ω-primality and asymptotic ω-primality on numerical semigroups. Computation and properties

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- We give an algorithm to compute from a presentation of a finitely generated atomic monoid, the ω-primality of any of its elements.
- For finitely generated quasi-Archimedean cancellative monoids, we give an explicit formulation of the asymptotic ω-primality of its elements.

S, numerical semigroup

Preliminaries

- S, f.g. monoid $\Longrightarrow S \simeq \mathbb{N}^p/\sigma$, σ a congruence on $\mathbb{N}^p$.
  $a \in S$, $a = [\gamma]_\sigma$, $\gamma \in \mathbb{N}^p$.
- $a, b \in S$, $a|b$, if there exists $c \in S$ such that $a + c = b$.
- The elements $a, b \in S$ are **associated** if $a|b$ and $b|a$.
- $a \in S$ is a **unit**, if there exists $b \in S$ such that $a + b = 0$.
  $S^\times = \{x \in S : x \text{ is a unit}\}$.

- $x \in S$ is an **atom** if $x \notin S^{\times}$ and if $a|x$, then either $a \in S^{\times}$ or a and x are associated. $\mathcal{A}(S)$
- If the semigroup $S \setminus S^{\times}$ is generated by its set of atoms $\mathcal{A}(S)$, the monoid S is called an **atomic monoid**.
  It is known that every non-group finitely generated cancellative monoid is atomic (R,G-S,G-G, 2004).
  Atomic monoid $\equiv$ commutative cancellative semigroup with identity element such that every non-unit may be expressed as a sum of finitely many atoms (irreducible elements).
- A subset I of a monoid S is an ideal if $I + S \subseteq I$.
  $a \in S$, the set $a + S = \{a + c \,|\, c \in S\} = \{s \in S \,|\, a \text{ divides } s\}$ is an ideal of S.

Definition (Anderson, Chapman, Kaplan, Torkornoo, 11)

Let S be an atomic monoid with set of units $S^\times$ and set of irreducibles $\mathcal{A}(S)$. For $x \in S \setminus S^\times$, we define $\omega(x) = n$ if n is the smallest positive integer with the property that whenever $x | a_1 + \cdots + a_t$, where each $a_i \in \mathcal{A}(S)$, there is a $T \subseteq \{1, 2, \ldots, t\}$ with $|T| \leq n$ such that $x | \sum_{k \in T} a_k$. If no such n exists, then $\omega(s) = \infty$. For $x \in S^\times$, we define $\omega(x) = 0$.

If $\omega(x) = 3$ and $x | (a_1 + a_2 + a_3 + a_4 + a_5) \Rightarrow$
$\exists i_1, i_2, i_3 \subset \{1, \ldots, 5\}$ such that $x | (a_{i_1} + a_{i_2} + a_{i_3})$.

n is prime $\iff \omega(n) = 1$.

Example

$S = \langle 3, 5 \rangle$,
$15 = 5 + 5 + 5 = 3 + 3 + 3 + 3 + 3$, then $\omega(15) = 5$.

Computing the ω-primality in atomic monoids

$S \simeq \mathbb{N}^p/\sigma$, $\varphi : \mathbb{N}^p \to \mathbb{N}^p/\sigma$ the projection map.

$A \subset \mathbb{N}^p/\sigma$, denote by $\mathrm{E}(A)$ the set $\varphi^{-1}(A)$.

For every $a \in S$, $\mathrm{E}(a+S)$ is an ideal of $\mathbb{N}^p$.

Proposition (Blanco, García-Sánchez, Geroldinger, 11)

Let $S = \mathbb{N}^p/\sigma$ be a finitely generated atomic monoid and $a \in S$. Then $\omega(a)$ is equal to max$\{\|\delta\| : \delta \in \mathrm{Minimals}_{\leq}(\mathrm{E}(a+S))\}$.

[Anderson,Chapman,Kaplan,Torkornoo, 10]: numerical semigroups.

[O'Neill, Pelayo, 14]: bullets.

[Rosales, García-Sánchez, García-García, 01]: $\mathrm{Minimals}_{\leq}(I)$, I ideal in S

Algorithm

Input: *A finite presentation of* $S = \mathbb{N}^p/\sigma$ *and* γ *an element of* $\mathbb{N}^p$ *verifying that* $a = [\gamma]_\sigma$.

Output: $\omega(a)$.

(1) *Compute the set* $\Delta = \mathrm{Minimals}_{\leq}(\mathrm{E}([\gamma]_\sigma + S))$ *using [R,G-S, G-G, 01].*

(2) *Set* $\Psi = \{\|\mu\| : \mu \in \Delta\}$.

(3) *Return* $\max \Psi$.

Example (R, G-S, G-G, 01)

$S \cong \mathbb{N}^4/\sigma$,

$\sigma = \langle\{((5,0,0,0),(0,7,0,0)),((0,0,6,0),(0,0,1,0))\}\rangle$,

S is atomic, but non-cancellative .

$a = [(3,3,6,5)]_\sigma \in S$,

$\mathrm{Minimals}_\leq \mathrm{E}(a+S) = \{(8,0,1,5),(0,10,1,5),(3,3,1,5)\}$.

$\omega(a) = \max\{\|(8,0,1,5)\|, \|(0,10,1,5)\|, \|(3,3,1,5)\|\} = 16$.

Software

- `OmegaPrimality`: Groebner Basis Calculations

  J. I. García-García, A. Vigneron-Tenorio. *OmegaPrimality, a package for computing the omega primality of finitely generated atomic monoids.*

  Handle: `http://hdl.handle.net/10498/15961` (2014)

- `numericalsgps GAP`: Construction of Apéry set.

  M. Delgado, P. A. García-Sánchez, J. Morais, *"NumericalSgps": a `GAP` package for numerical semigroups,*

  `http://www.gap-system.org/Packages/numericalsgps.html`

Comparison (milliseconds)

S	$\omega(n)$	OP	GAP
$\langle 115, 212, 333, 571\rangle$	$\omega(10000)$	22	1389
$\langle 115, 212, 333, 571\rangle$	$\omega(s_i)$	496	1888
$\langle 10, \ldots, 19\rangle$	$\omega(S)$	3779	125
$\langle 101, 111, 121, 131, 141, 151, 161, 171, 181, 191\rangle$	$\omega(S)$	135081	383949

We conclude:
the larger are the elements or generators, the better performance one gets with OP. But, if there are many generators and *small*, then one should use the Apéry method.

Asymptotic ω-primality

Definition (Anderson-Chapman, 10)

1. Let S be an atomic monoid and $x \in S$, define:
   - $\overline{\omega}(x) = \lim_{n\to+\infty} \frac{\omega(nx)}{n}$ the asymptotic ω-primality of x.
   - Asymptotic ω-primality of S is defined as $\overline{\omega}(S) = \sup\{\overline{\omega}(x) | x \text{ is irreducible}\}$.
2. $S = \langle s_1, \ldots, s_p \rangle$, then $\overline{\omega}(S) =$max$\{\overline{\omega}(s_i) | i = 1, \ldots, p\}$.

Asymptotic ω-primality in monoids generated by two elements

S cancelative, reduced. minimally generated by two elements $\Longrightarrow$ atomic.

$S \cong \mathbb{N}^2/\sigma$

Lemma

A non-free monoid S is cancellative, reduced and minimally generated by two elements if and only if $S \cong \mathbb{N}^2/\sigma$ with $\sigma = \langle((\alpha, 0), (0, \beta))\rangle$ and $\alpha, \beta > 1$.

S, numerical semigroup

$[\gamma]_\sigma \in S$,

Lemma

Let $S = \mathbb{N}^2/\sigma$ *with* $\sigma = \langle((\alpha, 0), (0, \beta))\rangle$ *and* $\alpha, \beta > 1$*. Then for all* $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^2$*, we have:*

$$\mathrm{E}([\gamma]_\sigma) = \{\gamma + \lambda(\alpha, -\beta) | \lambda \in \mathbb{Z}, -\lfloor \tfrac{\gamma_1}{\alpha} \rfloor \leq \lambda \leq \lfloor \tfrac{\gamma_2}{\beta} \rfloor\},$$

$$\begin{aligned} &\mathrm{Minimals}_\leq(\mathrm{E}([\gamma]_\sigma + S)) \\ = &\mathrm{Minimals}_\leq(\mathrm{E}([\gamma]_\sigma) \cup \{(0, \gamma_2 + (\lfloor \frac{\gamma_1}{\alpha} \rfloor + 1)\beta), (\gamma_1 + (\lfloor \frac{\gamma_2}{\beta} \rfloor + 1)\alpha, 0)\}) \end{aligned}$$

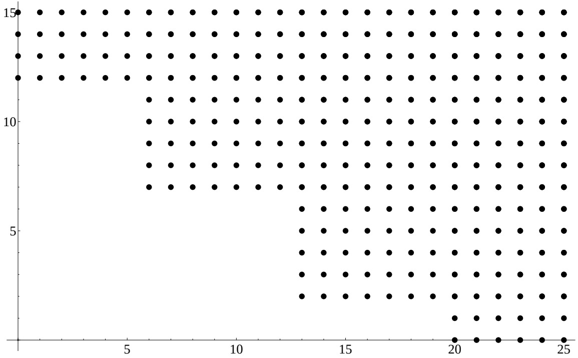
and

$$\omega([\gamma]_\sigma) = \max\{\gamma_2 + (\lfloor \tfrac{\gamma_1}{\alpha} \rfloor + 1)\beta, \gamma_1 + (\lfloor \tfrac{\gamma_2}{\beta} \rfloor + 1)\alpha\}.$$

Example

$S \cong \mathbb{N}^2/\sigma$, $\sigma = \langle((7,0),(0,5))\rangle$, $\gamma = (6,7) \in \mathbb{N}^2$.

$\mathrm{E}([(6,7)]_\sigma + \mathbb{N}^2/\sigma) = \langle(0,12),(6,7),(13,2),(20,0)\rangle$.



$\omega([(6,7)]_\sigma) = \max\{0+12, 6+7, 13+2, 20+0\} = 20$.

Proposition

Let $S = \mathbb{N}^2/\sigma$ *with* $\sigma = \langle ((\alpha, 0), (0, \beta)) \rangle$ *and* $\alpha, \beta > 1$. *Then:*

- *If* $\alpha \geq \beta$, *then* $\overline{\omega}([(\gamma_1, \gamma_2)]_\sigma) = \gamma_1 + \frac{\alpha}{\beta}\gamma_2$.
- *If* $\alpha < \beta$, *then* $\overline{\omega}([(\gamma_1, \gamma_2)]_\sigma) = \frac{\beta}{\alpha}\gamma_1 + \gamma_2$.

Corollary

Let $S = \mathbb{N}^2/\sigma$ *with* $\sigma = \langle ((\alpha, 0), (0, \beta)) \rangle$ *and* $\alpha, \beta > 1$. *Then:*

- *If* $\alpha \geq \beta$, *then* $\overline{\omega}([e_1]_\sigma) = 1$ *and* $\overline{\omega}(S) = \overline{\omega}([e_2]_\sigma) = \frac{\alpha}{\beta}$.
- *If* $\alpha < \beta$, *then* $\overline{\omega}([e_2]_\sigma) = 1$ *and* $\overline{\omega}(S) = \overline{\omega}([e_1]_\sigma) = \frac{\beta}{\alpha}$.

Asymptotic $\overline{\omega}$-primality in Archimedean semigroups

Definition

- An element $x \neq 0$ of a monoid S is **archimedean** if for all $y \in S \setminus \{0\}$ there exists a positive integer k such that $y|kx$.
- S is **quasi-archimedean** if the zero element is not archimedean and the rest of elements in S are archimedean.

S, numerical semigroups are quasi-archimedean

S monoid is finitely generated, cancellative and quasi-archimedean $\Longrightarrow$ for all $x, y \in S \setminus \{0\}$, there exist positive integers p and q such that $px = qy$.

$S = \langle s_1, \ldots, s_p \rangle$ quasi-archimedean cancellative monoid. There exists $k_1 \geq \cdots \geq k_p \in \mathbb{N} \setminus \{0\}$ s.t. $k_1[e_1]_\sigma = \cdots = k_p[e_p]_\sigma$.
In this way some elements of S can be expressed using only the generator $[e_1]_\sigma$.

Theorem

Let $S = \mathbb{N}^p/\sigma = \langle s_1, \dots, s_p \rangle$ *be a cancellative monoid with* σ *a congruence, let* $k_1 \geq \cdots \geq k_p \in \mathbb{N}$ *be such that* $k_1 s_1 = \cdots = k_p s_p$ *and let* $\gamma \in \mathbb{N}^p$*. Then every element* $x = (x_1, \dots, x_p) \in \mathbb{N}^p \setminus \{0\}$ *fulfilling that*

$$\sum_{i=1}^{p} \frac{k_1 \cdots k_p}{k_i} x_i \geq (p-1) k_1 \cdots k_p + \sum_{i=1}^{p} \frac{k_1 \cdots k_p}{k_i} \gamma_i$$

belongs to $\mathrm{E}([\gamma]_\sigma + S)$.

Theorem

Let $S = \mathbb{N}^p/\sigma$ *be a quasi-archimedean cancellative reduced monoid. There exists a rearrange* $\{t_1, \dots, t_p\}$ *of the set* $\{1, \dots, p\}$ *such that* $\overline{\omega}(a) = \gamma_{t_1} + \sum_{i=2}^{p} \frac{k_{t_1} \gamma_{t_i}}{k_{t_i}}$, $a = [(\gamma_1, \dots, \gamma_p)]_\sigma \in S$.

Corollary

Let $S = \mathbb{N}^p/\sigma$ *be a quasi-archimedean cancellative reduced monoid. There exist* $k_1, \ldots, k_p \in \mathbb{N}$ *such that* $\overline{\omega}([e_i]_\sigma) = \frac{\max\{k_1,\ldots,k_p\}}{k_i}$ *for all* $i = 1, \ldots, p$.

Corollary

Let S *be a numerical monoid minimally generated by* $\langle s_1 < s_2 < \cdots < s_p \rangle$. *For every* $s \in S$*, we have that* $\overline{\omega}(s) = \frac{s}{s_1}$.

Thanks for your attention!!