On the Hilbert function of one-dimensional semigroup rings

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Let $(R, \mathfrak{m})$ be a Noetherian local ring with $|R \setminus \mathfrak{m}| = \infty$.

$$gr(R) = \oplus_{i\geq 0}\mathfrak{m}^i/\mathfrak{m}^{i+1}$$

is the associated graded ring of R.

Definition

The Hilbert function of R is

$$H_R : \mathbb{N} \to \mathbb{N}, \quad H_R(i) = dim_k \mathfrak{m}^i/\mathfrak{m}^{i+1},$$

where $k = R/\mathfrak{m}$.

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Definition

$C \subseteq \mathbb{A}_k^n$ is an algebraic curve if

- $\exists\ I(C) \subseteq k[x_1, \ldots, x_n]$ *such that* $C = V(I(C))$*;*
- $dim_k \frac{k[x_1,\ldots,x_n]}{I(C)} = 1$.

Suppose there are some numbers $g_1, \ldots, g_n \in \mathbb{N}$ with $gcd(g_1, \ldots, g_n) = 1$, and an homomorphism $\psi : k[x_1, \ldots, x_n] \to k[t]$:

$$x_1 \mapsto t^{g_1}$$

$$\vdots$$

$$x_n \mapsto t^{g_n},$$

such that $I(C) = ker\psi$, then C is called monomial curve, denoted by $C = C(g_1, \ldots, g_n)$.

Let $C = C(g_1, \ldots, g_n)$ be a monomial curve determined by the homomorphism ψ.

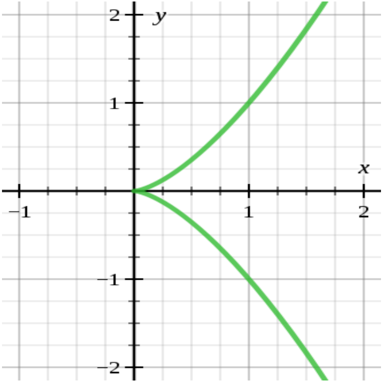
Then

1. $S = \langle g_1, \ldots, g_n \rangle$ is a numerical semigroup;
2. By extending ψ to $\hat{\psi} : k[[x_1, \ldots, x_n]] \to k[[t]]$, we get
$$Im(\hat{\psi}) = k[[t^S]],$$
the semigroup ring associated to S;
3. $k[[t^S]] \cong \frac{k[[x_1,\ldots,x_n]]}{I(C)^e}$ is the completion of the coordinate ring of C;
4. $gr(R) \cong \frac{k[x_1,\ldots,x_n]}{I(C)^*}$ is the coordinate ring of the tangent cone of C at 0.

Example

The cusp curve

$$\psi : k[x_1, x_2] \to k[t]$$
$$x_1 \mapsto t^2$$
$$x_2 \mapsto t^3$$
$$S = \langle 2, 3 \rangle,\ I(C) = (x_1^3 - x_2^2)$$
$$gr(R) \cong \frac{k[x_1, x_2]}{(x_2^2)}$$



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Question (1)

[Rossi's conjecture] Is the Hilbert function of one-dimensional Gorenstein local rings non-decreasing?

Answer:

- In general the problem is open.

Question (2)

Is the answer to the previous question affermative for rings associated to monomial curves?

Partial answers:

- If $gr(R)$ is Cohen-Macaulay, yes (A. Garcìa).
- Yes for some semigroups obtained by gluing (Arslan-Mete-Sahin, Jafari-Zarzuela Armengou).

Question (3)

Is the Hilbert function of rings associated to monomial curves non-decreasing for small embedding dimensions (e.g. $edim = 3, 4, 5$)?

Answer:

- $edim = 3$: Yes, more generally it is true for one-dimensional equicharacteristic rings (J. Elìas).
- $edim = 4$: Yes if the associated graded ring is Buchsbaum (Cortadellas Benitez-Jafari-Zarzuela Armengou).
  Open in general.
- $edim = 5, \ldots, 9$: The problem is totally open, the first counterexample is for $edim = 10$.

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$v : k((t))$	$\to$	$\mathbb{Z} \cup \{0\}$
$\sum_{h=i}^{\infty} r_h t^h, r_h \neq 0$	$\mapsto$	i

Semigroup rings		Semigroups
$R = k[[t^S]] = k[[t^{g_1}, \dots, t^{g_n}]]$	$\to$	$S = \langle g_1, \dots, g_n \rangle$
$\mathfrak{m} = (t^{g_1}, \dots, t^{g_n})$ maximal ideal of R	$\to$	$M = S \setminus \{0\}$ maximal ideal of S
$\mathfrak{m}^i$	$\to$	iM
$dim_k \mathfrak{m}^i / \mathfrak{m}^{i+1}$	$\to$	$\|iM \setminus (i+1)M\|$
$R' = \cup_i (\mathfrak{m}^i :_{Q(R)} \mathfrak{m}^i)$ blow-up of R	$\to$	$S' = \cup_i (iM -_{\mathbb{Z}} iM)$ blow-up of S
R Gorenstein	$\to$	S symmetric

$$H_R \text{ non-decreasing} \Leftrightarrow |iM \setminus (i+1)M| \leq |(i+1)M \setminus (i+2)M|, \quad \forall i$$

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Let $S = \langle g_1, \ldots, g_n \rangle$, where $g_1 < \ldots < g_n$ are the generators of the minimal system of generators.

Definition

The Apéry-set of S is the set

$$Ap(S) = \{\omega_0, \omega_1, \ldots, \omega_{g_1 - 1}\},$$

where $\omega_i = min\{s \in S \mid s \equiv i (mod \quad g_1)\}$.

Similarly, one can define the Apéry-set for $S' = \langle g_1, g_2 - g_1, \ldots, g_n - g_1 \rangle$

$$Ap(S') = \{\omega'_0, \omega'_1, \ldots, \omega'_{g_1 - 1}\},$$

where $\omega'_i = min\{s' \in S' \mid s' \equiv i (mod \quad g_1)\}$.

Definition

$a_i =$ *the positive number such that* $\omega_i = \omega'_i + a_i g_1, \quad i = 0, 1, \ldots, g_1 - 1$
$b_i = max\{l \mid \omega_i \in lM\}, \quad i = 0, 1, \ldots, g_1 - 1$

In general $a_i \geq b_i$ for every i.

Example

$R = \mathbb{Q}[[t^8, t^9, t^{12}, t^{13}, t^{19}]]$
$S = \langle 8, 9, 12, 13, 19 \rangle = \{0, 8, 9, 12, 13, 16, 17, 18, 19, 20, 21, 22, 24, \rightarrow\}$
$M \setminus 2M = \{8, 9, 12, 13, 19\}$
$2M \setminus 3M = \{16, 17, 18, 20, 21, 22\}$
$3M \setminus 4M = \{24, 25, 26, 27, 28, 29, 30, 31\}$
$4M \setminus 5M = \{32, 33, 34, 35, 36, 37, 38, 39\}$
reduction number = 4
$Ap(S) = \{0, 9, 18, 19, 12, 13, 22, 31\}$
$Ap(S') = \{0, 1, 2, 3, 4, 5, 6, 7\}$
$a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 2, a_4 = 1, a_5 = 1, a_6 = 2, a_7 = 3$
$b_0 = 0, b_1 = 1, b_2 = 2, b_3 = 1, b_4 = 1, b_5 = 1, b_6 = 2, b_7 = 3$

Let $R = k[[t^S]]$, where $S = \langle g_1, \ldots, g_n \rangle$, $g_1 < g_2 < \ldots < g_n$.

Proposition (A. Garcìa)

$gr(R)$ is Cohen-Macaulay if and only if t^{g_1} is a regular element.

Proposition (Barucci-Fröberg)

$gr(R)$ is Cohen-Macaulay if and only if $a_i = b_i$, for every i.

Definition

We call order of an element $s \in S$ the integer i such that $s \in iM \setminus (i+1)M$, denoted by $ord(s)$; we also say that s is on the i-th level.
An element s skips the level when adding g_1 if $ord(s + g_1) > ord(s) + 1$.

t^{g_1} is a zerodivisor in $R \Leftrightarrow \exists \ s \in S$ that skips the level when adding g_1.

Example

$S = \langle 8, 9, 12, 13, 19\rangle$
$M \setminus 2M = \{8, 9, 12, 13, 19\}$
$2M \setminus 3M = \{16, 17, 18, 20, 21, 22\}$
$3M \setminus 4M = \{24, 25, 26, 27, 28, 29, 30, 31\}$
$4M \setminus 5M = \{32, 33, 34, 35, 36, 37, 38, 39\}$
reduction number = 4
19 *skips the order when adding* 8*;* 18, 22, 27, 31 *do not come from the previous level.*

Definition

$D_i = \{s \in (i-1)M \setminus iM : s + g_1 \in (i+1)M\}, \quad i \geq 2.$
$C_i = \{s \in iM \setminus (i+1)M : s - g_1 \notin (i-1)M \setminus iM\}, \quad i \geq 1.$

H_R is non-decreasing $\Leftrightarrow |D_i| \leq |C_i|, \quad \forall i \in \{2, \ldots, r\}$

Example

$19 \in D_2$; $18, 22 \in C_2$, $27, 31 \in C_3$.

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Let $D = \cup_{i \geq 2} D_i$.

Proposition

For every index i there exists an element $s \in D$ such that $s \equiv i (mod\ g_1)$ if and only if $a_i > b_i$.

Example

$S = \langle 8, 9, 12, 13, 19 \rangle$
$M \setminus 2M = \{8, 9, 12, 13, 19\}$
$2M \setminus 3M = \{16, 17, 18, 20, 21, 22\}$
$3M \setminus 4M = \{24, 25, 26, 27, 28, 29, 30, 31\}$
$4M \setminus 5M = \{32, 33, 34, 35, 36, 37, 38, 39\}$
reduction number = 4
$Ap(S) = \{0, 9, 18, 19, 12, 13, 22, 31\}$
$Ap(S') = \{0, 1, 2, 3, 4, 5, 6, 7\}$
$a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 2, a_4 = 1, a_5 = 1, a_6 = 2, a_7 = 3$
$b_0 = 0, b_1 = 1, b_2 = 2, b_3 = 1, b_4 = 1, b_5 = 1, b_6 = 2, b_7 = 3$

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Lemma

For every index $i \geq 2$ there exists a function $\phi : D_i \to C_i$.

Proof sketch:

- Let $s \in D_i$, so $ord(s) = i - 1$ and $ord(s + g_1) > i$;
- Let $s + g_1 = g_{h_1} + \ldots + g_{h_i} + g_{h_{i+1}} + \ldots$ be the greatest among the maximal representations of $s + g_1$;
- Then $\phi(s) = g_{h_1} + \ldots + g_{h_i}$ is an element of C_i.

Theorem

If $|D_i| \leq i + 1$, then there exists an injective function $\hat{\phi} : D_i \to C_i$.

Hence,

$$\boxed{|D_i| \leq i + 1 \text{ for every } i \geq 2 \Rightarrow H_R \text{ non-decreasing.}}$$

Example (Molinelli-Tamone)

$S = \langle 13, 19, 24, 44, 49, 54, 55, 59, 60, 66 \rangle$

$M \setminus 2M = \{13, 19, 24, 44, 49, 54, 55, 59, 60, 66\}$
$2M \setminus 3M = \{26, 32, 37, 38, 43, 48, 68, 73, 79\}$
$3M \setminus 4M = \{39, 45, 50, 51, 56, 57, 61, 62, 67, 72, 92\}$

$\vdots$

H_R is decreasing: $|M \setminus 2M| = 10 > |2M \setminus 3M| = 9$.

$D_2 = \{44, 49, 54, 59\}$;
$C_2 = \{38, 43, 48\}$.

Here $|D_2| = 4$, *so the bound of the theorem cannot be improved uniformly.*

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Proposition

If H_R is decreasing, then there exists an index $h \geq 2$ such that $|C_i| \geq i + 1$, for every $i \leq h$.

In particular,

- The index h can be chosen as the index where the function decreases;
- One could estabilish that the function is non-decreasing without computing necessarily the cardinalities of all the levels.

Corollary

If H_R is decreasing, then

$$|\{\omega_i \in Ap(S) : b_i = 2\}| > 3$$

We then obtain the following results for small embedding dimensions.

Corollary

If H_R is decreasing at $h = 2$, then

$$edim(S) > 5$$

In general we know that if $g_1 - edim(S) \leq 2$ then H_R is non-decreasing.

Corollary

If $edim(S) = 4, 5$ and $g_1 \leq 8$, then H_R is non-decreasing.

Indice

1. Relate the numerical conditions on C_i and D_i to the properties for S symmetric.

2. Extend the partial answer given for embedding dimension 4.

3. Analize the Hilbert function in the case $gr(R)$ Buchsbaum.

Thank you!