Identifying torsion in the tensor product...

Micah Leamer

micahleamer@gmail.com
Notations and Definitions

Notation

- Throughout this talk \( \Gamma \) will denote a numerical semigroup;
- \( A \) and \( B \) will denote relative ideals of \( \Gamma \); and
- The dual of \( A \) is denoted by \( A^* = \Gamma - A = \{ z \in \mathbb{Z} \mid z + A \subseteq \Gamma \} \).

Definition

A splitting of \( A \) is a pair of relative ideals \( P \) and \( Q \) such that \( P \cup Q = A \).

Definition

\( A \) is said to be Huneke-Wiegand if either it is principal, or there exists a splitting \( P \cup Q = A \) such that

\[
(P \cap Q) + A^* \neq (P + A^*) \cap (Q + A^*)
\]

\( \subseteq \leftarrow \) This inclusion is automatic.
Notations and Definitions

Notation

- Throughout this talk $\Gamma$ will denote a numerical semigroup;
- $A$ and $B$ will denote relative ideals of $\Gamma$; and
- The dual of $A$ is denoted by $A^* = \Gamma - A = \{ z \in \mathbb{Z} | z + A \subseteq \Gamma \}$.

Definition

A splitting of $A$ is a pair of relative ideals $P$ and $Q$ such that $P \cup Q = A$.

Definition

$A$ is said to be Huneke-Wiegand if either it is principal, or there exists a splitting $P \cup Q = A$ such that

$$(P \cap Q) + A^* \neq (P + A^*) \cap (Q + A^*)$$

$\subseteq \leftarrow$ This inclusion is automatic.
Notations and Definitions

**Notation**
- Throughout this talk $\Gamma$ will denote a numerical semigroup;
- $A$ and $B$ will denote relative ideals of $\Gamma$; and
- The dual of $A$ is denoted by $A^* = \Gamma - A = \{ z \in \mathbb{Z} \mid z + A \subseteq \Gamma \}$.

**Definition**
A splitting of $A$ is a pair of relative ideals $P$ and $Q$ such that $P \cup Q = A$.

**Definition**
$A$ is said to be Huneke-Wiegand if either it is principal, or there exists a splitting $P \cup Q = A$ such that

\[
(P \cap Q) + A^* \neq (P + A^*) \cap (Q + A^*) \subseteq \leftarrow \text{This inclusion is automatic}
\]
Notations and Definitions

Notation
- Throughout this talk $\Gamma$ will denote a numerical semigroup;
- $A$ and $B$ will denote relative ideals of $\Gamma$; and
- The dual of $A$ is denoted by $A^* = \Gamma - A = \{ z \in \mathbb{Z} | z + A \subseteq \Gamma \}$.

Definition
A splitting of $A$ is a pair of relative ideals $P$ and $Q$ such that $P \cup Q = A$.

Definition
$A$ is said to be Huneke-Wiegand if either it is principal, or there exists a splitting $P \cup Q = A$ such that

\[(P \cap Q) + A^* \neq (P + A^*) \cap (Q + A^*) \subseteq \leftarrow \text{This inclusion is automatic}\]
Notations and Definitions

Notation

- Throughout this talk $\Gamma$ will denote a numerical semigroup;
- $A$ and $B$ will denote relative ideals of $\Gamma$; and
- The dual of $A$ is denoted by $A^* = \Gamma - A = \{z \in \mathbb{Z} | z + A \subseteq \Gamma\}$.

Definition

A splitting of $A$ is a pair of relative ideals $P$ and $Q$ such that $P \cup Q = A$

Definition

$A$ is said to be Huneke-Wiegand if either it is principal, or there exists a splitting $P \cup Q = A$ such that

$$(P \cap Q) + A^* \neq (P + A^*) \cap (Q + A^*)$$

$\subseteq \leftarrow$ This inclusion is automatic
Notations and Definitions

Notation
- Throughout this talk $\Gamma$ will denote a numerical semigroup;
- $A$ and $B$ will denote relative ideals of $\Gamma$; and
- The dual of $A$ is denoted by $A^* = \Gamma - A = \{z \in \mathbb{Z} \mid z + A \subseteq \Gamma\}$.

Definition
A splitting of $A$ is a pair of relative ideals $P$ and $Q$ such that $P \cup Q = A$.

Definition
$A$ is said to be Huneke-Wiegand if either it is principal, or there exists a splitting $P \cup Q = A$ such that

$$(P \cap Q) + A^* \neq (P + A^*) \cap (Q + A^*) \subseteq \iff \text{This inclusion is automatic}.$$
The Huneke-Wiegand Conjecture for Numerical Semigroups

Conjecture

All relative ideals are Huneke-Wiegand.

Recall: \( A \) is Huneke-Wiegand provided there exists a splitting \( P \cup Q = A \) such that \( (P \cap Q) + A^* \neq (P + A^*) \cap (Q + A^*) \)

Question

Why would we make this conjecture and where does it come from?

Answer

It is equivalent to a special case of the Huneke-Wiegand Conjecture, which is a well known conjecture in commutative algebra related to torsion and tensor products.
The Huneke-Wiegand Conjecture for Numerical Semigroups

**Conjecture**

All relative ideals are Huneke-Wiegand.

Recall: $A$ is Huneke-Wiegand provided there exists a splitting $P \cup Q = A$ such that $(P \cap Q) + A^* \neq (P + A^*) \cap (Q + A^*)$.

**Question**

Why would we make this conjecture and where does it come from?

**Answer**

It is equivalent to a special case of the Huneke-Wiegand Conjecture, which is a well known conjecture in commutative algebra related to torsion and tensor products.
The Huneke-Wiegand Conjecture for Numerical Semigroups

**Conjecture**

*All relative ideals are Huneke-Wiegand.*

Recall: $A$ is Huneke-Wiegand provided there exists a splitting $P \cup Q = A$ such that $(P \cap Q) + A^* \neq (P + A^*) \cap (Q + A^*)$.

**Question**

Why would we make this conjecture and where does it come from?

**Answer**

It is equivalent to a special case of the Huneke-Wiegand Conjecture, which is a well known conjecture in commutative algebra related to torsion and tensor products.
The Huneke-Wiegand Conjecture for Numerical Semigroups

**Conjecture**

*All relative ideals are Huneke-Wiegand.*

Recall: $A$ is Huneke-Wiegand provided there exists a splitting $P \cup Q = A$ such that $(P \cap Q) + A^* \neq (P + A^*) \cap (Q + A^*)$

**Question**

Why would we make this conjecture and where does it come from?

**Answer**

It is equivalent to a special case of the Huneke-Wiegand Conjecture, which is a well known conjecture in commutative algebra related to torsion and tensor products.
Introducing torsion and tensor products

**Notation**
- $R$ will denote a commutative Noetherian domain
- $M$ and $N$ will be $R$-modules

**Definition**
The torsion submodule of $M$ is

$$T(M) := \{ m \in M \mid r m = 0 \text{ for some } r \in R \setminus \{0\} \}$$

It is often the case that $T(M \otimes_R N) \neq 0$

**Example**
Suppose $R = k[\Gamma]$ is a numerical semigroup ring with monomial ideals $I$ and $J$. Then $T(I \otimes_R J)$ is the $k$-linear span of elements of the form

$$t^a \otimes t^b - t^c \otimes t^d \in I \otimes_R J \quad \text{where } a + b = c + d$$
Introducing torsion and tensor products

Notation
- $R$ will denote a commutative Noetherian domain
- $M$ and $N$ will be $R$-modules

Definition
The torsion submodule of $M$ is $T(M) := \{m \in M| \text{ rm = 0 for some } r \in R \setminus \{0\}\}$

It is often the case that $T(M \otimes_R N) \neq 0$

Example
Suppose $R = k[\Gamma]$ is a numerical semigroup ring with monomial ideals $I$ and $J$. Then $T(I \otimes_R J)$ is the $k$-linear span of elements of the form

$$t^a \otimes t^b - t^c \otimes t^d \in I \otimes_R J \quad \text{where} \quad a + b = c + d$$
Introducing torsion and tensor products

Notation
- \( R \) will denote a commutative Noetherian domain
- \( M \) and \( N \) will be \( R \)-modules

Definition
The torsion submodule of \( M \) is
\[
T(M) := \{ m \in M \mid rm = 0 \text{ for some } r \in R \setminus \{0\} \}
\]

It is often the case that \( T(M \otimes_R N) \neq 0 \)

Example
Suppose \( R = k[\Gamma] \) is a numerical semigroup ring with monomial ideals \( I \) and \( J \). Then \( T(I \otimes_R J) \) is the \( k \)-linear span of elements of the form
\[
t^a \otimes t^b - t^c \otimes t^d \in I \otimes_R J \quad \text{where } a + b = c + d
\]
Introducing torsion and tensor products

Notation
- $R$ will denote a commutative Noetherian domain
- $M$ and $N$ will be $R$-modules

Definition
The torsion submodule of $M$ is
$T(M) := \{ m \in M | rm = 0 \text{ for some } r \in R \setminus \{0\} \}$

It is often the case that $T(M \otimes_R N) \neq 0$

Example
Suppose $R = k[\Gamma]$ is a numerical semigroup ring with monomial ideals $I$ and $J$. Then $T(I \otimes_R J)$ is the $k$-linear span of elements of the form
$t^a \otimes t^b - t^c \otimes t^d \in I \otimes_R J$ where $a + b = c + d$
Introducing torsion and tensor products

Notation

- $R$ will denote a commutative Noetherian domain
- $M$ and $N$ will be $R$-modules

Definition

The torsion submodule of $M$ is
$$T(M) := \{ m \in M | rm = 0 \text{ for some } r \in R \setminus \{0\} \}$$

It is often the case that $T(M \otimes_R N) \neq 0$

Example

Suppose $R = k[\Gamma]$ is a numerical semigroup ring with monomial ideals $I$ and $J$. Then $T(I \otimes_R J)$ is the $k$-linear span of elements of the form
$$t^a \otimes t^b - t^c \otimes t^d \in I \otimes_R J \quad \text{where} \quad a + b = c + d$$
Introducing torsion and tensor products

**Notation**
- $R$ will denote a commutative Noetherian domain
- $M$ and $N$ will be $R$-modules

**Definition**
The torsion submodule of $M$ is

$$T(M) := \{ m \in M \mid rm = 0 \text{ for some } r \in R \setminus \{0\} \}$$

It is often the case that $T(M \otimes_R N) \neq 0$

**Example**
Suppose $R = k[\Gamma]$ is a numerical semigroup ring with monomial ideals $I$ and $J$. Then $T(I \otimes_R J)$ is the $k$-linear span of elements of the form

$$t^a \otimes t^b - t^c \otimes t^d \in I \otimes_R J \quad \text{where } a + b = c + d$$
Introducing torsion and tensor products

<table>
<thead>
<tr>
<th>Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>- $R$ will denote a commutative Noetherian domain</td>
</tr>
<tr>
<td>- $M$ and $N$ will be $R$-modules</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>The torsion submodule of $M$ is $T(M) := { m \in M \mid rm = 0 \text{ for some } r \in R \setminus {0} }$</td>
</tr>
</tbody>
</table>

It is often the case that $T(M \otimes_R N) \neq 0$

<table>
<thead>
<tr>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Suppose $R = k[\Gamma]$ is a numerical semigroup ring with monomial ideals $I$ and $J$. Then $T(I \otimes_R J)$ is the $k$-linear span of elements of the form $t^a \otimes t^b - t^c \otimes t^d \in I \otimes_R J$ where $a + b = c + d$</td>
</tr>
</tbody>
</table>
Examples

Example

Let $R = k[t^3, t^4, t^5]$

$I = t^3R + t^4R$ and $J = t^4R + t^5R$

Then $t^3 \otimes t^5 - t^4 \otimes t^4 \in T(M \otimes_R N) \neq 0$

\[ t^5(t^3 \otimes t^5 - t^4 \otimes t^4) = t^8 \otimes t^5 - t^4 \otimes t^9 \]
\[ = t^4 t^4 \otimes t^5 - t^4 \otimes t^9 \]
\[ = t^4 \otimes t^9 - t^4 \otimes t^9 = 0 \]

Example

Let $R = k[t^4, t^5, t^6]$

$I = t^4R + t^6R$ and $J = t^4R + t^5R$.

Then $T(I \otimes_R J) = 0$.

Because $t^a \otimes t^b = t^c \otimes t^d \in I \otimes_R J$, whenever $a + b = c + d$. 
Examples

Example

Let \( R = k[t^3, t^4, t^5] \)
\( I = t^3 R + t^4 R \) and \( J = t^4 R + t^5 R \)

Then \( t^3 \otimes t^5 - t^4 \otimes t^4 \in T(M \otimes_R N) \neq 0 \)

\[
t^5(t^3 \otimes t^5 - t^4 \otimes t^4) = t^8 \otimes t^5 - t^4 \otimes t^9
= t^4 t^4 \otimes t^5 - t^4 \otimes t^9
= t^4 \otimes t^9 - t^4 \otimes t^9 = 0
\]

Example

Let \( R = k[t^4, t^5, t^6] \)
\( I = t^4 R + t^6 R \) and \( J = t^4 R + t^5 R \).

Then \( T(I \otimes_R J) = 0 \).

Because \( t^a \otimes t^b = t^c \otimes t^d \in I \otimes_R J \), whenever \( a + b = c + d \).
Examples

Example

Let \( R = k[t^3, t^4, t^5] \)
\( I = t^3 R + t^4 R \) and \( J = t^4 R + t^5 R \)
Then \( t^3 \otimes t^5 - t^4 \otimes t^4 \in T(M \otimes_R N) \neq 0 \)

\[
t^5(t^3 \otimes t^5 - t^4 \otimes t^4) = t^8 \otimes t^5 - t^4 \otimes t^9 \\
= t^4 t^4 \otimes t^5 - t^4 \otimes t^9 \\
= t^4 \otimes t^9 - t^4 \otimes t^9 = 0
\]

Example

Let \( R = k[t^4, t^5, t^6] \)
\( I = t^4 R + t^6 R \) and \( J = t^4 R + t^5 R \)
Then \( T(I \otimes_R J) = 0. \)
Because \( t^a \otimes t^b = t^c \otimes t^d \in I \otimes_R J \), whenever \( a + b = c + d. \)
Examples

Example

Let \( R = k[t^3, t^4, t^5] \)
\( I = t^3R + t^4R \) and \( J = t^4R + t^5R \)
Then \( t^3 \otimes t^5 - t^4 \otimes t^4 \in T(M \otimes_R N) \neq 0 \)

\[
\begin{align*}
t^5(t^3 \otimes t^5 - t^4 \otimes t^4) &= t^8 \otimes t^5 - t^4 \otimes t^9 \\
&= t^4t^4 \otimes t^5 - t^4 \otimes t^9 \\
&= t^4 \otimes t^9 - t^4 \otimes t^9 = 0
\end{align*}
\]

Example

Let \( R = k[t^4, t^5, t^6] \)
\( I = t^4R + t^6R \) and \( J = t^4R + t^5R \).
Then \( T(I \otimes_R J) = 0. \)
Because \( t^a \otimes t^b = t^c \otimes t^d \in I \otimes_R J \), whenever \( a + b = c + d. \)
Examples

Example

Let $R = k[t^3, t^4, t^5]$

$I = t^3R + t^4R$ and $J = t^4R + t^5R$

Then $t^3 \otimes t^5 - t^4 \otimes t^4 \in T(M \otimes_R N) \neq 0$

$$t^5(t^3 \otimes t^5 - t^4 \otimes t^4) = t^8 \otimes t^5 - t^4 \otimes t^9$$

$$= t^4 t^4 \otimes t^5 - t^4 \otimes t^9$$

$$= t^4 \otimes t^9 - t^4 \otimes t^9 = 0$$

Example

Let $R = k[t^4, t^5, t^6]$

$I = t^4R + t^6R$ and $J = t^4R + t^5R$

Then $T(I \otimes_R J) = 0$.

Because $t^a \otimes t^b = t^c \otimes t^d \in I \otimes_R J$, whenever $a + b = c + d$. 
Examples

Example

Let $R = k[t^3, t^4, t^5]$
$I = t^3R + t^4R$ and $J = t^4R + t^5R$
Then $t^3 \otimes t^5 - t^4 \otimes t^4 \in T(M \otimes_R N)$ $\neq 0$

$$t^5(t^3 \otimes t^5 - t^4 \otimes t^4) = t^8 \otimes t^5 - t^4 \otimes t^9$$
$$= t^4 t^4 \otimes t^5 - t^4 \otimes t^9$$
$$= t^4 \otimes t^9 - t^4 \otimes t^9 = 0$$

Example

Let $R = k[t^4, t^5, t^6]$
$I = t^4R + t^6R$ and $J = t^4R + t^5R$
Then $T(I \otimes_R J) = 0$.
Because $t^a \otimes t^b = t^c \otimes t^d \in I \otimes_R J$, whenever $a + b = c + d$. 
Examples

Example

Let $R = k[t^3, t^4, t^5]$
$I = t^3R + t^4R$ and $J = t^4R + t^5R$
Then $t^3 \otimes t^5 - t^4 \otimes t^4 \in T(M \otimes_R N) \neq 0$

\[
t^5(t^3 \otimes t^5 - t^4 \otimes t^4) = t^8 \otimes t^5 - t^4 \otimes t^9
= t^4 t^4 \otimes t^5 - t^4 \otimes t^9
= t^4 \otimes t^9 - t^4 \otimes t^9 = 0
\]

Example

Let $R = k[t^4, t^5, t^6]$
$I = t^4R + t^6R$ and $J = t^4R + t^5R$
Then $T(I \otimes_R J) = 0$.
Because $t^a \otimes t^b = t^c \otimes t^d \in I \otimes_R J$, whenever $a + b = c + d$. 
Examples

Example

Let $R = k[t^3, t^4, t^5]$

$I = t^3 R + t^4 R$ and $J = t^4 R + t^5 R$

Then $t^3 \otimes t^5 - t^4 \otimes t^4 \in T(M \otimes_R N) \neq 0$

$$t^5(t^3 \otimes t^5 - t^4 \otimes t^4) = t^8 \otimes t^5 - t^4 \otimes t^9$$
$$= t^4 t^4 \otimes t^5 - t^4 \otimes t^9$$
$$= t^4 \otimes t^9 - t^4 \otimes t^9 = 0$$

Example

Let $R = k[t^4, t^5, t^6]$

$I = t^4 R + t^6 R$ and $J = t^4 R + t^5 R$.

Then $T(I \otimes_R J) = 0$.

Because $t^a \otimes t^b = t^c \otimes t^d \in I \otimes_R J$, whenever $a + b = c + d$. 
Examples

Example

Let \( R = k[t^3, t^4, t^5] \)
\( I = t^3 R + t^4 R \) and \( J = t^4 R + t^5 R \)
Then \( t^3 \otimes t^5 - t^4 \otimes t^4 \in T(M \otimes_R N) \neq 0 \)

\[
\begin{align*}
t^5(t^3 \otimes t^5 - t^4 \otimes t^4) &= t^8 \otimes t^5 - t^4 \otimes t^9 \\
&= t^4 t^4 \otimes t^5 - t^4 \otimes t^9 \\
&= t^4 \otimes t^9 - t^4 \otimes t^9 = 0
\end{align*}
\]

Example

Let \( R = k[t^4, t^5, t^6] \)
\( I = t^4 R + t^6 R \) and \( J = t^4 R + t^5 R \).
Then \( T(I \otimes_R J) = 0. \)
Because \( t^a \otimes t^b = t^c \otimes t^d \in I \otimes_R J \), whenever \( a + b = c + d \).
Examples

Example

Let \( R = k[t^3, t^4, t^5] \)
\( I = t^3R + t^4R \) and \( J = t^4R + t^5R \)
Then \( t^3 \otimes t^5 - t^4 \otimes t^4 \in T(M \otimes_R N) \neq 0 \)

\[
t^5(t^3 \otimes t^5 - t^4 \otimes t^4) = t^8 \otimes t^5 - t^4 \otimes t^9 = t^4 t^4 \otimes t^5 - t^4 \otimes t^9 = t^4 \otimes t^9 - t^4 \otimes t^9 = 0
\]

Example

Let \( R = k[t^4, t^5, t^6] \)
\( I = t^4R + t^6R \) and \( J = t^4R + t^5R \).
Then \( T(I \otimes_R J) = 0. \)
Because \( t^a \otimes t^b = t^c \otimes t^d \in I \otimes_R J \), whenever \( a + b = c + d \).
Conjectures

The Huneke-Wiegand Conjecture (HWC)

Let $R$ be a one-dimensional Gorenstein domain. Let $M$ be a finitely generated $R$ module such that $T(M) = T(M \otimes_R \text{Hom}_R(M, R)) = 0$, then $M$ is projective.

- HWC is around 30 years old and is well known.
- Proving HWC would imply the Auslander-Reiten Conjecture is true for Gorenstein domains of any dimension.
- The Auslander-Reiten Conjecture is one of the most sought after results in commutative algebra.
- HWC is known to be true when $R$ is a hyper-surface
- HWC is open when $R$ is complete intersection with $\text{codim}(R) \geq 2$
- HWC is open when $M$ is a 2-generated monomial ideal in a NSGR.
Conjectures

The Huneke-Wiegand Conjecture (HWC)

Let $R$ be a one-dimensional Gorenstein domain. Let $M$ be a finitely generated $R$ module such that $T(M) = T(M \otimes_R \text{Hom}_R(M, R)) = 0$, then $M$ is projective.

- HWC is around 30 years old and is well known.
- Proving HWC would imply the Auslander-Reiten Conjecture is true for Gorenstein domains of any dimension.
- The Auslander-Reiten Conjecture is one of the most sought after results in commutative algebra.
- HWC is known to be true when $R$ is a hyper-surface.
- HWC is open when $R$ is complete intersection with $\text{codim}(R) \geq 2$.
- HWC is open when $M$ is a 2-generated monomial ideal in a NSGR.

Micah Leamer

Identifying torsion in the tensor product...
Conjectures

The Huneke-Wiegand Conjecture (HWC)

Let $R$ be a one-dimensional Gorenstein domain. Let $M$ be a finitely generated $R$ module such that $T(M) = T(M \otimes_R \text{Hom}_R(M, R)) = 0$, then $M$ is projective.

- HWC is around 30 years old and is well known.
- Proving HWC would imply the Auslander-Reiten Conjecture is true for Gorenstein domains of any dimension.
- The Auslander-Reiten Conjecture is one of the most sought after results in commutative algebra.
- HWC is known to be true when $R$ is a hyper-surface.
- HWC is open when $R$ is complete intersection with $\text{codim}(R) \geq 2$.
- HWC is open when $M$ is a 2-generated monomial ideal in a NSGR.
Conjectures

The Huneke-Wiegand Conjecture (HWC)

Let $R$ be a one-dimensional Gorenstein domain. Let $M$ be a finitely generated $R$ module such that $T(M) = T(M \otimes_R \text{Hom}_R(M, R)) = 0$, then $M$ is projective.

- HWC is around 30 years old and is well known.
- Proving HWC would imply the Auslander-Reiten Conjecture is true for Gorenstein domains of any dimension.
- The Auslander-Reiten Conjecture is one of the most sought after results in commutative algebra.
- HWC is known to be true when $R$ is a hyper-surface.
- HWC is open when $R$ is complete intersection with $\text{codim}(R) \geq 2$.
- HWC is open when $M$ is a 2-generated monomial ideal in a NSGR.
Conjectures

The Huneke-Wiegand Conjecture (HWC)

Let $R$ be a one-dimensional Gorenstein domain. Let $M$ be a finitely generated $R$ module such that $T(M) = T(M \otimes_R \text{Hom}_R(M, R)) = 0$, then $M$ is projective.

- HWC is around 30 years old and is well known.
- Proving HWC would imply the Auslander-Reiten Conjecture is true for Gorenstein domains of any dimension.
- The Auslander-Reiten Conjecture is one of the most sought after results in commutative algebra.
- HWC is known to be true when $R$ is a hyper-surface
- HWC is open when $R$ is complete intersection with $\text{codim}(R) \geq 2$
- HWC is open when $M$ is a 2-generated monomial ideal in a NSGR.
Conjectures

The Huneke-Wiegand Conjecture (HWC)

Let $R$ be a one-dimensional Gorenstein domain. Let $M$ be a finitely generated $R$ module such that $T(M) = T(M \otimes_R \text{Hom}_R(M, R)) = 0$, then $M$ is projective.

- HWC is around 30 years old and is well known.
- Proving HWC would imply the Auslander-Reiten Conjecture is true for Gorenstein domains of any dimension.
- The Auslander-Reiten Conjecture is one of the most sought after results in commutative algebra.
- HWC is known to be true when $R$ is a hyper-surface.
  - HWC is open when $R$ is complete intersection with $\text{codim}(R) \geq 2$.
  - HWC is open when $M$ is a 2-generated monomial ideal in a NSGR.
Conjectures

The Huneke-Wiegand Conjecture (HWC)

Let $R$ be a one-dimensional Gorenstein domain. Let $M$ be a finitely generated $R$ module such that $T(M) = T(M \otimes_R \text{Hom}_R(M, R)) = 0$, then $M$ is projective.

- HWC is around 30 years old and is well known.
- Proving HWC would imply the Auslander-Reiten Conjecture is true for Gorenstein domains of any dimension.
- The Auslander-Reiten Conjecture is one of the most sought after results in commutative algebra.
- HWC is known to be true when $R$ is a hyper-surface.
- HWC is open when $R$ is complete intersection with $\text{codim}(R) \geq 2$.
- HWC is open when $M$ is a 2-generated monomial ideal in a NSGR.
Conjectures

The Huneke-Wiegand Conjecture (HWC)

Let $R$ be a one-dimensional Gorenstein domain. Let $M$ be a finitely generated $R$ module such that $T(M) = T(M \otimes_R \text{Hom}_R(M, R)) = 0$, then $M$ is projective.

- HWC is around 30 years old and is well known.
- Proving HWC would imply the Auslander-Reiten Conjecture is true for Gorenstein domains of any dimension.
- The Auslander-Reiten Conjecture is one of the most sought after results in commutative algebra.
- HWC is known to be true when $R$ is a hyper-surface
- HWC is open when $R$ is complete intersection with $\text{codim}(R) \geq 2$
- HWC is open when $M$ is a 2-generated monomial ideal in a NSGR.
Theorem

Let $I$ and $J$ be monomial ideals of $k[\Gamma]$ and $A := \deg(I)$, $B := \deg(J)$. Then $T(I \otimes_{k[\Gamma]} J) = 0 \iff (P \cap Q) + B = (P + B) \cap (Q + B)$ for every splitting $P \cup Q = A$.

Let $K$ be the total quotient ring of $R$. Then $\text{Hom}_R(I, R) \simeq (R :_K I)$ and $\deg((R :_K I)) = \Gamma - \deg(I) = A^*$.

Corollary

Let $I$ be a monomial ideal in $k[\Gamma]$ and $\deg(I) = A$. Then $T(I \otimes_R \text{Hom}(I, R)) \neq 0 \iff \exists$ a splitting $P \cup Q = A$ such that $(P \cap Q) + A^* \neq (P + A^*) \cap (Q + A^*)$.

Theorem (G-S,L)

Let $R = k[\Gamma]$ be a complete intersection numerical semigroup ring and $I$ a 2-generated monomial ideal of $R$. Then $T(I \otimes_R \text{Hom}_R(I, R)) \neq 0$. 
Theorem

Let $I$ and $J$ be monomial ideals of $k[\Gamma]$ and $A := \deg(I)$, $B := \deg(J)$. Then $T(I \otimes_{k[\Gamma]} J) = 0 \iff (P \cap Q) + B = (P + B) \cap (Q + B)$ for every splitting $P \cup Q = A$.

Let $K$ be the total quotient ring of $R$. Then $\text{Hom}_R(I, R) \cong (R :_K I)$ and $\deg((R :_K I)) = \Gamma - \deg(I) = A^*$.

Corollary

Let $I$ be a monomial ideal in $k[\Gamma]$ and $\deg(I) = A$. Then $T(I \otimes_R \text{Hom}(I, R)) \neq 0 \iff \exists$ a splitting $P \cup Q = A$ such that $(P \cap Q) + A^* \neq (P + A^*) \cap (Q + A^*)$.

Theorem (G-S,L)

Let $R = k[\Gamma]$ be a complete intersection numerical semigroup ring and $I$ a 2-generated monomial ideal of $R$. Then $T(I \otimes_R \text{Hom}_R(I, R)) \neq 0$. 
**Theorem**

Let $I$ and $J$ be monomial ideals of $k[\Gamma]$ and $A := \deg(I)$, $B := \deg(J)$. Then $T(I \otimes_{k[\Gamma]} J) = 0 \iff (P \cap Q) + B = (P + B) \cap (Q + B)$ for every splitting $P \cup Q = A$.

Let $K$ be the total quotient ring of $R$. Then $\text{Hom}_R(I, R) \simeq (R :_K I)$ and $\deg((R :_K I)) = \Gamma - \deg(I) = A^*$.

**Corollary**

Let $I$ be a monomial ideal in $k[\Gamma]$ and $\deg(I) = A$. Then $T(I \otimes_R \text{Hom}(I, R)) \neq 0 \iff \exists$ a splitting $P \cup Q = A$ such that $(P \cap Q) + A^* \neq (P + A^*) \cap (Q + A^*)$.

**Theorem (G-S,L)**

Let $R = k[\Gamma]$ be a complete intersection numerical semigroup ring and $I$ a 2-generated monomial ideal of $R$. Then $T(I \otimes_R \text{Hom}_R(I, R)) \neq 0$. 

Micah Leamer

Identifying torsion in the tensor product...
Theorem

Let $I$ and $J$ be monomial ideals of $k[\Gamma]$ and $A := \deg(I)$, $B := \deg(J)$. Then $T(I \otimes_{k[\Gamma]} J) = 0 \iff (P \cap Q) + B = (P + B) \cap (Q + B)$ for every splitting $P \cup Q = A$.

Let $K$ be the total quotient ring of $R$. Then $\text{Hom}_R(I, R) \simeq (R : K I)$ and $\deg((R : K I)) = \Gamma - \deg(I) = A^*$.

Corollary

Let $I$ be a monomial ideal in $k[\Gamma]$ and $\deg(I) = A$. Then $T(I \otimes_R \text{Hom}(I, R)) \neq 0 \iff \exists$ a splitting $P \cup Q = A$ such that $(P \cap Q) + A^* \neq (P + A^*) \cap (Q + A^*)$.

Theorem (G-S,L)

Let $R = k[\Gamma]$ be a complete intersection numerical semigroup ring and $I$ a 2-generated monomial ideal of $R$. Then $T(I \otimes_R \text{Hom}_R(I, R)) \neq 0$. 
**Theorem**

Let $I$ and $J$ be mononomial ideals of $k[\Gamma]$ and $A := \deg(I)$, $B := \deg(J)$. Then $T(I \otimes_{k[\Gamma]} J) = 0 \iff (P \cap Q) + B = (P + B) \cap (Q + B)$ for every splitting $P \cup Q = A$.

Let $K$ be the total quotient ring of $R$. Then $\text{Hom}_R(I, R) \cong (R :_K I)$ and $\deg((R :_K I)) = \Gamma - \deg(I) = A^*$.

**Corollary**

Let $I$ be a monomial ideal in $k[\Gamma]$ and $\deg(I) = A$. Then $T(I \otimes_R \text{Hom}(I, R)) \neq 0 \iff \exists$ a splitting $P \cup Q = A$ such that $(P \cap Q) + A^* \neq (P + A^*) \cap (Q + A^*)$.

**Theorem (G-S,L)**

Let $R = k[\Gamma]$ be a complete intersection numerical semigroup ring and $I$ a 2-generated monomial ideal of $R$. Then $T(I \otimes_R \text{Hom}_R(I, R)) \neq 0$. 

Micah Leamer
Identifying torsion in the tensor product... 7 / 10
Theorem

Let $I$ and $J$ be monomial ideals of $k[\Gamma]$ and $A := \deg(I)$, $B := \deg(J)$. Then
\[ T(I \otimes_{k[\Gamma]} J) = 0 \iff (P \cap Q) + B = (P + B) \cap (Q + B) \text{ for every splitting } P \cup Q = A. \]

Let $K$ be the total quotient ring of $R$. Then
\[ \text{Hom}_R(I, R) \simeq (R :_K I) \text{ and } \deg((R :_K I)) = \Gamma - \deg(I) = A^*. \]

Corollary

Let $I$ be a monomial ideal in $k[\Gamma]$ and $\deg(I) = A$. Then
\[ T(I \otimes_R \text{Hom}(I, R)) \neq 0 \iff \exists \text{ a splitting } P \cup Q = A \text{ such that } (P \cap Q) + A^* \neq (P + A^*) \cap (Q + A^*). \]

Theorem (G-S,L)

Let $R = k[\Gamma]$ be a complete intersection numerical semigroup ring and $I$ a 2-generated monomial ideal of $R$. Then
\[ T(I \otimes_R \text{Hom}_R(I, R)) \neq 0. \]
Theorem

Let $I$ and $J$ be monomial ideals of $k[\Gamma]$ and $A := \deg(I)$, $B := \deg(J)$. Then $T(I \otimes_{k[\Gamma]} J) = 0 \iff (P \cap Q) + B = (P + B) \cap (Q + B)$ for every splitting $P \cup Q = A$.

Let $K$ be the total quotient ring of $R$. Then $\text{Hom}_R(I, R) \simeq (R :_K I)$ and $\deg((R :_K I)) = \Gamma - \deg(I) = A^*$.

Corollary

Let $I$ be a monomial ideal in $k[\Gamma]$ and $\deg(I) = A$. Then $T(I \otimes_R \text{Hom}(I, R)) \neq 0 \iff \exists$ a splitting $P \cup Q = A$ such that $(P \cap Q) + A^* \neq (P + A^*) \cap (Q + A^*)$.

Theorem (G-S,L)

Let $R = k[\Gamma]$ be a complete intersection numerical semigroup ring and $I$ a 2-generated monomial ideal of $R$. Then $T(I \otimes_R \text{Hom}_R(I, R)) \neq 0$. 
**Theorem**

Let $I$ and $J$ be monomial ideals of $k[\Gamma]$ and $A := \deg(I)$, $B := \deg(J)$. Then $T(I \otimes_{k[\Gamma]} J) = 0 \iff (P \cap Q) + B = (P + B) \cap (Q + B)$ for every splitting $P \cup Q = A$.

Let $K$ be the total quotient ring of $R$. Then $\text{Hom}_R(I, R) \cong (R :_K I)$ and $\deg((R :_K I)) = \Gamma - \deg(I) = A^*$.  

**Corollary**

Let $I$ be a monomial ideal in $k[\Gamma]$ and $\deg(I) = A$. Then $T(I \otimes_R \text{Hom}(I, R)) \neq 0 \iff \exists$ a splitting $P \cup Q = A$ such that $(P \cap Q) + A^* \neq (P + A^*) \cap (Q + A^*)$.

**Theorem (G-S,L)**

Let $R = k[\Gamma]$ be a complete intersection numerical semigroup ring and $I$ a 2-generated monomial ideal of $R$. Then $T(I \otimes_R \text{Hom}_R(I, R)) \neq 0$. 

Micah Leamer
Identifying torsion in the tensor product...
Theorem

Let $I$ and $J$ be monomial ideals of $k[\Gamma]$ and $A := \deg(I)$, $B := \deg(J)$. Then $T(I \otimes_{k[\Gamma]} J) = 0 \iff (P \cap Q) + B = (P + B) \cap (Q + B)$ for every splitting $P \cup Q = A$.

Let $K$ be the total quotient ring of $R$. Then $\text{Hom}_R(I, R) \simeq (R :_K I)$ and $\deg((R :_K I)) = \Gamma - \deg(I) = A^*$.

Corollary

Let $I$ be a monomial ideal in $k[\Gamma]$ and $\deg(I) = A$. Then $T(I \otimes_R \text{Hom}(I, R)) \neq 0 \iff \exists$ a splitting $P \cup Q = A$ such that $(P \cap Q) + A^* \neq (P + A^*) \cap (Q + A^*)$.

Theorem (G-S,L)

Let $R = k[\Gamma]$ be a complete intersection numerical semigroup ring and $I$ a 2-generated monomial ideal of $R$. Then $T(I \otimes_R \text{Hom}_R(I, R)) \neq 0$. 
Theorem

Let $I$ and $J$ be monomial ideals of $k[\Gamma]$ and $A := \deg(I)$, $B := \deg(J)$. Then $T(I \otimes_{k[\Gamma]} J) = 0 \iff (P \cap Q) + B = (P + B) \cap (Q + B)$ for every splitting $P \cup Q = A$.

Let $K$ be the total quotient ring of $R$. Then $\text{Hom}_R(I, R) \cong (R :_K I)$ and $\deg((R :_K I)) = \Gamma - \deg(I) = A^*$.

Corollary

Let $I$ be a monomial ideal in $k[\Gamma]$ and $\deg(I) = A$. Then $T(I \otimes_R \text{Hom}(I, R)) \neq 0 \iff \exists$ a splitting $P \cup Q = A$ such that $(P \cap Q) + A^* \neq (P + A^*) \cap (Q + A^*)$.

Theorem (G-S,L)

Let $R = k[\Gamma]$ be a complete intersection numerical semigroup ring and $I$ a 2-generated monomial ideal of $R$. Then $T(I \otimes_R \text{Hom}_R(I, R)) \neq 0$. 
2-generated ideals

Remark

If $A = (a_1, a_2)$ is two generated. Then there is only one non-trivial splitting $P = a_1 + \Gamma$ and $Q = a_2 + \Gamma$.

Hence $A$ is Huneke-Wiegand $\iff (a_1) \cap (a_2) + A^* \neq (a_1 + A^*) \cap (a_2 + A^*)$

Subtracting $a_1 + a_2$ from both sides we get $A^* + A^* \neq (A + A)^*$.

$A^*$ corresponds to pairs in $\Gamma$ differing by $s = a_2 - a_1$. $(A + A)^*$ corresponds to triples in $\Gamma$ differing by $s$.

Lemma

Let $I = (t^a, t^{a+s})$ be an ideal in a numerical semigroup ring $R = k[\Gamma]$. Then the following numbers are equal.

- $\lambda(T(I \otimes_R \text{Hom}_R(I, R)))$;
- $|(A + A)^* \setminus (A^* + A^*)|$;
- The number of sets of the form $\{x, x + s, x + 2s\} \subset \Gamma$ that do not factor as a sum of sets $\{y, y + s\} + \{z, z + s\}$ also in $\Gamma$. 
2-generated ideals

Remark

If $A = (a_1, a_2)$ is two generated. Then there is only one non-trivial splitting $P = a_1 + \Gamma$ and $Q = a_2 + \Gamma$. Hence $A$ is Huneke-Wiegand $\iff (a_1) \cap (a_2) + A^* \neq (a_1 + A^*) \cap (a_2 + A^*)$.

Subtracting $a_1 + a_2$ from both sides we get $A^* + A^* \neq (A + A)^*$. $A^*$ corresponds to pairs in $\Gamma$ differing by $s = a_2 - a_1$. $(A + A)^*$ corresponds to triples in $\Gamma$ differing by $s$.

Lemma

Let $I = (t^a, t^{a+s})$ be an ideal in a numerical semigroup ring $R = k[\Gamma]$. Then the following numbers are equal.

- $\lambda(T(I \otimes_R \text{Hom}_R(I, R)))$;
- $|(A + A)^* \setminus (A^* + A^*)|$;
- The number of sets of the form $\{x, x + s, x + 2s\} \subset \Gamma$ that do not factor as a sum of sets $\{y, y + s\} + \{z, z + s\}$ also in $\Gamma$. 

Micah Leamer
Identifying torsion in the tensor product...
2-generated ideals

Remark

If $A = (a_1, a_2)$ is two generated. Then there is only one non-trivial splitting $P = a_1 + \Gamma$ and $Q = a_2 + \Gamma$.
Hence $A$ is Huneke-Wiegand $\iff (a_1) \cap (a_2) + A^* \neq (a_1 + A^*) \cap (a_2 + A^*)$
Subtracting $a_1 + a_2$ from both sides we get $A^* + A^* \neq (A + A)^*$.

Lemma

Let $I = (t^a, t^{a+s})$ be an ideal in a numerical semigroup ring $R = k[\Gamma]$.
Then the following numbers are equal.

- $\lambda(T(I \otimes_R \text{Hom}_R(I, R)))$;
- $|(A + A)^* \setminus (A^* + A^*)|$;
- The number of sets of the form $\{x, x + s, x + 2s\} \subset \Gamma$ that do not factor as a sum of sets $\{y, y + s\} + \{z, z + s\}$ also in $\Gamma$. 
### Remark

If $A = (a_1, a_2)$ is two generated. Then there is only one non-trivial splitting $P = a_1 + \Gamma$ and $Q = a_2 + \Gamma$.

Hence $A$ is Huneke-Wiegand $\iff (a_1) \cap (a_2) + A^* \neq (a_1 + A^*) \cap (a_2 + A^*)$.

Subtracting $a_1 + a_2$ from both sides we get $A^* + A^* \neq (A + A)^*$.

$A^*$ corresponds to pairs in $\Gamma$ differing by $s = a_2 - a_1$. $(A + A)^*$ corresponds to triples in $\Gamma$ differing by $s$.

### Lemma

Let $I = (t^a, t^{a+s})$ be an ideal in a numerical semigroup ring $R = k[\Gamma]$. Then the following numbers are equal.

- $\lambda(\bigwedge(I \otimes_R \text{Hom}_R(I, R)))$;
- $|(A + A)^* \setminus (A^* + A^*)|$;
- The number of sets of the form $\{x, x + s, x + 2s\} \subset \Gamma$ that do not factor as a sum of sets $\{y, y + s\} + \{z, z + s\}$ also in $\Gamma$. 

Micah Leamer
Identifying torsion in the tensor product... 8 / 10
2-generated ideals

**Remark**

If $A = (a_1, a_2)$ is two generated. Then there is only one non-trivial splitting $P = a_1 + \Gamma$ and $Q = a_2 + \Gamma$.

Hence $A$ is Huneke-Wiegand $\iff (a_1) \cap (a_2) + A^* \neq (a_1 + A^*) \cap (a_2 + A^*)$

Subtracting $a_1 + a_2$ from both sides we get $A^* + A^* \neq (A + A)^*$.

$A^*$ corresponds to pairs in $\Gamma$ differing by $s = a_2 - a_1$. $(A + A)^*$ corresponds to triples in $\Gamma$ differing by $s$.

**Lemma**

Let $I = (t^a, t^{a+s})$ be an ideal in a numerical semigroup ring $R = k[\Gamma]$. Then the following numbers are equal.

- $\lambda(T(I \otimes_R \text{Hom}_R(I, R)))$;
- $|(A + A)^* \setminus (A^* + A^*)|$;
- The number of sets of the form $\{x, x + s, x + 2s\} \subset \Gamma$ that do not factor as a sum of sets $\{y, y + s\} + \{z, z + s\}$ also in $\Gamma$. 
2-generated ideals

Remark

If \( A = (a_1, a_2) \) is two generated. Then there is only one non-trivial splitting \( P = a_1 + \Gamma \) and \( Q = a_2 + \Gamma \).

Hence \( A \) is Huneke-Wiegand \( \iff (a_1) \cap (a_2) + A^* \neq (a_1 + A^*) \cap (a_2 + A^*) \)

Subtracting \( a_1 + a_2 \) from both sides we get \( A^* + A^* \neq (A + A)^* \).

\( A^* \) corresponds to pairs in \( \Gamma \) differing by \( s = a_2 - a_1 \).

\( (A + A)^* \) corresponds to triples in \( \Gamma \) differing by \( s \).

Lemma

Let \( I = (t^a, t^{a+s}) \) be an ideal in a numerical semigroup ring \( R = k[\Gamma] \).

Then the following numbers are equal.

- \( \lambda(T(I \otimes_R \text{Hom}_R(I, R))) \);
- \( |(A + A)^* \setminus (A^* + A^*)| \);
- The number of sets of the form \( \{x, x + s, x + 2s\} \subset \Gamma \) that do not factor as a sum of sets \( \{y, y + s\} + \{z, z + s\} \) also in \( \Gamma \).
2-generated ideals

Remark

If $A = (a_1, a_2)$ is two generated. Then there is only one non-trivial splitting $P = a_1 + \Gamma$ and $Q = a_2 + \Gamma$.

Hence $A$ is Huneke-Wiegand $\iff (a_1) \cap (a_2) + A^* \neq (a_1 + A^*) \cap (a_2 + A^*)$

Subtracting $a_1 + a_2$ from both sides we get $A^* + A^* \neq (A + A)^*$.

$A^*$ corresponds to pairs in $\Gamma$ differing by $s = a_2 - a_1$.

$(A + A)^*$ corresponds to triples in $\Gamma$ differing by $s$.

Lemma

Let $I = (t^a, t^{a+s})$ be an ideal in a numerical semigroup ring $R = k[\Gamma]$. Then the following numbers are equal.

- $\lambda(T(I \otimes_R \text{Hom}_R(I, R)))$;
- $|(A + A)^* \setminus (A^* + A^*)|$;
- The number of sets of the form $\{x, x + s, x + 2s\} \subset \Gamma$ that do not factor as a sum of sets $\{y, y + s\} + \{z, z + s\}$ also in $\Gamma$. 
2-generated ideals

Remark

If \( A = (a_1, a_2) \) is two generated. Then there is only one non-trivial splitting \( P = a_1 + \Gamma \) and \( Q = a_2 + \Gamma \).

Hence \( A \) is Huneke-Wiegand \( \iff (a_1) \cap (a_2) + A^* \neq (a_1 + A^*) \cap (a_2 + A^*) \)

Subtracting \( a_1 + a_2 \) from both sides we get \( A^* + A^* \neq (A + A)^* \).

\( A^* \) corresponds to pairs in \( \Gamma \) differing by \( s = a_2 - a_1 \). \( (A + A)^* \) corresponds to triples in \( \Gamma \) differing by \( s \).

Lemma

Let \( I = (t^a, t^{a+s}) \) be an ideal in a numerical semigroup ring \( R = k[\Gamma] \).

Then the following numbers are equal.

- \( \lambda(T(I \otimes_R \text{Hom}_R(I, R))) \);
- \( |(A + A)^* \setminus (A^* + A^*)| \);
- The number of sets of the form \( \{x, x + s, x + 2s\} \subset \Gamma \) that do not factor as a sum of sets \( \{y, y + s\} + \{z, z + s\} \) also in \( \Gamma \).
2-generated ideals

Remark

If \( A = (a_1, a_2) \) is two generated. Then there is only one non-trivial splitting \( P = a_1 + \Gamma \) and \( Q = a_2 + \Gamma \).

Hence \( A \) is Huneke-Wiegand \( \iff (a_1) \cap (a_2) + A^* \neq (a_1 + A^*) \cap (a_2 + A^*) \).

Subtracting \( a_1 + a_2 \) from both sides we get \( A^* + A^* \neq (A + A)^* \).

\( A^* \) corresponds to pairs in \( \Gamma \) differing by \( s = a_2 - a_1 \).
\( (A + A)^* \) corresponds to triples in \( \Gamma \) differing by \( s \).

Lemma

Let \( I = (t^a, t^{a+s}) \) be an ideal in a numerical semigroup ring \( R = k[\Gamma] \).

Then the following numbers are equal.

- \( \lambda(T(I \otimes_R \text{Hom}_R(I, R))) \);
- \( |(A + A)^* \setminus (A^* + A^*)| \);
- The number of sets of the form \( \{x, x + s, x + 2s\} \subset \Gamma \) that do not factor as a sum of sets \( \{y, y + s\} + \{z, z + s\} \) also in \( \Gamma \).
2-generated ideals

Remark
If $A = (a_1, a_2)$ is two generated. Then there is only one non-trivial splitting $P = a_1 + \Gamma$ and $Q = a_2 + \Gamma$.
Hence $A$ is Huneke-Wiegand $\iff (a_1) \cap (a_2) + A^* \neq (a_1 + A^*) \cap (a_2 + A^*)$.
Subtracting $a_1 + a_2$ from both sides we get $A^* + A^* \neq (A + A)^*$.
$A^*$ corresponds to pairs in $\Gamma$ differing by $s = a_2 - a_1$. $(A + A)^*$
corresponds to triples in $\Gamma$ differing by $s$.

Lemma
Let $I = (t^a, t^{a+s})$ be an ideal in a numerical semigroup ring $R = k[\Gamma]$. Then the following numbers are equal.

- $\lambda( T(I \otimes_R \text{Hom}_R(I, R)))$;
- $|(A + A)^* \setminus (A^* + A^*)|$;
- The number of sets of the form $\{x, x + s, x + 2s\} \subset \Gamma$ that do not factor as a sum of sets $\{y, y + s\} + \{z, z + s\}$ also in $\Gamma$. 
Monoids of arithmetic sets

Definition

Given a numerical semigroup $\Gamma$ and an integer $s \in \mathbb{N} \setminus \Gamma$, let $S^s_\Gamma$ denote the monoid containing $\{0\}$ along with all arithmetic sets $\{x, x + s, \ldots, x + ns\} \subseteq \Gamma$. Where the operation on $S^s_\Gamma$ is setwise addition.

- The HWC for 2-generated monomial ideals over $k[\Gamma]$ is equivalent to the property that $S^s_\Gamma$ has an atom $\{x, x + s, x + 2s\}$ of length 2 for all $s \in \mathbb{N} \setminus \Gamma$.
- P. G-S. and I show that this second property is closed under gluings $\Gamma = a_1\Gamma_1 + a_2\Gamma_2$.
- Since then another group wrote a paper studying factorization invariants of these monoids and dubbing them Leamer Monoids.
- They conjecture that $\Delta(S^s_\Gamma)$ is always of the form $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. 
Monoids of arithmetic sets

**Definition**

Given a numerical semigroup $\Gamma$ and an integer $s \in \mathbb{N} \setminus \Gamma$, let $S_{\Gamma}^s$ denote the monoid containing $\{0\}$ along with all arithmetic sets $\{x, x + s, \ldots, x + ns\} \subseteq \Gamma$. Where the operation on $S_{\Gamma}^s$ is setwise addition.

- The HWC for 2-generated monomial ideals over $k[\Gamma]$ is equivalent to the property that $S_{\Gamma}^s$ has an atom $\{x, x + s, x + 2s\}$ of length 2 for all $s \in \mathbb{N} \setminus \Gamma$.

- P. G-S. and I show that this second property is closed under gluings $\Gamma = a_1\Gamma_1 + a_2\Gamma_2$.

- Since then another group wrote a paper studying factorization invariants of these monoids and dubbing them Leamer Monoids.

- They conjecture that $\Delta(S_{\Gamma}^s)$ is always of the form $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.
Monoids of arithmetic sets

Definition
Given a numerical semigroup \( \Gamma \) and an integer \( s \in \mathbb{N} \setminus \Gamma \), let \( S^s_\Gamma \) denote the monoid containing \( \{0\} \) along with all arithmetic sets \( \{x, x + s, \ldots, x + ns\} \subseteq \Gamma \). Where the operation on \( S^s_\Gamma \) is setwise addition.

- The HWC for 2-generated monomial ideals over \( k[\Gamma] \) is equivalent to the property that \( S^s_\Gamma \) has an atom \( \{x, x + s, x + 2s\} \) of length 2 for all \( s \in \mathbb{N} \setminus \Gamma \).
- P. G-S. and I show that this second property is closed under gluings \( \Gamma = a_1 \Gamma_1 + a_2 \Gamma_2 \).
- Since then another group wrote a paper studying factorization invariants of these monoids and dubbing them Leamer Monoids.
- They conjecture that \( \Delta(S^s_\Gamma) \) is always of the form \( \{1, 2, \ldots, n\} \) for some \( n \in \mathbb{N} \).
Monoids of arithmetic sets

**Definition**

Given a numerical semigroup $\Gamma$ and an integer $s \in \mathbb{N} \setminus \Gamma$, let $S_{\Gamma}^s$ denote the monoid containing $\{0\}$ along with all arithmetic sets $\{x, x+s, \ldots, x+ns\} \subseteq \Gamma$. Where the operation on $S_{\Gamma}^s$ is setwise addition.

- The HWC for 2-generated monomial ideals over $k[\Gamma]$ is equivalent to the property that $S_{\Gamma}^s$ has an atom $\{x, x+s, x+2s\}$ of length 2 for all $s \in \mathbb{N} \setminus \Gamma$
- P. G-S. and I show that this second property is closed under gluings $\Gamma = a_1 \Gamma_1 + a_2 \Gamma_2$.
- Since then another group wrote a paper studying factorization invariants of these monoids and dubbing them Leamer Monoids.
- They conjecture that $\Delta(S_{\Gamma}^s)$ is always of the form $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.
Monoids of arithmetic sets

Definition

Given a numerical semigroup $\Gamma$ and an integer $s \in \mathbb{N} \setminus \Gamma$, let $S^s_{\Gamma}$ denote the monoid containing $\{0\}$ along with all arithmetic sets $\{x, x + s, \ldots, x + ns\} \subseteq \Gamma$. Where the operation on $S^s_{\Gamma}$ is setwise addition.

- The HWC for 2-generated monomial ideals over $k[\Gamma]$ is equivalent to the property that $S^s_{\Gamma}$ has an atom $\{x, x + s, x + 2s\}$ of length 2 for all $s \in \mathbb{N} \setminus \Gamma$.
- P. G-S. and I show that this second property is closed under gluings $\Gamma = a_1 \Gamma_1 + a_2 \Gamma_2$.
- Since then another group wrote a paper studying factorization invariants of these monoids and dubbing them Leamer Monoids.
- They conjecture that $\Delta(S^s_{\Gamma})$ is always of the form $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.  

Micah Leamer

Identifying torsion in the tensor product...
Identifying torsion in the tensor product...

Micah Leamer

Thank You!

micahleamer@gmail.com