Almost Gorenstien rings

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Joint work with Shiro Goto and Tran Thi Phuong
History

- **1997** The notion of almost Gorenstein rings was introduced by Valentina Barucci-Ralf Fröberg (analytically unramified case) with a result about the Gorenstein property of \( m : m = \{ \alpha \in Q(R) \mid \alpha m \subseteq m \}. \)

- **2009** A counterexample for a result about \( m : m \) was given by Barucci. (But their result is true!)

- **2013** A new definition of almost Gorenstein rings of dimension one was given and repair the proof of the Gorenstein property of \( m : m \).
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Classes of local rings

regular $\Rightarrow$ complete-intersection $\Rightarrow$ Gorenstein $\Rightarrow$ Cohen-Macaulay $\Rightarrow$ Buchsbaum

Any numerical semigroup ring is CM

Cohen-Macaulay (CM)

Gorenstein

symmetric
Classes of local rings

regular $\Rightarrow$ complete-intersection $\Rightarrow$ Gorenstein $\Rightarrow$ almost Gorenstein
$\Rightarrow$ Cohen-Macaulay $\Rightarrow$ Buchsbaum
1 Preliminaries (Hilbert coefficients, existence of canonical ideals)

2 Definition of almost Gorenstein rings

3 The Gorenstein property of $m : m$

4 Almost Gorenstein rings obtained by idealization

5 Recent researches

$$k[[H]] = k[[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}]] \subseteq k[[t]] \text{ for } H = \langle a_1, a_2, \ldots, a_\ell \rangle.$$
§1. Preliminaries

Let \((R, \mathfrak{m})\) a CM local ring, \(\dim R = 1\), \(I\) an \(\mathfrak{m}\)-primary ideal

\[ \exists e_0(I), e_1(I) \in \mathbb{Z} \text{ such that} \]

\[ \ell_R(R/I^{n+1}) = e_0(I) \binom{n+1}{1} - e_1(I) \quad (\forall n \gg 0). \]

We call \(e_0(I)\) the multiplicity of \(R\) w.r.t. \(I\) and \(e_1(I)\) the first Hilbert coefficient of \(R\) w.r.t. \(I\).

How to compute \(e_0(I)\) and \(e_1(I)\)?
Assume \( \exists a \in I \) such that \( Q = (a) \) is a reduction of \( I \) (i.e., \( \exists n \geq 0, \ I^{n+1} = QI^n \))

For any \( n > 0 \), put \( \frac{I^n}{a^n} = \{ \frac{x}{a^n} \mid x \in I^n \} \subseteq Q(R) \)

Let \( S = R[\frac{I}{a}] \subseteq Q(R) \).

\[ S = \bigcup_{n>0} \frac{I^n}{a^n} = \frac{I^r}{a^r} \]

where \( r = \text{red}_Q(I) = \min\{n \geq 0 \mid I^{n+1} = QI^n \} \).

Hence

\[
\ell_R(R/I^{n+1}) = \ell_R(R/Q^{n+1}) - \ell_R(I^{n+1}/Q^{n+1})
\]

\[
= \ell_R(R/Q)\binom{n+1}{1} - \ell_R(S/R) \quad \text{if } n \geq r - 1
\]

\[
\parallel \quad e_0(I) \quad \parallel \quad e_1(I)
\]
Theorem

\[ e_0(I) = \ell_R(R/Q), \quad e_1(I) = \ell_R(S/R). \]

Corollary

\[ \mu_R(I/Q) \leq \ell_R(I/Q) \leq e_1(I) \]

1. \( \mu_R(I/Q) = \ell_R(I/Q) \iff mI \subseteq Q \) (i.e. \( mI = mQ \))
2. \( \ell_R(I/Q) = e_1(I) \iff I^2 = QI \) (i.e. \( \text{red}_Q(I) \leq 1 \))
The case $H = \langle 3, 4, 5 \rangle$

**Example**

Let $H = \langle 3, 4, 5 \rangle$ and $R = k[[H]] = k[[t^3, t^4, t^5]]$ ($k$ a field). Take $I = (t^3, t^4)$ and $Q = (t^3)$, then $Q$ is a reduction of $I$. In fact, $I^3 = QI^2$. Hence $S = \frac{l^2}{t^6}$.

For $e_0(I)$:

\[
\begin{array}{ccc}
0 & 1 & 2 \\
3 & 4 & 5 \\
6 & 7 & 8 \\
9 & 10 & 11 \\
\vdots & & \\
\end{array}
\quad
\begin{array}{ccc}
0 & 1 & 2 \\
3 & 4 & 5 \\
6 & 7 & 8 \\
9 & 10 & 11 \\
\vdots & & \\
\end{array}
\quad
\begin{array}{ccc}
0 & 1 & 2 \\
3 & 4 & 5 \\
6 & 7 & 8 \\
9 & 10 & 11 \\
\vdots & & \\
\end{array}
\]

$\Rightarrow e_0(I) = \ell_R(R/Q) = 3$
The case $H = \langle 3, 4, 5 \rangle$

Example

Let $H = \langle 3, 4, 5 \rangle$ and $R = k[[H]] = k[[t^3, t^4, t^5]]$ ($k$ a field). Take $I = (t^3, t^4)$ and $Q = (t^3)$, then $Q$ is a reduction of $I$. In fact, $I^3 = QI^2$. Hence $S = \frac{I^2}{t^6}$.

For $e_1(I)$:

\[
\begin{array}{ccc}
0 & 1 & 2 \\
3 & 4 & 5 \\
6 & 7 & 8 \\
9 & 10 & 11 \\
\ldots \\
\end{array}
\quad
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0 & 1 & 2 \\
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0 & 1 & 2 \\
3 & 4 & 5 \\
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\ldots \\
\end{array}
\]

$e_1(I) = \ell_R(S/R) = 2$
Existence of canonical ideals

- Let $K_R$ denote the canonical module of $R$.
- $\exists K_R \iff R \cong$ a Gorenstein ring / \sim.

**Definition**

We say that $I \subsetneq R$ is a **canonical ideal** of $R$ if $I \cong K_R$.

When $\exists I \subsetneq R$ a canonical ideal?
**Theorem (Herzog-Kunz)**

TFAE

1. \( \exists I \subsetneq R \) a canonical ideal of \( R \).
2. \( \mathbb{Q}(\hat{R}) \) is a Gorenstein ring.

Hence if \( R \) is analytically unramified then \( \exists I \) a canonical ideal of \( R \).

**Corollary**

Suppose that \( \mathbb{Q}(\hat{R}) \) is Gorenstein. If \( |R/m| = \infty \), then \( R \subseteq \exists K \subseteq \overline{R} \) such that \( K \cong K_R \) where \( \overline{R} \) is the integral closure of \( R \).

**Proof.**

\( \exists a \in I \) such that \( Q = (a) \) is a reduction of \( I \). Put \( K = \frac{I}{a} \)
Example

Let $R = k[[X, Y, Z]]/(X, Y) \cap (Y, Z) \cap (Z, X)$. Then

$I = (x + y, y + z) \cong K_R$.

If $k = \mathbb{Z}/(2)$, then $\forall a \in I$, $(a)$ is not a reduction of $I$. 
Proposition

Let $k = R/m$ and $\tilde{k}/k$ an extension of fields. then

$\exists \varphi : (R, m) \to (\tilde{R}, \tilde{m})$ a flat homomorphism of local rings such that

1. $\tilde{m} = m\tilde{R}$
2. $\tilde{R}/\tilde{m} \cong \tilde{k}$ as $k$-algebras.

Moreover we have the following

(a) $Q(\tilde{R})$ is Gorenstein $\iff Q(\tilde{R})$ is Gorenstein. In this case, $\forall I$ a canonical ideal of $R$, $I\tilde{R}$ is a canonical ideal of $\tilde{R}$ and $e_1(I\tilde{R}) = e_1(I)$.

(b) $m : m$ is Gorenstein $\iff \tilde{m} : \tilde{m}$ is Gorenstein.

Problem

$? R$ is analytically unramified $\iff \tilde{R}$ is analytically unramified
§2. Definition of almost Gorenstein rings

**Definition**

We say that $R$ is an almost Gorenstein ring, if

1. $Q(\hat{R})$ is Gorenstein. Hence $\exists I \subsetneq R$ a canonical ideal of $R$.
2. $e_1(I) \leq r(R)$ (the Cohen-Macaulay type of $R$) = $\mu_R(I)$.

**Remark**

Let $I, J \subsetneq R$ canonical ideals, then $e_1(I) = e_1(J)$.

$R$ is Gorenstein $\Rightarrow r(R) = 1$ and $I$ is a parameter ideal. Hence $e_1(I) = 0 \leq r(R)$. Thus $R$ is almost Gorenstein.
Examples of almost Gorenstein rings

Example

1. $R = k[[t^3, t^4, t^5]] \subseteq k[[t]]$ (r($R$) = 2; an integral domain)
2. $R = k[[X, Y, Z]]/(X, Y) \cap (Y, Z) \cap (Z, X)$ (r($R$) = 2; a reduced ring)
3. $R = k[[X, Y, Z, W]]/(Y^2, Z^2, W^2,YW,ZW,XW-YZ)$ (r($R$) = 3; not a reduced ring)
4. Let $3 \leq a \in \mathbb{Z}$ and $R = k[[t^a, t^{a+1}, t^{2-a-1}]]$.
   Let $I$ be a canonical ideal of $R \Rightarrow e_1(I) = \frac{a(a-1)}{2} - 1$, r($R$) = 2.
   Hence $R$ is an almost Gorenstein ring $\Leftrightarrow$ $a = 3$. On the other hand, $R \cong k[[x, y, z]]/I_2 \left(\begin{array}{ccc} x & y^{a-2} & z \\ y & z & x^{a-1} \end{array}\right)$. Hence, by Nari-Numata-Watanabe, $R$ is almost Gorenstein $\Leftrightarrow a - 2 = 1$. 
Settings

- $R \subseteq \exists K \subseteq \bar{R}$ an $R$-submodule such that $K \cong K_R$.
- Choose a NZD $a \in \mathfrak{m}$ such that $I = aK \subsetneq R$.
  Hence $Q = (a)$ is a reduction of $I$.
- $S = R[\frac{I}{a}] = R[K]$.
- $c = R : S := \{\alpha \in Q(R) \mid \alpha S \subseteq R\} \subseteq R$.

Definition (BF)

$R$ is an almost Gorenstein ring (in the sense of [BF]) if $\mathfrak{m}K \subseteq R$. 
Lemma

\[ r(R) - 1 = \mu_R(I/Q) \leq \ell_R(I/Q) \leq e_1(I) = \ell_R(I/Q) + \ell_R(R/c) \]

Proof

- \( e_1(I) = \ell_R(S/R) = \ell_R(S/K) + \ell_R(K/R) \).
- Since \( \ell_R(K/R) = \ell_R(I/Q) \), it is enough to show that \( \ell_R(S/K) = \ell_R(R/c) \).
- \( K : S = K : KS = (K : K) : S = R : S = c \).
- \( \ell_R(S/K) = \ell_R(K : K/K : S) = \ell_R(R/c) \).
Characterization of Gorenstein rings

Theorem

TFAE

1. $R$ is a Gorenstein ring.
2. $K = R.$
3. $K = S.$
4. $R = S.$
5. $\ell_R(S/R) = \ell_R(R/c).$
6. $I^2 = QI.$
7. $e_1(I) = 0.$
8. $e_1(I) = r(R) - 1.$
Characterization of almost Gorenstein rings

Theorem

$R$ is an almost Gorenstein ring $\iff mK \subseteq R$ (i.e. $ml \subseteq Q$)

When this is the case, $mS \subseteq R$.

This means two definitions of almost Gorenstein property coincide.

$$
r(R) - 1 \leq \ell_R(I/Q) \leq e_1(I)
$$

almost Gorenstein $\iff$ Gorenstein

$$
l^2 = Ql
$$

$m/l \subseteq Q \iff l^2 = Ql$
Proof

**Theorem**

*R is an almost Gorenstein ring* \(\iff\) \(\mathfrak{m}K \subseteq R\) (i.e. \(\mathfrak{m}I \subseteq Q\))

*When this is the case, \(\mathfrak{m}S \subseteq R\)*

\[
r(R) - 1 \leq \ell_R(I/Q) \leq e_1(I)
\]

- \(\Rightarrow\) is easy.
- \(\Leftarrow\) We may assume \(R\) is not Gorenstein.
- Put \(J = Q :_R \mathfrak{m}\). Then \(I \subseteq J\) and \(J^2 = QJ\).
- We have \(R \subseteq S = R[\frac{I}{a}] \subseteq R[\frac{J}{a}] = \frac{J}{a}\).
- Hence \(e_1(I) = \ell_R(S/R) \leq \ell_R(R[\frac{J}{a}]/R) = \ell_R(J/Q) = r(R)\).
Corollary

TFAE

1. $R$ is almost Gorenstein but not Gorenstein.
2. $e_1(I) = r(R)$.
3. $e_1(I) = e_0(I) - \ell_R(R/I) + 1$ (Sally’s equality).
4. $S = K : m$.
5. $\ell_R(I^2/QI) = 1$.
6. $m : m = S$ and $R$ is not a DVR.

When this is the case,

(a) $\text{red}_Q(I) = 2$.
(b) Put $G = \text{gr}_I(R)$ and $M = mG + G_+$. Then $G$ is Buchsbaum and $H^0_M(G) = [H^0_M(G)]_0 \cong R/m$. Hence $\mathcal{I}(G) = 1$. 
Proof of (1) \iff (3)

1. *R* is almost Gorenstein but not Gorenstein.
2. \(e_1(I) = e_0(I) - \ell_R(R/I) + 1\) (Sally’s equality).

\[
e_1(I) - e_0(I) = (\ell_R(R/c) + \ell_R(I/Q)) - \ell_R(R/Q) = \ell_R(R/c) - \ell_R(R/I).
\]

Hence \(e_1(I) = e_0(I) - \ell_R(R/I) + \ell_R(R/c)\).

(3) \iff \(m = c(= S : R) \iff S \neq R\) and \(mS \subseteq R \iff (1)\)
1. \( e_1(I) = r(R) - 1 \iff R \text{ is Gorenstein.} \)
2. \( e_1(I) = r(R) \iff R \text{ is almost Gorenstein but not Gorenstein.} \)

**Problem**

*When \( e_1(I) = r(R) + 1 \)?*

**Theorem**

\( e_1(I) \neq r(R) + 1 \)
Preliminaries

Definition

$m : m$

Idealization

Recent

1. $e_1(I) = r(R) - 1 \iff R \text{ is Gorenstein.}$
2. $e_1(I) = r(R) \iff R \text{ is almost Gorenstein but not Gorenstein.}$

Problem

When $e_1(I) = r(R) + 1$?

Theorem

$e_1(I) \neq r(R) + 1$
§3. The Gorenstein property of $m : m$

Theorem (Barucci-Fröberg)

TFAE

1. $A = m : m$ is a Gorenstein ring.

2. $R$ is an almost Gorenstein ring and $v(R) = e(R)$.
   $v(R)$ the embedding dimension of $R$,
   $e(R)$ the multiplicity of $R$

When $R = k[[H]]$ is a numerical semigroup ring of $H = \langle a_1, a_2, \ldots, a_\ell \rangle$, $v(R) = \ell$ and $e(R) = \min\{a_i \mid 1 \leq i \leq \ell\}$. 
How to prove the theorem

- We may assume $R/m$ is algebraically closed.
- Thanks to this assumption, we get the following claim.

**Claim**

$$\ell_A(X) = \ell_R(X) \text{ for } \forall X \text{ A-modules.}$$

- Then Barucci and Fröberg's argument works well.

**Problem (again)**

$$? \text{ R is analytically unramified } \iff \tilde{R} \text{ is analytically unramified}$$

We now have no proof in special cases, but we can prove in full generality.
§4. Almost Gorenstein rings obtained by idealization

**Theorem**

**TFAE**

1. $R \ltimes m$ is an almost Gorenstein ring.
2. $R$ is an almost Gorenstein ring.

*When this is the case, $\nu(R \ltimes m) = 2\nu(R)$.***
Example

\(\forall n \geq 0, \text{ put}
\)

\[ R_n = \begin{cases} 
R & (n = 0) \\
R \times m & (n = 1) \\
[R_{n-1}]_1 & (n \geq 2) 
\end{cases} \]

1. If \( R \) is Gorenstein, then \( R_n \) is almost Gorenstein \( (\forall n \geq 0) \).
2. \( R_n \) is not Gorenstein \( (\forall n \geq 2) \).

Example

\[ k[[X, Y, Z, W]]/(Y^2, Z^2, W^2, YW, ZW, XW - YZ) \]
\[ \cong k[[X, Y]]/(Y^2) \times (X, Y)/(Y^2) \]
Recent researches

GTT Shiro Goto, Ryo Takahashi, and Naoki Taniguchi gave a possible definition of **higher-dimensional** or **graded** almost Gorenstein rings in terms of $C = \text{Coker}(0 \to R \to K_R)$.

MM Satoshi Murai and I consider the graded almost Gorenstein property for Stanley-Reisner rings following the definition by [GTT]. (Preprint, arXiv:1405.7438).