When the catenary degree meets the tame degree in embedding dimension three numerical semigroups

Caterina Viola

Cortona - September 2014

based on

P. A. García-Sánchez, C. Viola,
When the catenary degree meets the tame degree in embedding dimension three numerical semigroups, to appear in *Involve*.

S. T. Chapman, P. A. García-Sánchez, Z. Tripp, C. Viola,
ω-primality in embedding dimension three numerical semigroups, preprint.

M. Delgado, P. A. García-Sánchez, J. J. Morais,
GAP pakage numericalsgps

Setup

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- The length of x is $|x| = x_1 + \cdots + x_p$.
- Given another factorization $y = (y_1, \ldots, y_p)$, the distance between x and y is
$\mathrm{d}(x, y) = \max\{|x - \gcd(x, y)|, |y - \gcd(x, y)|\}$,
where $\gcd(x, y) = (\min\{x_1, y_1\}, \ldots, \min\{x_p, y_p\})$.

$66 \in S = \langle 6, 9, 11 \rangle$, $\mathsf{c}(S) = 4$

The factorizations of $66 \in \langle 6, 9, 11 \rangle$ are

$$\mathrm{F}(66) = \{(0,0,6),(1,3,3),(2,6,0),(4,1,3),(5,4,0),(8,2,0),(11,0,0)\}$$

The distance between $(11,0,0)$ and $(0,0,6)$ is 11.

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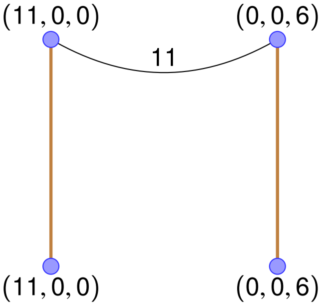

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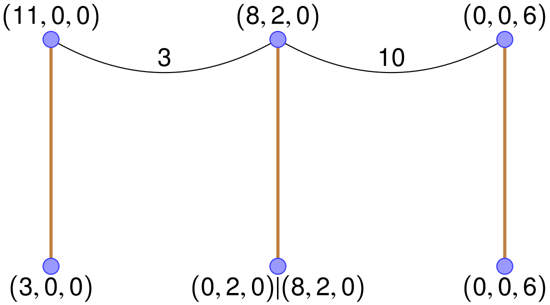

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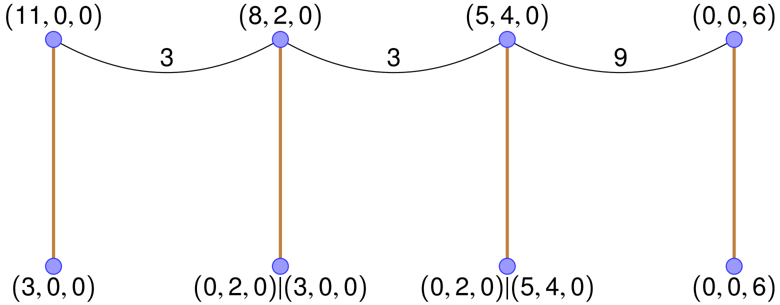

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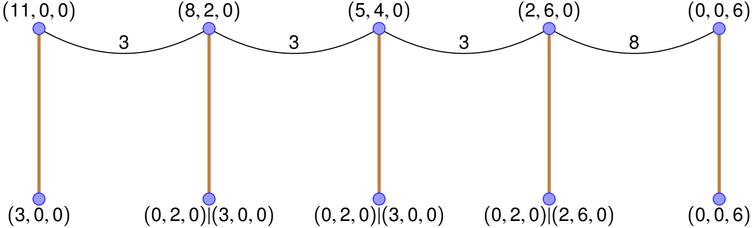

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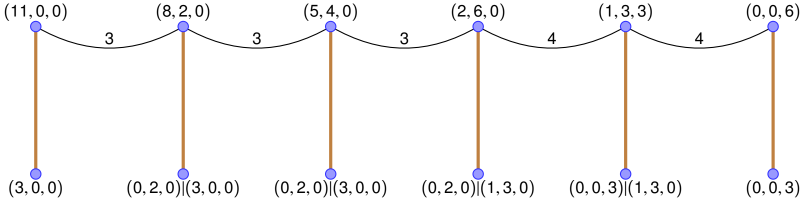


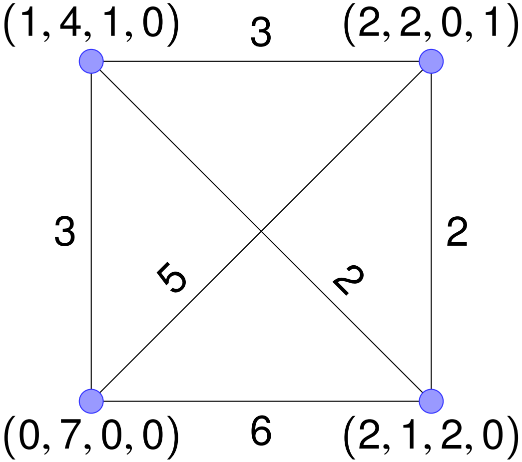
The catenary degree

The catenary degree of $s \in S$, $\mathrm{c}(s)$, is the minimum nonnegative integer N such that for any two factorizations x and y of s, there exists a sequence of factorizations $x_1, \ldots, x_t$ of s such that

- $x_1 = x$, $x_t = y$,
- for all $i \in \{1, \ldots, t-1\}$, $\mathrm{d}(x_i, x_{i+1}) \leq N$.

The catenary degree of S, $\mathrm{c}(S)$, is the supremum (maximum) of the catenary degrees of the elements of S.

The catenary degree of $77 \in \langle 10, 11, 23, 35 \rangle$



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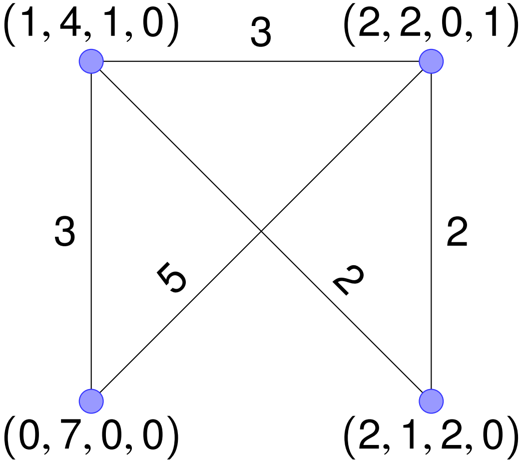

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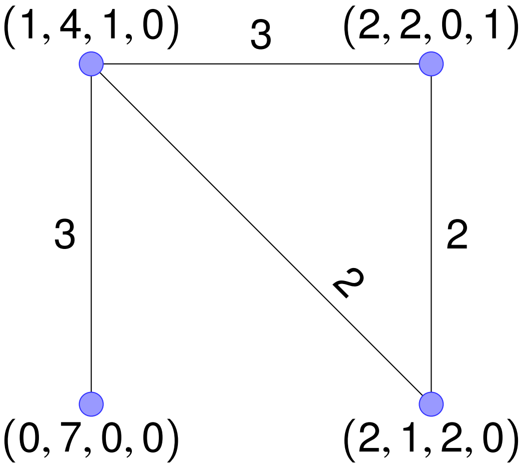

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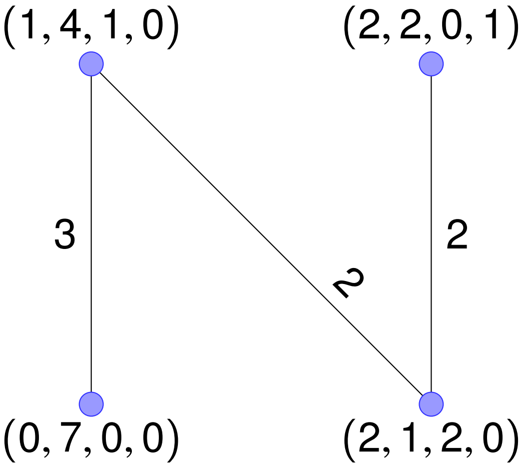

$66 \in S = \langle 6, 9, 11 \rangle$, $\mathrm{t}(S) = 7$

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Besides, 9 *divides* 66

$$(11, 0, 0)$$

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and 11 also *divides* 66

$$\begin{array}{c} (8, 2, 0) \\ 3 \big| \\ (11, 0, 0) \end{array}$$

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$$(8,2,0)$$
$$3 \;|$$
$$(11,0,0)$$
$$7 \;|$$
$$(4,1,3)$$

The tame degree

The **tame degree** of S, $\mathrm{t}(S)$, is defined as the minimum N such that for any $s \in S$ and any factorization x of s, if $s - n_i \in S$ for some $i \in \{1, \dots, p\}$, then there exists another factorization y of s such that $\mathrm{d}(x, y) \leq N$ and the ith coordinate of y is nonzero (n_i “occurs” in this factorization).

The catenary degree of S is less than or equal to the tame degree of S.

$$\mathrm{c}(S) \le \mathrm{t}(S)$$

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It is known that in some cases both coincide (for instance for monoids with a generic presentation).

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We want to characterize when the equality holds if the embedding dimension of S is three.

Embedding dimension three numerical semigroups

Let $S = \langle n_1 < n_2 < n_3 \rangle$ be a numerical semigroup with embedding dimension 3.

Define

$$c_i = \min\{k \in \mathbb{N} \setminus \{0\} \mid kn_i \in \langle n_j, n_k \rangle, \{i, j, k\} = \{1, 2, 3\}\}.$$

Then, for all $\{i, j, k\} = \{1, 2, 3\}$, there exist some $r_{ij}, r_{ik} \in \mathbb{N}$ such that

$$c_i n_i = r_{ij} n_j + r_{ik} n_k.$$

Embedding dimension three numerical semigroups

We know that

$$\mathrm{Betti}(S) = \{c_1 n_1, c_2 n_2, c_3 n_3\}.$$

Hence $1 \leq \#\,\mathrm{Betti}(S) \leq 3$.

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Herzog proved that S is symmetric if and only if $r_{ij} = 0$ for some $i, j \in \{1, 2, 3\}$, or equivalently, $\#\,\mathrm{Betti}(S) \in \{1, 2\}$.

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Herzog proved that S is symmetric if and only if $r_{ij} = 0$ for some $i, j \in \{1, 2, 3\}$, or equivalently, $\# \mathrm{Betti}(S) \in \{1, 2\}$. Therefore, S is nonsymmetric if and only if $\# \mathrm{Betti}(S) = 3$.

The nonsymmetric case

Let S be a numerical semigroup minimally generated by $\{n_1, n_2, n_3\}$ with $n_1 < n_2 < n_3$.

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For this reason we focus henceforth in the case S is symmetric, and thus $\#\,\mathrm{Betti}(S) \in \{1, 2\}$.

When S has two Betti elements

When S has two Betti elements, we distinguish the three subcases:

- $c_1 n_1 = c_2 n_2 \neq c_3 n_3$;
- $c_1 n_1 = c_3 n_3 \neq c_2 n_2$;
- $c_1 n_1 \neq c_2 n_2 = c_3 n_3$;

The case $c_1 n_1 = c_2 n_2 \neq c_3 n_3$

Proposition

Let $S = \langle n_1, n_2, n_3 \rangle$ *with* $n_1 < n_2 < n_3$ *and* $c_1 n_1 = c_2 n_2 \neq c_3 n_3$. *Then* $\mathrm{c}(S) < \mathrm{t}(S)$.

Example

$S = \langle 4, 6, 7 \rangle$ $\mathrm{c}(S) = 3 < \mathrm{t}(S) = 5$

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Example

$S = \langle 4, 5, 6 \rangle$ $\mathrm{c}(S) = 3 < \mathrm{t}(S) = 4$

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Example

$S = \langle 5, 8, 12 \rangle$ $\mathrm{c}(S) = 4 < \mathrm{t}(S) = 6$

The case $c_1 n_1 \neq c_2 n_2 = c_3 n_3$

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Let $S = \langle n_1, n_2, n_3 \rangle$ *with* $n_1 < n_2 < n_3$ *and* $c_1 n_1 \neq c_2 n_2 = c_3 n_3$. *If* $c_2 n_2 \mid c_1 n_1$, *then* $\mathrm{c}(S) = \mathrm{t}(S)$.

Example

$S = \langle 12, 14, 21 \rangle$ $\mathrm{c}(S) = \mathrm{t}(S) = 7$

When S has a single Betti element

Proposition

Let $S = \langle n_1, n_2, n_3 \rangle$ *with* $n_1 < n_2 < n_3$ *and* $c_1 n_1 = c_2 n_2 = c_3 n_3$*.*
Then $\mathrm{c}(S) = \mathrm{t}(S)$*.*

Example

$S = \langle 6, 10, 15 \rangle$ $\mathrm{c}(S) = \mathrm{t}(S) = 5$

Main result

Theorem

Let S be an embedding dimension three numerical semigroup minimally generated by $\{n_1, n_2, n_3\}$. For every $\{i, j, k\} = \{1, 2, 3\}$, define

$$c_i = \min\{k \in \mathbb{N} \setminus \{0\} \mid kn_i \in \langle n_j, n_k \rangle\}.$$

Then $\mathrm{c}(S) = \mathrm{t}(S)$ if and only if

- *either $\#\,\mathrm{Betti}(S) \neq 2$,*
- *or $c_1 n_1 \neq c_2 n_2 = c_3 n_3$ and $c_2 n_2$ divides $c_1 n_1$.*

ω-primality, the definition

The ω-primality function assigns to each element $n \in S$ the value $\omega(n) = m$ if m is the smallest positive integer with the property that whenever $\sum_{i=1}^{p} a_i n_i - n \in S$ for $|a| > m$, there exists $b = (b_1, \ldots, b_p) \in \mathbb{N}^p$ with $b \leq a$ (with the usual partial ordering on $\mathbb{N}^p$) such that $\sum_{i=1}^{p} b_i n_i - n \in S$ and $|b| \leq m$.

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We set

$$\omega(S) = \sup\{\omega(S, n_i) \mid i \in \{1, \ldots, p\}\}.$$

ω-primality

By definition, an element $b \in S$ is a prime element if and only if $\omega(S, b) = 1$, and S is factorial if and only if $\omega(S) = 1$.

Numerical semigroups other than $\mathbb{N}$ have no prime elements.

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Numerical semigroups other than $\mathbb{N}$ have no prime elements.

$$\mathrm{c}(S) \leq \omega(S) \leq \mathrm{t}(S)$$

We are interested in comparing the ω-primality with the catenary degree in embedding dimension three numerical semigroups.

Comparing the ω-primality with the catenary degree

Theorem

Let $S = \langle n_1, n_2, n_3\rangle$ with $n_1 < n_2 < n_3$ be a numerical semigroup with embedding dimension three.

(a) *If* $\#\,\mathrm{Betti}(S) = 3$, *then* $\mathrm{c}(S) = \omega(S) = \mathrm{t}(S)$.

(b) *If* $c_1 n_1 = c_2 n_2 \neq c_3 n_3$, *then* $\mathrm{c}(S) < \omega(S)$.

(c) *If* $c_1 n_1 = c_3 n_3 \neq c_2 n_2$, *then* $\mathrm{c}(S) < \omega(S)$.

(d) *If* $c_1 n_1 \neq c_2 n_2 = c_3 n_3$ *and* $c_2 n_2 \mid c_1 n_1$, *then* $\mathrm{c}(S) = \omega(S) = \mathrm{t}(S)$.

(e) *If* $c_1 n_1 = c_2 n_2 = c_3 n_3$, *then* $\mathrm{c}(S) = \omega(S) = \mathrm{t}(S)$.

Comparing ω-primality with the catenary degree

In the case $c_1 n_1 \neq c_2 n_2 = c_3 n_3$ and $c_2 n_2 \nmid c_1 n_1$, with some examples we show that we cannot say more than we state in the theorem above.

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Example

- $S = \langle 19, 350, 490 \rangle$ in which $c_1 n_1 \neq c_2 n_2 = c_3 n_3$ and $c_2 n_2 \nmid c_1 n_1$. We have $\mathrm{c}(S) = \omega(S) = 70$.

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- $S = \langle 62, 63, 147 \rangle$ in which $c_1 n_1 \neq c_2 n_2 = c_3 n_3$ and $c_2 n_2 \nmid c_1 n_1$. We have $\mathrm{c}(S) = 21 < \omega(S) = 23$.

Questions

- When $\omega(S) = \mathrm{t}(S)$ in embedding dimension three numerical semigroup with two Betti elements?

Questions

- When $\omega(S) = \mathrm{t}(S)$ in embedding dimension three numerical semigroup with two Betti elements?
- When $\mathrm{c}(S) = \mathrm{t}(S)$ for embedding dimension four numerical semigroup?

Thank you