IMNS- 2014
Syzygies of GS monomial curves and smoothability.

Grazia Tamone

Dima - University of Genova - Italy

International meeting on numerical semigroups

Cortona September 8-12, 2014 (Joint work with Anna Oneto)

Introduction.

TOPIC . Some aspects of the study of

$$R = k[x_0, \ldots, x_n]/I = k[S]$$

S semigroup generated by a generalized arithmetic sequence (GS semigroup), k field of characteristic 0:

bideterminantal shape of the ideal I
and
minimal free resolution of I

$$\downarrow$$

determinantal description of the first syzygy module of I
and
smoothability of certain GS curve X of $\mathbb{A}_k^{n+1}$ defined by I.

1. Setting, syzygies, Betti numbers.

Let S be a numerical semigroup generated by a generalized arithmetic sequence, $S = \sum_{0 \leq i \leq n} \mathbb{N} m_i$, with

$$m_i = \eta m_0 + i\, d \ (\eta \geq 1,\ 1 \leq i \leq n),\ and\ GCD(m_0, d) = 1.$$

Let $a, b, \mu \in \mathbb{N}$ be such that

$$m_0 = an + b\ , \quad a \geq 1, \quad 1 \leq b \leq n, \quad \mu := a\eta + d.$$

Let $P := k[x_0, \ldots, x_n]$ (k field), with $weight(x_i) := m_i$, let

$$k[S] = k[t^s,\ s \in S]$$

and consider the monomial curve (shortly GS curve)

$$X = Spec(k[S]) \subseteq \mathbb{A}_k^{n+1}$$

associated to S. When $\eta = 1$, S is generated by an arithmetic sequence (AS semigroup, resp AS curve).

The defining ideal $I \subseteq P$ of X is generated by the 2×2 minors of the following two matrices:

$$A := \begin{pmatrix} x_0^\eta & x_1 & \dots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & \dots & x_{n-1} & x_n \end{pmatrix}, \; A' := \begin{pmatrix} x_n^a & x_0^\eta & \dots & x_{n-b} \\ x_0^\mu & x_b & \dots & x_n \end{pmatrix}$$

and a minimal set of generators for I is union of

- the $\binom{n}{2}$ maximal minors $\{f_1, \dots, f_{\binom{n}{2}}\}$ of the matrix A

call $\mathfrak{c}$ the ideal generated by these elements (if $\eta = 1$ it defines the cone over the rational normal curve in $\mathbb{P}_k^n$)

- the $n-b+1$ minors of the matrix A' containing the first column :

$$\left[\begin{array}{ll} g_j = x_n^a x_{n-j} - x_0^\mu x_{n-b-j} & \quad (j = 0, \dots, n-b-1) \\ g_{n-b} = x_n^a x_b - x_0^{\mu+\eta} & \\ with\ weights \quad \delta_j = a m_n + m_{n-j} & \end{array} \right.$$

Starting from
- the Eagon-Northcott free resolution for the ideal

$$\mathfrak{c} = \left(f_1, \ldots, f_{\binom{n}{2}}\right):$$

$$\mathbb{E}: \quad 0 \longrightarrow E_{n-1} \longrightarrow \ \ldots \ \longrightarrow E_1 \longrightarrow E_0 \longrightarrow P/\mathfrak{c} \longrightarrow 0$$

where $E_0 \simeq P$, and for $1 \leq s \leq n-1$,

$$E_s = \wedge^{s+1} P^n \bigotimes \left(Sym_{s-1}(P^2)\right)^* \simeq P^{\beta_s}(-s-1),$$

with basis $\quad \underset{(1 \leq i_1 < i_2 < \cdots < i_{s+1} \leq n, \quad v_0 + v_1 = s-1)}{e_{i_1} \wedge \cdots \wedge e_{i_{s+1}} \otimes \lambda_0^{v_0} \lambda_1^{v_1}},$

- the Koszul complex $\mathbb{K}$, minimal free resolution for

$$P/(x_1, \ldots, x_n):$$

$$\mathbb{K}: \quad 0 \longrightarrow K_n \longrightarrow \ \ldots \ \longrightarrow K_1 \longrightarrow K_0 \longrightarrow P/(x_1, \ldots, x_n) \longrightarrow 0$$

(with $K_s = \wedge^s P^n, \quad s \geq 1$).

A minimal free homogeneous resolution of the ideal I can be obtained via iterated mapping cone by adapting to GS the technique used by Gimenez, Sengupta, Srinivasan in the case AS $(\eta = 1)$ [4], Theorem 3.8: it is the complex

$$\mathcal{R}: \quad 0 \longrightarrow R_n \longrightarrow \quad \dots \quad \longrightarrow R_2 \longrightarrow R_1 \longrightarrow P$$

where (respectively if $b = n$, or $1 \leq b < n$),
$R_s =$

- $E_{s-1}(-\delta_0) \oplus E_s, \qquad (b = n)$
- $\Big(K_{s-1}(-\delta_{n-b}) \oplus ... \oplus K_{s-1}(-\delta_1) \oplus E_{s-1}(-\delta_0) \oplus E_s\Big) \Big/ D_s$

$D_s \subseteq K_{s-1}(-\delta_{n-b}) \oplus ... \oplus K_{s-1}(-\delta_1) \oplus E_{s-1}(-\delta_0), \quad (b < n).$

In particular:

$$R_1 = K_0(-\delta_{n-b}) \oplus ... \oplus K_0(-\delta_1) \oplus K_0(-\delta_0) \oplus E_1$$

and if

$$b < n : \left[\begin{array}{l} R_2 = K_1(-\delta_{n-b}) \oplus ... \oplus K_1(-\delta_1) \oplus E_2 \\ dim\, R_2 = (n-b)n \,+\, 2\binom{n}{3} \end{array} \right.$$

$$b = n : \left[\begin{array}{l} R_2 = E_1(-\delta_0) \oplus E_2 \\ dim\, R_2 = \binom{n}{2} + \, 2\binom{n}{3}\,. \end{array} \right.$$

From this resolution one deduces the Betti numbers of R:

$$\beta_s = \left[\begin{array}{ll} (n-b+2-s)\binom{n}{s-1} + s\binom{n}{s+1}, & if\ \ 1 \le s < n-b+2 \\ \\ (s-1-n+b)\binom{n}{s} + s\binom{n}{s+1}, & if\ \ n-b+2 \le s \le n\,. \end{array} \right.$$

Corollary

Let $R = P/I$ be the coordinate ring of a GS curve. Then:

(1) *The Betti numbers of R depend only on the values of n and b.*

(2) *The Betti numbers of R are maximal: in fact they are equal to the Betti numbers of the associated graded ring G of R with respect to the maximal ideal $(x_0, \ldots, x_n)$ (as computed by Sharifan and Zaare-Nahandi).*

By the knowledge of the above resolution we obtain also a

"determinantal " description of the first syzygies module of R:

Corollary

The first syzygies of the generating ideal I of a GS curve can be described as follows:

(1) *The $2\binom{n}{3}$ syzygies concerning the ideal $\mathfrak{C}$ are given as determinants of the 3×3 minors obtained by doubling a row in the matrix A. [Kurano, 1989]*

(2) *If $b = n$ the remaining $\binom{n}{2}$ syzygies are trivial:* $f_i g_0 - f_i g_0 = 0$.

(3) *If $1 \leq b \leq n-1$ the remaining $(n-b)n$ syzygies can be written by expanding the determinants of the following matrices along the first column and the third row:*

$$\begin{cases} 1 \leq h < n-b \\ 2 \leq i \leq n \end{cases} \quad : \quad \begin{pmatrix} x_n^a & x_{n-b-h} & x_{n-b-h+1} \\ x_0^\mu & x_{n-h} & x_{n-h+1} \\ 0 & x_{i-1} & x_i \end{pmatrix};$$

$$\begin{cases} h = n-b \\ 2 \leq i \leq n \end{cases} \quad : \quad \begin{pmatrix} x_n^a & x_0^\eta & x_1 \\ x_0^\mu & x_b & x_{b+1} \\ 0 & x_{i-1} & x_i \end{pmatrix};$$

$$\begin{cases} 1 \leq h < n-b \\ (i = 1) \end{cases} \quad : \quad \begin{pmatrix} x_n^a & x_{n-b-h} & x_{n-b-h+1} \\ x_0^\mu & x_{n-h} & x_{n-h+1} \\ 0 & x_0^\eta & x_1 \end{pmatrix};$$

$$\begin{cases} h = n-b \\ i = 1 \end{cases} \quad : \quad \begin{pmatrix} x_n^a & x_0^\eta & x_1 \\ x_0^\mu & x_b & x_{b+1} \\ 0 & x_0^\eta & x_1 \end{pmatrix}$$

This "determinantal property" can be seen directly from the definition of the map $d_2 : R_2 \longrightarrow R_1$ of the complex $\mathcal{R}$. It will be very useful in the following, as we shall see.

2. Weierstrass semigroups and smoothability.

A numerical semigroup S, is Weierstrass if S is not ordinary and there exist a smooth projective curve C and a closed point Q on C such that

$S = \{h \in \mathbb{N} \mid$ there exists $f \in k(C)$, f regular outside Q with at most one pole of order h at $Q\}$.

An important property of Weierstrass semigroups is their connection with the theory of algebraic-geometric codes (AG codes).

The problem of classifying Weierstrass semigroups is still open and difficult. It is known that there are non-Weierstrass semigroups (examples of Buchweitz (1980) , Kim, Komeda, Torres and others).

On the other hand, it is known that several semigroups S are Weierstrass, in particular

(1) S minimally 3-generated [Shaps]

(2) S with multiplicity ≤ 5 [Maclachlan, Komeda]

(3) S with genus $g \leq 8$, or $g = 9$, in particular cases [Komeda]

(4) if the curve $X = Spec(k[S]))$ is a complete intersection,

(5) if $X \subseteq \mathbb{A}^q$ is defined by the $l \times l$ minors of a $m \times n$ matrix, $codim\, X = (m - l + 1)(n - l + 1)$ and or $m = n = l$, or $l = 1$, or $q < (m - l + 2)(n - l + 2)$. [Shaps]

A fundamental result is the following

Theorem (Pinkham)

Let k be an algebraically closed field , $char(k) = 0$:

A semigroup S is Weierstrass if and only if the curve $X = Spec\ (k[S])$ is smoothable.

“Smoothability” for a scheme X means the existence of flat deformations

$$\begin{array}{ccc} \pi^{-1}(0) \simeq X & \hookrightarrow & Y \\ \downarrow & & \downarrow \pi \;\; (flat) \\ \{0\} & \hookrightarrow & \Sigma \; (base\ space) \end{array}$$

with Σ integral scheme of finite type, such that π admits non-singular fibres.

3. Smoothability of Arf-GS curves

The notion of Arf semigroup comes from the classical one given by Lipman [11] for a semi-local ring R. We recall one of the equivalent definitions of such semigroups. See [Barucci, Dobbs, Fontana].

Given a numerical semigroup S minimally generated by $m_0 < m_1 < ... < m_n$, the *blowing-up* (or *Lipman semigroup*) $L(S)$ of S along the maximal ideal $M = S \setminus \{0\}$ is defined as $L(S) := \cup_{h\geq 1}\big(hM - hM\big)$ and it is well-known that

(a) $L(S) =< m_0, m_1 - m_0, ..., m_n - m_0 >$.

(b) There exists a finite sequence of blowing-ups :
$S \subseteq S_1 = L(S) \subseteq ... \subseteq S_m = L(S_{m-1}) = \mathbb{N}$.

Definition

A numerical semigroup S is called an Arf semigroup if in the sequence of its blowing-up

$$S_0 = S \subseteq S_1 \subseteq ... \subseteq S_m = \mathbb{N}$$

the S_i have maximal embedding dimension $\forall i = 0, ..., m$.

By the above Theorem of Shaps on the smoothability of determinantal schemes we get:

Proposition

Let $S =< m_0, ..., m_n >$ $(m_0 = an + b, \quad m_i = \eta m_0 + i\, d)$ be a GS semigroup. If S is Arf, then the associated monomial curve $X = Spec\, k[S]$ is smoothable.

Proof. Assume S is a *GS* and *Arf* semigroup: then $n + 1 = embdim(S) = e(S) = m_0$. Therefore $a = b = 1$, and so the defining ideal I is determinantal generated by the 2×2 minors of the matrix A'. Then the curve X is smoothable since the assumptions of Shaps are satisfied.

We recall some characterizations of *Arf-GS* semigroups given by [Matthews and T.].

Proposition

(1) *A numerical GS semigroup $S \neq \mathbb{N}$ is Arf if and only if either S has multiplicity $e(S) = 2$, or $d = 1$, or $d = 2$.*

(2) *Given a semigroup of maximal embedding dimension minimally generated by* $m_0 < m_1 < ... < m_n$,
if $m_1 \equiv 1 \ (mod\, m_0)$, then
S is Arf if and only if it is GS $(with\ d = 1)$.

4. Smoothability of certain GS curves

When $X = Spec(k[S])$ is a GS monomial curve,

$S =< m_0, \ldots, m_n >, \quad m_i = \eta m_0 + id, \quad m_0 = an + b$,

we already know that the curve X is smoothable:

- if $b = 1$, by [Shaps], since the ideal I is determinantal.
- if $\eta = 1$ and ($b = n$, or $n \leq 4$), by recent papers [Oneto,T.].

If $\eta = 1$, i.e. for AS curves, we are able to prove the smoothability in several new subcases.

In the following we shall assume $b < n$.

In order to construct deformations of an AS curve X, by the particular bideterminantal shape of the defining ideal I it is quite natural to approach the problem finding suitable

compatible deformations of the matrices A, A'

$$A := \begin{pmatrix} x_0 & x_1 & \dots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & \dots & x_{n-1} & x_n \end{pmatrix}, \ A' := \begin{pmatrix} x_n^a & x_0 & \dots & x_{n-b} \\ x_0^\mu & x_b & \dots & x_n \end{pmatrix}$$

By this way the needed flatness of the induced morphism is immediately obtained thanks to the determinantal description of the first syzygies (which naturally lift to the set of deformed generators).

Successively, to prove the smoothability of X, we apply the following version of the classical Bertini's theorem

Theorem (Bertini - Kleiman 1997)

Let Z be an integral scheme over an algebraically closed field k of characteristic 0. Let D be a finite dimensional linear system.

Then:

Almost every element of D, considered as a closed subscheme of Z, is non singular outside the base points of D and the singular points of Z.

Theorem (1)

Assume $n \leq 2b$. *Deform the matrices* A *and* A' *respectively as*

$$A_{def} = \begin{pmatrix} x_0 & \dots & x_{n-b-1} & x_{n-b} & \dots & x_{n-1} \\ x_1 & \dots & x_{n-b} - V & x_{n-b-1} & \dots & x_n \end{pmatrix}$$

$$A'_{def} = \begin{pmatrix} x_n^a & x_0 & \dots & x_{n-b} - V \\ x_0^\mu - U & x_b & \dots & x_n \end{pmatrix}.$$

Let $Y \subseteq \mathbb{A}^{n+3}$ *be the variety defined by the union of the* 2×2 *minors of* A_{def} *and* A'_{def}. *Then*

(1) *The ideal* $I_Y \subseteq k[x_0, \dots, x_n, U, V]$ *is minimally generated by the* 2×2 *minors* $\{F_1, \dots, F_{\binom{n}{2}}\}$ *of* A_{def} *and by the minors* $\{G_0, \dots, G_{n-b}\}$ *of* A'_{def} *containing the first column.*

(2) *The induced morphism* $\pi : Y \longrightarrow Spec\, k[U, V]$ *is a deformation, with smooth fibres, of the monomial curve* X.

Proof. (1) One can verify that the 2×2 minors of the matrix A'_{def} not containing the first column belong to the ideal generated by the 2×2 minors of the matrix A_{def} (this means "compatibility" among the minors of the matrices A_{def}, A'_{def}).

(2) According to this "compatibility" and the determinantal shape of the syzygies of generators of I, we obtain that these relations lift naturally to those among generators of I_Y. Therefore there exists a flat morphism:

$$\pi : Y \longrightarrow Spec\, k[U, V]$$

with the curve X as special fibre. It remains to verify that the deformation has smooth fibres, equivalently that the rank of the jacobian matrix of the generic fibre is n at every point.

For this : we fix $V = V_0 \neq 0$ and we first obtain that the $two\text{-}dimensional$ variety Z defined by the minors of A_{def}, (with $V = V_0$), is non singular (Z is a deformation of the cone on the rational normal curve).

Now apply Bertini's theorem to Z and to the divisor D on Z defined by the element $G_0 = x_n^{a+1} - (x_0^\mu - U)(x_{n-b} - V_0)$: a fortiori the generic fibre X' of π is smooth outside the fixed points of D.

Finally, by choosing other suitable generators of X', we shall deduce the regularity of X' at the above fixed points. $\diamond$

When $2b < n \leq 3b$, to obtain compatible deformations of the matrices defining X, we need some more technical trick.
First we consider the following the matrix A'' :

$$\left(\begin{array}{ccccc|ccc} x_n^{a-1} & x_0 & x_1 & \dots & x_{n-2b} & x_{n-2b+1} & \dots & x_{n-b} \\ x_0^{\mu} & x_{n-b}x_{2b} & x_{n-b}x_{2b+1} & \dots & x_{n-b}x_n & x_{n-b+1}x_n & \dots & x_n^2 \end{array} \right)$$

and prove that the 2×2 minors of A, A'' are another system of generators for the ideal I.

By deforming A, A'' we can prove the smoothability of X:

Theorem (2)

Assume $2b < n \leq 3b$.
Consider the deformed matrices A_{def}, A''_{def}

$$\begin{pmatrix} x_0 & \dots & x_{n-b-2} & x_{n-b-1} & x_{n-b} & x_{n-b+1} & \dots & x_{n-1} \\ x_1 & \dots & x_{n-b-1} & x_{n-b} - V & x_{n-b+1} & x_{n-b+2} & \dots & x_n \end{pmatrix}$$

$$\left(\begin{array}{cccc|ccc} x_n^{a-1} & x_0 & \dots & x_{n-2b} & x_{n-2b+1} & \dots & x_{n-b} - V \\ x_0^{\mu} - U & x_{n-b}x_{2b} & \dots & x_{n-b}x_n & x_{n-b+1}x_n & \dots & x_n^2 \end{array} \right)$$

Then the 2×2 minors of these matrices define a deformation of X with smooth fibres.

By recalling the above cited known results, we deduce the following

Corollary

(1) *The AS semigroups with $n \leq 3b$ are Weierstrass.*

(2) *In particular every AS semigroup with embedding dimension less or equal to seven and every semigroup with $b \neq 2$ and embedding dimension ≤ 10 have this property.*

5. Example

We show the procedure in a particular example.

Let $S =< 13, 17, 21, 25, 29, 33 >$ $(m_0 = 13, n = 5, d = 4)$.
$m_0 = 2n + 3$, and so, $a = 2, b = 3, \mu = 6, n - b = 2$.
The defining ideal I of X is generated by the binomials:

$$x_i x_j - x_{i+1} x_{j-1} \ {\scriptstyle (0 \le i < j \le 5)},$$

$$x_5{}^2 x_{3+i} - x_0{}^6 x_i \ {\scriptstyle (0 \le i \le 2)}.$$

Consider the deformed matrices :

$$A_{def} = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ x_1 & x_2 - V & x_3 & x_4 & x_5 \end{pmatrix}$$

$$A'_{def} = \begin{pmatrix} x_5^2 & x_0 & x_1 & x_2 - V \\ x_0{}^6 - U & x_3 & x_4 & x_5 \end{pmatrix}.$$

Let Z be the variety defined by the minors $F_1, \ldots, F_{10}$ of A_{def}, (lexicographically ordered),with $V = V_0 \neq 0$ and let Y be the variety defined by the union of the 2×2 minors of A_{def} and A'_{def}. Consider the Jacobian matrix J:

$$\begin{pmatrix} x_2 - V_0 & -2x_1 & x_0 & 0 & 0 & 0 \\ x_3 & -x_2 & -x_1 & x_0 & 0 & 0 \\ x_4 & -x_3 & 0 & -x_1 & x_0 & 0 \\ x_5 & -x_4 & 0 & 0 & -x_1 & x_0 \\ 0 & x_3 & -2x_2 + V_0 & x_1 & 0 & 0 \\ 0 & x_4 & -x_3 & -x_2 + V_0 & x_1 & 0 \\ 0 & x_5 & -x_4 & 0 & -x_2 + V_0 & x_1 \\ \ldots & & \ldots & & \ldots & \\ -7x_0^6 + U & 0 & 0 & x_5^2 & 0 & 2x_3x_5 \\ \ldots & & \ldots & & \ldots & \end{pmatrix}$$

The jacobian submatrix J_Z of J has rank $4 (= n - 1)$: if $x_0 \neq 0$ a non vanishing minor is formed by the rows corresponding to the elements $F_1, \ldots, F_4$ and columns $C_3, \ldots, C_6$.

The points belonging to Z with $x_0 = 0$ are

$$P(0,0,0,0,0,0,x_5), \qquad Q(0,0,V_0,x_3,x_4,x_5).$$

In both cases there exists a non vanishing minor with size 4 in J_Z (e.g. at the points P the red entries are the diagonal of the minor). Now apply Bertini's theorem to Z and to the linear system D defined by $G_0 = x_5^3 - (x_0^6 - U)(x_2 - V_0)$.

It remains to verify that the generic fibre X' is smooth at the fixed points of D, which are $R(x_0,0,V_0,0,0,0)$.
Again, one can exhibit a non vanishing minor with size 5 of J.

Question

- *For AS curves with $n > 3b$ we cannot find compatible deformations of the defining matrices (do they exist?).*

- *Moreover for GS non-AS curves with $b \geq 2$ the above algorithm doesn't work, because the so obtained deformations haven't smooth fibres.*

THANKS FOR THE ATTENTION.

Bibliography.

(1) V. Barucci, D.E. Dobbs, M. Fontana "*Maximality properties in numerical semigroups and applications to one-dimensional analitically irreducible local domains*" *Lecture Notes in Pure and Appl. Math* vol. 153, *Dekker, New York* (1994).

(2) R.O. Buchweitz, "*On Deformations of Monomial Curves*", in: Sminaire sur les Singularits des Surfaces, Centre de Math. de l'Ecole Polytechnique, Palaiseau, 1976/77 (ed. par M. Demazure et al.), 205-220, Springer Lecture Notes in Mathematics 777, Springer Verlag (1980).

(3) P. Gimenez, I. Sengupta, H. Srinivasan "*Minimal free resolutions for certain affine monomial curves*" *Commutative Algebra and Its connections to Geometry Contemporary Mathematics* Vol. 555, AMS (2011).

(4) P. Gimenez, I. Sengupta, H. Srinivasan "*Minimal graded free resolutions for monomial curves defined by arithmetic*

sequences"s *Journal of Algebra*, vol.388, pp.294-310 (2013).

(5) S.J. Kim "*Semigroups which are not Weiestrass semigroups*" *Bull Korean Math. Soc.*, vol. 33, n.2 pp. 187-191, (1996).

(6) J. Komeda "*On Weierstrass points whose first non gaps are four*" *J. Reine Angew Math* vol. 341, pp. 68-86, (1983).

(7) J. Komeda "*On the existence of Weierstrass points whose first non gaps are five*" *Manuscripta Math* vol. 76, (1992).

(8) J. Komeda "*On the existence of Weierstrass gap sequences on curves of genus* ≤ 8" *JPAA* vol. 97, pp. 51-71, (1994).

(9) J. Komeda "*Existence of the primitive Weiestrass gap sequences on curve of genus 9*" *Boll. Soc Brasil. Math* vol. 30.2, pp. 125-137, (1999).

(10) K. Kurano “*The First Syzygies of Determinantal Ideals*” *Journal of Algebra* vol. 124, pp. 414-436, (1989).

(11) J. Lipman “*Stable ideals and Arf rings*” *American J. of Math.* vol.9 , pp. 649-685, (1971).

(12) G.L. Matthews “*On numerical semigroups generated by generalized arithmetic sequences*” *Comm. Algebra* 32 no. 9, pp. 34593469, (2004).

(13) A. Oneto, G. Tamone “*On some invariants in numerical semigroups and estimations of the order bound*” *Semigroup Forum,* Volume 81, Issue 3, pp. 483-509, (2010).

(14) A. Oneto, G. Tamone “*Smoothability and order bound for AS semigroups.*” *Semigroup Forum,* (To appear) (2012).

(15) H.C. Pinkham “*Deformations of algebraic varieties with G_m action*” *Asterisque* vol. 20, (1974).

(16) M. Schaps “*Nonsingular deformations of a determinantal scheme*” *Pac. J. Math.*, vol. 65 n.1, pp. 209-215, (1976).

(17) I. Sengupta "*A Grobner basis for certain affine monomial curves*" *Comm. Algebra*, vol. 31, no. 3, pp 1113-1129 (2003).

(18) G. Tamone "*Blowing-up and glueings in one-dimensional rings - Commutative algebra (Trento, 1981)* ", pp. 321337, *Lecture Notes in Pure and Appl. Math.*, 84, Dekker, New York, (1983).