On the Abhyankar-Moh inequality

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In this talk we present some results of


We study semigroups of integers appearing in connection with the Abhyankar-Moh inequality which is the main tool in proving the famous embedding line theorem.
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Since the Abhyankar-Moh inequality can be stated in terms of semigroups associated with the branch at infinity of a plane algebraic curve it is natural to consider the semigroups for which such an inequality holds.
A subset $G$ of $\mathbb{N}$ is a **semigroup** if it contains $0$ and it is closed under addition.

Let $G$ be a nonzero semigroup and let $n \in G$, $n > 0$. There exists a unique sequence $(v_0, \ldots, v_h)$ such that

- $v_0 = n$,
- $v_k = \min(G \setminus v_0\mathbb{N} + \cdots + v_{k-1}\mathbb{N})$ for $1 \leq k \leq h$ and
- $G = v_0\mathbb{N} + \cdots + v_h\mathbb{N}$.

We call the sequence $(v_0, \ldots, v_h)$ the **$n$-minimal system of generators of $G$**.

If $n = \min(G \setminus \{0\})$ then we say that $(v_0, \ldots, v_h)$ is the **minimal system of generators of $G$**.
Characteristic sequences

A sequence of positive integers \((\overline{b}_0, \ldots, \overline{b}_h)\) will be called a \textit{characteristic sequence} if satisfies

- Set \(e_k = \gcd(\overline{b}_0, \ldots, \overline{b}_k)\) for \(0 \leq k \leq h\). Then \(e_k < e_{k-1}\) for \(1 \leq k \leq h\) and \(e_h = 1\).

- \(e_{k-1} \overline{b}_k < e_k \overline{b}_{k+1}\) for \(1 \leq k \leq h - 1\).

Put \(n_k = \frac{e_{k-1}}{e_k}\) for \(1 \leq k \leq h\). Therefore \(n_k > 1\) for \(1 \leq k \leq h\) and \(n_h = e_{h-1}\).

**Examples**

If \(h = 0\) there is exactly one characteristic sequence \((\overline{b}_0) = (1)\).

If \(h = 1\) then the sequence \((\overline{b}_0, \overline{b}_1)\) is a characteristic sequence if and only if \(\gcd(\overline{b}_0, \overline{b}_1) = 1\).
Characteristic sequences

Proposition

Let $G = \overline{b_0} \mathbb{N} + \cdots + \overline{b_h} \mathbb{N}$, where $(\overline{b_0}, \ldots, \overline{b_h})$ is a characteristic sequence. Then

1. the sequence $(\overline{b_0}, \ldots, \overline{b_h})$ is the $\overline{b_0}$-minimal system of generators of the semigroup $G$.
2. $\min(G \setminus \{0\}) = \min(\overline{b_0}, \overline{b_1})$.
3. The minimal system of generators of $G$ is $(\overline{b_0}, \ldots, \overline{b_h})$ if $\overline{b_0} < \overline{b_1}$, $(\overline{b_1}, \overline{b_0}, \overline{b_2}, \ldots, \overline{b_h})$ if $\overline{b_0} > \overline{b_1}$ and $\overline{b_0} \not\equiv 0 \pmod{\overline{b_1}}$ and $(\overline{b_1}, \overline{b_2}, \ldots, \overline{b_h})$ if $\overline{b_0} \equiv 0 \pmod{\overline{b_1}}$.
4. Let $c = \sum_{k=1}^{h} (n_k - 1) \overline{b_k} - \overline{b_0} + 1$. Then $c$ is the conductor of $G$, that is the smallest element of $G$ such that all integers bigger than or equal to it are in $G$. 
A semigroup $G \subseteq \mathbb{N}$ will be called an \textit{Abhyankar-Moh semigroup of degree $n > 1$} if it is generated by a characteristic sequence $(\overline{b}_0 = n, \overline{b}_1, \ldots, \overline{b}_h)$, satisfying the Abhyankar-Moh inequality

\[(\text{AM}) \quad e_{h-1} \overline{b}_h < n^2.\]
Let $G \subseteq \mathbb{N}$ be a semigroup generated by a characteristic sequence, which minimal system of generators is $(\beta_0, \ldots, \beta_g)$.

**Proposition**

$G$ is an Abhyankar-Moh semigroup of degree $n > 1$ if and only if $\epsilon_{g-1} \beta_g < n^2$ and $n = \beta_1$ or $n = l \beta_0$, where $l$ is an integer such that $1 < l < \beta_1 / \beta_0$ and $\epsilon_{g-1} = \gcd(\beta_0, \ldots, \beta_{g-1})$. 
Theorem (Barrolleta-GB-Płoski)

Let $G$ be an Abhyankar-Moh semigroup of degree $n > 1$ and let $c$ be the conductor of $G$. Then $c \leq (n-1)(n-2)$.

Moreover if $G$ is generated by the characteristic sequence $(\overline{b}_0 = n, \overline{b}_1, \ldots, \overline{b}_h)$ satisfying (AM) then $c = (n-1)(n-2)$ if and only if $\overline{b}_k = \frac{n^2}{e_{k-1}} - e_k$ for $1 \leq k \leq h$, where $e_k = \gcd(\overline{b}_0, \ldots, \overline{b}_k)$. 
Abhyankar-Moh semigroups

Let $n > 1$ be an integer. A sequence of integers $(e_0, \ldots, e_h)$ will be called a sequence of divisors of $n$ if $e_k$ divides $e_{k-1}$ for $1 \leq k \leq h$ and $n = e_0 > e_1 > \cdots > e_{h-1} > e_h = 1$.

Lemma

If $(e_0, \ldots, e_h)$ is a sequence of divisors of $n > 1$ then the sequence

$$\left( n, n - e_1, \frac{n^2}{e_1} - e_2, \ldots, \frac{n^2}{e_{k-1}} - e_k, \ldots, \frac{n^2}{e_{h-1}} - 1 \right)$$

(2.1)

is a characteristic sequence satisfying the Abhyankar-Moh inequality (AM).

Let $G(e_0, \ldots, e_h)$ be the semigroup generated by the sequence (2.1).
Abhyankar-Moh semigroups

Proposition (Barrolleta-GB-Płoski)

A semigroup $G \subseteq \mathbb{N}$ is an Abhyankar-Moh semigroup of degree $n > 1$ with $c = (n - 1)(n - 2)$ if and only if $G = G(e_0, \ldots, e_h)$ where $(e_0, e_1, \ldots, e_h)$ is a sequence of divisors of $n$. 
Corollary

Let $G$ be an Abhyankar-Moh semigroup of degree $n > 1$ with $c = (n - 1)(n - 2)$ and let $n' = \min(G \setminus \{0\})$. Then $n - n'$ divides $n$. 
Abhyankar-Moh semigroups

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Corollary

Let $G$ be an Abhyankar-Moh semigroup of degree $n > 1$ with $c = (n - 1)(n - 2)$ and let $(\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g)$ be the minimal system of generators of the semigroup $G$. Then $n = \overline{\beta}_1$ or $n = 2\overline{\beta}_0$.

If $n = \overline{\beta}_1$ then $G = G(n, \epsilon_1, \ldots, \epsilon_g)$.

If $n = 2\overline{\beta}_0$ then $G = G(n, \epsilon_0, \ldots, \epsilon_g)$. 
Let $K$ be an algebraically closed field of arbitrary characteristic.

A projective plane curve $C$ defined over $K$ has one branch at infinity if there is a line (line at infinity) intersecting $C$ in only one point $O$, and $C$ has only one branch centered at this point. In what follows we denote by $n$ the degree of $C$, by $n'$ the multiplicity of $C$ at $O$ and we put $d := \gcd(n, n')$.

We call $C$ permissible if $d \not\equiv 0 \pmod{\text{char } K}$.
Plane curves with one branch at infinity

Theorem (Abhyankar-Moh inequality)

Assume that $C$ is a permissible curve of degree $n > 1$. Then the semigroup $G_O$ of the unique branch at infinity of $C$ is an Abhyankar-Moh semigroup of degree $n$.


Theorem (Abhyankar-Moh Embedding Line Theorem)

Assume that $C$ is a rational projective irreducible curve of degree $n > 1$ with one branch at infinity and such that the center of the branch at infinity $O$ is the unique singular point of $C$. Suppose that $C$ is permissible and let $n'$ be the multiplicity of $C$ at $O$. Then $n - n'$ divides $n$. 

Proof [Barrolleta-Gb-Płoski]

By Theorem (Abhyankar-Moh inequality) the semigroup $G_O$ of the branch at infinity is an Abhyankar-Moh semigroup of degree $n$. Let $c$ be the conductor of the semigroup $G_O$. Using the Noether formula for the genus of projective plane curve we get $c = (n - 1)(n - 2)$. Then the theorem follows from Corollary.
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Response to Teissier’s question on maximal contact

Let $\overline{\beta}_0 = n', \overline{\beta}_1, \cdots$ be the minimal system of generators of the semigroup $G_O$.

From the first characterization of A-M semigroups it follows that the line at infinity $L$ has maximal contact with $C$, that is intersects $C$ with multiplicity $\overline{\beta}_1$ if and only if $n \not\equiv 0 \pmod{n'}$. What happens if $n \equiv 0 \pmod{n'}$? Using the main result on the approximate roots in [GB-Płoski] one proves that if $n \equiv 0 \pmod{n'}$ then there is an irreducible curve $C'$ of degree $n/n'$ intersecting $C$ with multiplicity $\overline{\beta}_1$. In particular, if $C$ is rational then by last Corollary we get $n/n' = 2$ if $n \equiv 0 \pmod{n'}$ and $C'$ is a nonsingular curve of degree 2.
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- In particular, if $C$ is rational then by last Corollary we get $n/n' = 2$ (if $n \equiv 0 \pmod{n'}$) and $C'$ is a nonsingular curve of degree 2.
An affine curve $\Gamma \subseteq \mathbb{K}^2$ is a *coordinate line* if there is a polynomial automorphism $F : \mathbb{K}^2 \rightarrow \mathbb{K}^2$ such that $F(\Gamma) = \{0\} \times \mathbb{K}$. 

Theorem (GB-Gwoździewicz-Płoski)

Let $G \subseteq \mathbb{N}$ be a semigroup with conductor $c$. Then the following two conditions are equivalent:

(I) $G \in {\mathbb{A}}_{\mathbb{Q}}$ and $c = (n-1)(n-2)$,

(II) there exists a coordinate line $\Gamma \subseteq \mathbb{K}^2$ (char $\mathbb{K}$ is arbitrary) with a unique branch at infinity $\gamma$ such that $G(\gamma) = G$. 

Geometrically characterization of Abhyankar-Moh semigroups with maximum conductor
Introduction Abhyankar-Moh semigroups Plane curves with one branch at infinity

Geometrically characterization of Abhyankar-Moh semigroups with maximum conductor

An affine curve $\Gamma \subseteq K^2$ is a *coordinate line* if there is a polynomial automorphism $F : K^2 \rightarrow K^2$ such that $F(\Gamma) = \{0\} \times K$.

**Theorem (GB-Gwoździewicz-Płoski)**

*Let $G \subseteq N$ be a semigroup with conductor $c$. Then the following two conditions are equivalent:*

1. $G \in AM(n)$ and $c = (n - 1)(n - 2)$,
2. there exists a coordinate line $\Gamma \subseteq K^2$ (char $K$ is arbitrary!) with a unique branch at infinity $\gamma$ such that $G(\gamma) = G$. 