Example**

$S = \langle 5, 8, 12 \rangle$ $c(S) = 4 < t(S) = 6$
The case $c_1 n_1 \neq c_2 n_2 = c_3 n_3$

**Proposition**

Let $S = \langle n_1, n_2, n_3 \rangle$ with $n_1 < n_2 < n_3$ and $c_1 n_1 \neq c_2 n_2 = c_3 n_3$. If $c_2 n_2 \nmid c_1 n_1$, then $c(S) < t(S)$.

**Example**

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**Proposition**

Let $S = \langle n_1, n_2, n_3 \rangle$ with $n_1 < n_2 < n_3$ and $c_1 n_1 \neq c_2 n_2 = c_3 n_3$. If $c_2 n_2 | c_1 n_1$, then $c(S) = t(S)$.

**Example**

$S = \langle 12, 14, 21 \rangle$  
$c(S) = t(S) = 7$
When $S$ has a single Betti element

Proposition

Let $S = \langle n_1, n_2, n_3 \rangle$ with $n_1 < n_2 < n_3$ and $c_1 n_1 = c_2 n_2 = c_3 n_3$. Then $c(S) = t(S)$.

Example

$S = \langle 6, 10, 15 \rangle$ $c(S) = t(S) = 5$
Main result

**Theorem**

*Let S be an embedding dimension three numerical semigroup minimally generated by \( \{ n_1, n_2, n_3 \} \). For every \( \{ i, j, k \} = \{ 1, 2, 3 \} \), define* 

\[
c_i = \min\{ k \in \mathbb{N} \setminus \{ 0 \} \mid kn_i \in \langle n_j, n_k \rangle \}.
\]

*Then \( c(S) = t(S) \) if and only if*

- *either \( \# \text{Betti}(S) \neq 2 \),*
- *or \( c_1 n_1 \neq c_2 n_2 = c_3 n_3 \) and \( c_2 n_2 \) divides \( c_1 n_1 \).*
The \( \omega \)-primality function assigns to each element \( n \in S \) the value \( \omega(n) = m \) if \( m \) is the smallest positive integer with the property that whenever \( \sum_{i=1}^{p} a_i n_i - n \in S \) for \( |a| > m \), there exists \( b = (b_1, \ldots, b_p) \in \mathbb{N}^p \) with \( b \leq a \) (with the usual partial ordering on \( \mathbb{N}^p \)) such that \( \sum_{i=1}^{p} b_i n_i - n \in S \) and \( |b| \leq m \).
$\omega$-primality, the definition

The $\omega$-primality function assigns to each element $n \in S$ the value $\omega(n) = m$ if $m$ is the smallest positive integer with the property that whenever $\sum_{i=1}^{p} a_i n_i - n \in S$ for $|a| > m$, there exists $b = (b_1, \ldots, b_p) \in \mathbb{N}^p$ with $b \leq a$ (with the usual partial ordering on $\mathbb{N}^p$) such that $\sum_{i=1}^{p} b_i n_i - n \in S$ and $|b| \leq m$.

We set

$$\omega(S) = \sup\{\omega(S, n_i) \mid i \in \{1, \ldots, p\}\}.$$
By definition, an element $b \in S$ is a prime element if and only if $\omega(S, b) = 1$, and $S$ is factorial if and only if $\omega(S) = 1$.

Numerical semigroups other than $\mathbb{N}$ have no prime elements.
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Numerical semigroups other than $\mathbb{N}$ have no prime elements.

$$c(S) \leq \omega(S) \leq t(S)$$

We are interested in comparing the $\omega$-primality with the catenary degree in embedding dimension three numerical semigroups.
Comparing the $\omega$-primality with the catenary degree

Theorem

Let $S = \langle n_1, n_2, n_3 \rangle$ with $n_1 < n_2 < n_3$ be a numerical semigroup with embedding dimension three.

(a) If $\# \text{Betti}(S) = 3$, then $c(S) = \omega(S) = t(S)$.

(b) If $c_1 n_1 = c_2 n_2 \neq c_3 n_3$, then $c(S) < \omega(S)$.

(c) If $c_1 n_1 = c_3 n_3 \neq c_2 n_2$, then $c(S) < \omega(S)$.

(d) If $c_1 n_1 \neq c_2 n_2 = c_3 n_3$ and $c_2 n_2 \mid c_1 n_1$, then $c(S) = \omega(S) = t(S)$.

(e) If $c_1 n_1 = c_2 n_2 = c_3 n_3$, then $c(S) = \omega(S) = t(S)$.
Comparing $\omega$-primality with the catenary degree

In the case $c_1 n_1 \neq c_2 n_2 = c_3 n_3$ and $c_2 n_2 \nmid c_1 n_1$, with some examples we show that we cannot say more than we state in the theorem above.

Example $S = \langle 19, 350, 490 \rangle$ in which $c_1 n_1, c_2 n_2 = c_3 n_3$ and $c_2 n_2 \nmid c_1 n_1$. We have $c(S) = \omega(S) = 70$.

Example $S = \langle 62, 63, 147 \rangle$ in which $c_1 n_1, c_2 n_2 = c_3 n_3$ and $c_2 n_2 \nmid c_1 n_1$. We have $c(S) = 21 < \omega(S) = 23$. 
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Questions

- When $\omega(S) = t(S)$ in embedding dimension three numerical semigroup with two Betti elements?
Questions

- When $\omega(S) = t(S)$ in embedding dimension three numerical semigroup with two Betti elements?
- When $c(S) = t(S)$ for embedding dimension four numerical semigroup?
Thank you