On free resolutions of some semigroup rings

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Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical semigroup, i.e., n_i are positive integers with greatest common divisor 1, and $S = \{\sum_{i=1}^{k} a_i n_i \mid a_i \geq 0\}$. Let $PF(S) = \{n \in \mathbb{Z} : n + s \in S, s \in S, s > 0\} \setminus S$. The elements in $PF(S)$ are called the pseudofrobenius numbers of S. Since S is a numerical semigroup, $\mathbb{N} \setminus S$ is finite.

The largest integer $g(S) \notin S$ belongs to $PF(S)$ and is called the Frobenius number of S. If $PF(S) = \{g(S)\}$, S is called symmetric, since then, for each $n \in \mathbb{Z}$, exactly one of n and $g(S) - n$ lies in S. If $PF(S) = \{g(S)/2, g(S)\}$, S is called pseudosymmetric.

Let K be a field and $K[S] = K[t^{n_1}, \dots, t^{n_k}]$ be the semigroup ring of S, then

$$K[S] \simeq K[x_1, \dots, x_k]/I_S$$

where I_S is the kernel of the surjection

$$K[x_1, \dots, x_k] \xrightarrow{\phi_0} K[t],$$

where $x_i \mapsto t^{n_i}$. If $\deg(x_i) = n_i$, this map is homogeneous of degree 0. We will in the sequel denote $K[x_1, \dots, x_k]$ by A.

If you have a graded ring $R = A/I$ (I generated by homogeneous elements), one is interested in the A-resolution of R.

Example If $R = k[t^a, t^b]$, $(a, b) = 1$, then the map $A \longrightarrow R$ has kernel $(x_1^b - x_2^a)$ and

$$0 \longrightarrow A \xrightarrow{x_1^b - x_2^a} A \longrightarrow R \longrightarrow 0$$

is exact. This is the resolution.

Example If $R = k[t^4, t^5, t^6]$, the kernel of the map is $(x_1x_3 - x_2^2, x_1^3 - x_3^2) = (f_1, f_2)$. Now there is a relation between these generators and

$$0 \longrightarrow A \xrightarrow{\phi_1} A^2 \xrightarrow{(f_1,f_2)} A \longrightarrow R \longrightarrow 0,$$

where $\phi_1 = (f_2, -f_1)^t$ is exact.

The semigroup $\langle 4, 5, 6 \rangle$ is symmetric, and all symmetric 3-generated semigroups have a resolution like this. A k-generated semigroup has a resolution of length $k - 1$.

For some numerical semigroup rings of small embedding dimension, namely those of embedding dimension 3, and symmetric or pseudosymmetric of embedding dimension 4, presentations has been determined in the literature. We extend these results to whole graded minimal resolutions explicitly. Then we use these resolutions to determine some invariants of the semigroups and certain interesting relations among them.

For completeness we start with 3-generated not symmetric semigroups. We will use Herzog's result.

Theorem 1 (Herzog) *Let* $(n_1, n_2, n_3) = 1$*. Let* α_i*,* $1 \leq i \leq 3$ *be the smallest positive integer such that* $\alpha_i n_i \in \langle n_k, n_l \rangle$*,* $\{i, k, l\} = \{1, 2, 3\}$*, and let* $\alpha_i n_i = \alpha_{ik} n_k + \alpha_{il} n_l$*. Then* $S = \langle n_1, n_2, n_3 \rangle$ *is 3-generated not symmetric if and only if* $\alpha_{ik} > 0$ *for all* i, k*,* $\alpha_{21} + \alpha_{31} = \alpha_1, \alpha_{12} + \alpha_{32} = \alpha_2, \alpha_{13} + \alpha_{23} = \alpha_3$*. Then*

$$K[S] = K[\langle n_1, n_2, n_3 \rangle] = K[x_1, x_2, x_3]/(f_1, f_2, f_3)$$

where

$$(f_1, f_2, f_3) = (x_1^{\alpha_1} - x_2^{\alpha_{12}} x_3^{\alpha_{13}}, x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}}, x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}}).$$

Denham gave a minimal graded A-resolution of $K[S]$, where $A = K[x_1, x_2, x_3]$.

Theorem 2 (Denham) *If S is a 3-generated semigroup which is not symmetric. Then $K[S] = K[x_1, x_2, x_3]/I_S = A/I_S$ has a minimal graded A-resolution*

$$0 \longrightarrow A^2 \xrightarrow{\phi_2} A^3 \xrightarrow{\phi_1} A \longrightarrow 0,$$

where $\phi_1 = (x_1^{\alpha_1} - x_2^{\alpha_{12}} x_3^{\alpha_{13}}, x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}}, x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}}) = (f_1, f_2, f_3)$, *and* $\phi_2 = \begin{pmatrix} x_3^{\alpha_{23}} & x_2^{\alpha_{32}} \\ x_1^{\alpha_{31}} & x_3^{\alpha_{13}} \\ x_2^{\alpha_{12}} & x_1^{\alpha_{21}} \end{pmatrix}$.

Next we look at 4-generated symmetric but not complete intersection semigroups. We will use a theorem by Bresinsky.

Theorem 3 (Bresinsky) $S = \langle n_1, n_2, n_3, n_4 \rangle$ *is 4-generated symmetric not a complete intersection if and only if there are integers* α_i, $1 \leq i \leq 4$,

$\alpha_{ij}, ij \in \{21, 31, 32, 42, 13, 43, 14, 24\}$, *such that* $0 < \alpha_{ij} < \alpha_i$, *for all* i, j, $\alpha_1 = \alpha_{21} + \alpha_{31}, \alpha_2 = \alpha_{32} + \alpha_{42}, \alpha_3 = \alpha_{13} + \alpha_{43}$,

$\alpha_4 = \alpha_{14} + \alpha_{24}$ *and* $n_1 = \alpha_2\alpha_3\alpha_{14} + \alpha_{32}\alpha_{13}\alpha_{24}$, $n_2 = \alpha_3\alpha_4\alpha_{21} + \alpha_{31}\alpha_{43}\alpha_{24}$, $n_3 = \alpha_1\alpha_4\alpha_{32} + \alpha_{14}\alpha_{42}\alpha_{31}$, $n_4 = \alpha_1\alpha_2\alpha_{43} + \alpha_{42}\alpha_{21}\alpha_{13}$, $(n_1, n_2, n_3, n_4) = 1$. *Then*

$$K[S] = K[x_1, x_2, x_3, x_4]/(f_1, f_2, f_3, f_4, f_5)$$

where $f_1 = x_1^{\alpha_1} - x_3^{\alpha_{13}} x_4^{\alpha_{14}}, f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}, f_3 = x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}}, f_4 = x_4^{\alpha_4} - x_2^{\alpha_{42}} x_3^{\alpha_{43}}, f_5 = x_3^{\alpha_{43}} x_1^{\alpha_{21}} - x_2^{\alpha_{32}} x_4^{\alpha_{14}}$.

We now give the whole minimal A-resolution of $K[S]$.

Theorem 4 *In case S is 4-generated symmetric, not a complete intersection, then the following is a minimal resolution of $K[S]$:*

$$0 \longrightarrow A \xrightarrow{\phi_3} A^5 \xrightarrow{\phi_2} A^5 \xrightarrow{\phi_1} A \longrightarrow 0$$

where $\phi_1 = (f_1, f_2, f_3, f_4, f_5)$

$$\phi_2 = \begin{pmatrix} x_2^{\alpha_{32}} & x_3^{\alpha_{43}} & x_4^{\alpha_{24}} & 0 & 0 \\ 0 & 0 & x_1^{\alpha_{31}} & x_4^{\alpha_{14}} & x_3^{\alpha_{43}} \\ x_1^{\alpha_{21}} & x_4^{\alpha_{14}} & x_2^{\alpha_{42}} & 0 & 0 \\ 0 & 0 & x_3^{\alpha_{13}} & x_1^{\alpha_{21}} & x_2^{\alpha_{32}} \\ -x_3^{\alpha_{13}} & -x_1^{\alpha_{31}} & 0 & x_2^{\alpha_{42}} & x_4^{\alpha_{24}} \end{pmatrix}$$

and $\phi_3 = (-f_4, -f_2, -f_5, f_3, f_1)^t$.

Next we look at 4-generated pseudosymmetric semigroups. We will use a theorem by Komeda.

Theorem 5 (Komeda) $S = \langle n_1, n_2, n_3, n_4 \rangle$ *is 4-generated pseudosymmetric if and only if there are integers* $\alpha_i > 1$, $1 \leq i \leq 4$, *and* α_{21}, $1 < \alpha_{21} < \alpha_1$, *such that* $n_1 = \alpha_2\alpha_3(\alpha_4 - 1)$, $n_2 = \alpha_{21}\alpha_3\alpha_4 + (\alpha_1 - \alpha_{21} - 1)(\alpha_3 - 1) + \alpha_4$, $n_3 = \alpha_1\alpha_4 + (\alpha_1 - \alpha_{21} - 1)(\alpha_2 - 1)(\alpha_4 - 1) - \alpha_4 + 1$, $n_4 = \alpha_1\alpha_2(\alpha_3 - 1) + \alpha_{21}(\alpha_2 - 1) + \alpha_2$,

$(n_1, n_2, n_3, n_4) = 1$. *Then,*

$$K[S] = K[x_1, x_2, x_3, x_4]/(f_1, f_2, f_3, f_4, f_5)$$

where $f_1 = x_1^{\alpha_1} - x_3^{\alpha_3 - 1}x_4^{\alpha_4 - 1}, f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}}x_4, f_3 = x_3^{\alpha_3} - x_1^{\alpha_1 - \alpha_{21} - 1}x_2, f_4 = x_4^{\alpha_4} - x_1x_2^{\alpha_2 - 1}x_3^{\alpha_3 - 1}, f_5 = x_3^{\alpha_3 - 1}x_1^{\alpha_{21} + 1} - x_2x_4^{\alpha_4 - 1}$.

We now give the whole minimal A-resolution of $K[S]$.

Theorem 6 *In case S is 4-generated pseudosymmetric, then the following is a minimal resolution of $K[S]$:*

$$0 \longrightarrow A^2 \xrightarrow{\phi_3} A^6 \xrightarrow{\phi_2} A^5 \xrightarrow{\phi_1} A \longrightarrow 0$$

where $\phi_1 = (f_1, f_2, f_3, f_4, f_5)$

$$\phi_2 =$$

$$\begin{pmatrix} x_2 & 0 & x_3^{\alpha_3-1} & 0 & x_4 & 0 \\ 0 & f_3 & 0 & x_1x_3^{\alpha_3-1} & x_1^{\alpha_1-\alpha_{21}} & x_4^{\alpha_4-1} \\ x_1^{\alpha_{21}+1} & -f_2 & x_4^{\alpha_4-1} & 0 & x_1x_2^{\alpha_2-1} & 0 \\ 0 & 0 & 0 & x_2 & x_3 & x_1^{\alpha_{21}} \\ -x_3 & 0 & -x_1^{\alpha_1-\alpha_{21}-1} & x_4 & 0 & x_2^{\alpha_2-1} \end{pmatrix}$$

and $\phi_3 =$

$$\begin{pmatrix} x_4 & -x_1 & 0 & x_3 & -x_2 & 0 \\ -x_2^{\alpha_2-1}x_3^{\alpha_3-1} & x_4^{\alpha_4-1} & f_2 & -x_1^{\alpha_1-1} & x_1^{\alpha_{21}}x_3^{\alpha_3-1} & f_3 \end{pmatrix}^t.$$

In all proofs we use the following theorem by Buchsbaum-Eisenbud adopted to our situation.

Theorem 7 (Eisenbud-Buchsbaum) *Let*

$$0 \longrightarrow F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$$

be a complex of free modules. Let rank(ϕ_i) *be the size of the largest nonzero minor in the matrix describing ϕ_i, and let $I(\phi_i)$ be the ideal generated by the minors of* rank(ϕ_i) *in ϕ_i. Then the complex is exact if and only if for all i*

(a) rank(ϕ_{i+1}) + rank(ϕ_i) = rank(F_i) *and*

(b) $I(\phi_i)$ *contains an A-sequence of length i.*

In all theorems it is an easy, but sometimes tedious, task to check that we have complexes.

Applications

We will use the following well known facts: If S is generated by k elements, and $A = K[x_1, \ldots, x_k]$, then the free minimal A-resolution of $K[S]$ has length codim$(K[S]) = k - 1$ since $K[S]$ is a 1-dimensional Cohen-Macaulay ring:

$$0 \longrightarrow A^{\beta_{k-1}} \stackrel{\phi_{k-1}}{\longrightarrow} A^{\beta_{k-2}} \stackrel{\phi_{k-2}}{\longrightarrow} \ldots \stackrel{\phi_2}{\longrightarrow} A^{\beta_0} \longrightarrow K[S] \longrightarrow 0$$

The alternating sum of the β_i's, the Betti numbers is zero. The Betti numbers of A/I are

$$\beta_i = \dim_K H_i(\mathbf{F}_* \otimes K) = \dim_K \mathrm{Tor}_i^A(R, K).$$

This gives us an alternative way to define the Betti numbers, since also $\mathrm{Tor}_i^A(R, K) = H_i(\mathbf{G} \otimes R)$, where $\mathbf{G}$ is a minimal A-resolution of K (the Koszul complex). If R is Cohen-Macaulay, the highest nonzero Betti number is called the CM-type of R.

The ring is homogeneous if we set $\deg(x_i) = n_i$. If we concentrate $\mathbf{F}_*$ above to a certain degree d, we get an exact sequence of vector spaces

$$0 \longrightarrow \oplus_j A_{d-j}^{\beta_{k-1,j}} \longrightarrow \cdots \longrightarrow \oplus_j A_{d-j}^{\beta_{1,j}} \longrightarrow A_d \longrightarrow (A/I)_d$$

where the $\beta_{i,j}$ are the graded Betti numbers of $K[S]$.

The alternating sum of the dimensions of these vector spaces is 0. Multiplying each dimension with z^d and summing for $d \geq 0$, we get

$$\mathsf{Hilb}_{A/I}(z) = \mathsf{Hilb}_A(z)(1 + \sum_{i=1}^{k-1} \sum_j (-1)^i \beta_{i,j} z^j).$$

If $\deg(x_i) = n_i$, then

$\mathsf{Hilb}_{K[x_1,\ldots,x_k]}(z) = 1/\prod_{i=1}^{k}(1-z^{n_i})$. Letting $\mathcal{K}_S = 1 + \sum_{i=1}^{k-1} \sum_j (-1)^i \beta_{i,j} z^j$, we observe that

$$\mathsf{Hilb}_{K[S]}(z) = \frac{\mathcal{K}_S(z)}{\prod_{i=1}^{k}(1 - z^{n_i})} = \sum_{s \in S} z^s.$$

Recall that the set of pseudofrobenius numbers of a numerical semigroup S is $PF(S) = \{n \in \mathbb{Z} \setminus S; n+s \in S$ for all $s \in S \setminus \{0\}\}$ and its cardinality is by definition the type of the semigroup S. It is known that the type of S coincides with the CM-type of the semigroup ring $K[S]$. This can be made more strict.

Lemma 8 *Let* $S = \langle n_1, \ldots, n_k \rangle$*,* $0 \neq s \in S$*, and* $K[S] = K[t^{n_1}, \ldots, t^{n_k}]$*. Then* $n \in PF(S)$ *if and only if* $0 \neq \overline{t^{n+s}} \in \mathrm{Soc}(K[S]/(t^s))$*.*

Proposition 9 *Let* $S = \langle n_1, \ldots, n_k \rangle$ *and let* $\beta_{i,j}$ *be the graded Betti numbers of* $K[S]$*. Then* $n \in PF(S)$ *if and only if* $\beta_{k-1,n+N} \neq 0$ *(in fact* $\beta_{k-1,n+N} = 1$*), where* $N = \sum_{i=1}^{k} n_i$*. In particular, if* S *is symmetric if and only if* $\beta_{k-1} = \beta_{k-1,g(S)+N}$*.*

Example 10 *The semigroup* $S = \langle 7, 9, 8, 13 \rangle$ *is symmetric and not complete intersection by Theorem 3, thus the ring* $R = K[S]$ *is Gorenstein and not a complete intersection. Set* $\bar{R} = R/(t^7)$*. The dimension of Soc*$(\bar{R})$ *is one since* $\bar{R}$ *is also a Gorenstein ring and by Lemma 8 it is generated by* $\overline{t^{g(S)+7}} = \overline{t^{26}}$*. Since* $\mathbf{G}_*$ *is the Koszul complex of length* $k - 1 = 3$ *in the three variables* x_2, x_3, x_4 *of degrees* n_2, n_3, n_4*, the vector space* $H_3(\mathbf{G}_* \otimes R)$ *is nonzero only in degree* $(g(S) + n_1) + (n_2 + n_3 + n_4) = (19 + 7) + (9 + 8 + 13) = 56$*.*

Corollary 11 *In the notation of Theorem 1, if* $S = \langle n_1, n_2, n_3 \rangle$ *is not symmetric, then* $PF(S) = \{\alpha_1 n_1 + \alpha_{23} n_3 - N, \alpha_1 n_1 + \alpha_{32} n_2 - N\}$*, where* $\sum n_i = N$*.*

This corollary extends the result by Rosales and Garcia-Sanchez, where the Frobenius number of 3-generated semigroups is determined.

Example 12 *Let* $S = \langle 7, 9, 10 \rangle$. *Then* $\alpha_1 = 4, \alpha_{12} = 2, \alpha_{13} = 1, \alpha_2 = 3, \alpha_{21} = 1, \alpha_{23} = 2, \alpha_3 = 3, \alpha_{31} = 3, \alpha_{32} = 1$, *and thus* S *is 3-generated not symmetric. We have, by Theorem 2,* $\beta_{1,i} \neq 0$ *(in fact* $\beta_{1,i} = 1$*) only if* $i \in \{\alpha_1 n_1, \alpha_2 n_2, \alpha_3 n_3\} = \{28, 27, 30\}$, *and* $\beta_{2,i} \neq 0$ *(in fact* $\beta_{2,i} = 1$*) only if* $i \in \{\alpha_1 n_1 + \alpha_{23} n_3 - N, \alpha_1 n_1 + \alpha_{32} n_2 - N\} = \{28 + 20, 28 + 9\} = \{48, 37\}$. *Thus* $PF(S) = \{48 - N, 37 - N\} = \{22, 11\}$. *By usingthe* $\beta_{i,j}$ *given in Theorem 2, we obtain the* $\mathcal{K}$*-polynomial as*

$$\mathcal{K}_s = 1 - z^{28} - z^{27} - z^{30} + z^{48} + z^{37}$$

so that

$$\sum_{s \in S} z^s = \frac{\mathcal{K}_S(z)}{(1 - z^7)(1 - z^9)(1 - z^{10})}.$$

Corollary 13 *In the notation of Theorem 3, if* $S = \langle n_1, \ldots, n_4 \rangle$ *is 4-generated symmetric, not a complete intersection and* $N = \sum_{i=1}^{4} n_i$*, then* $g(S) = \alpha_1 n_1 + \alpha_{32} n_2 + \alpha_4 - N$*.*

Corollary 14 *If S is 4-generated symmetric, not a complete intersection, we always have*

$$A = \alpha_1 n_1 + \alpha_{32} n_2 = \alpha_3 n_3 + \alpha_{21} n_1 =$$

$$\alpha_{32} n_2 + \alpha_{13} n_3 + \alpha_{14} n_4 + \alpha_{32} n_2$$

$$B = \alpha_1 n_1 + \alpha_{43} n_3 = \alpha_3 n_3 + \alpha_{14} n_4 =$$

$$\alpha_{32} n_2 + \alpha_{14} n_4 + \alpha_{31} n_1$$

$$C = \alpha_1 n_1 + \alpha_{24} n_4 = \alpha_2 n_2 + \alpha_{31} n_1 =$$

$$\alpha_3 n_3 + \alpha_{42} n_2 = \alpha_4 n_4 + \alpha_{13} n_3$$

$$D = \alpha_2 n_2 + \alpha_{14} n_4 = \alpha_4 n_4 + \alpha_{21} n_1 =$$

$$\alpha_{21} n_1 + \alpha_{43} n_3 + \alpha_{42} n_2$$

$$E = \alpha_2 n_2 + \alpha_{43} n_3 = \alpha_4 n_4 + \alpha_{32} n_2 =$$

$$\alpha_{21} n_1 + \alpha_{43} n_3 + \alpha_{24} n_4$$

and

$$A + \alpha_4 n_4 = B + \alpha_2 n_2 = C + \alpha_{21} n_1 + \alpha_{43} n_3 = D + \alpha_3 n_3 = E + \alpha_1 n_1.$$

This follows from the different ways to determine the degrees of $H_2(\mathbf{F})$ and $H_3(\mathbf{F})$ in the resolution $\mathbf{F}$.

Example 15 *Let* $S = \langle 7, 9, 8, 13 \rangle$*. Then* $\alpha_1 = 3, \alpha_{13} = \alpha_{14} = 1, \alpha_2 = 3, \alpha_{21} = 2, \alpha_{24} = 1, \alpha_3 = 2, \alpha_{31} = \alpha_{32} = 1, \alpha_4 = \alpha_{42} = 2, \alpha_{43} = 1$, *and thus* S *is 4-generated symmetric. We get* $g(S) = \alpha_1 n_1 + \alpha_{32} n_2 + \alpha_4 n_4 - N = 21 + 9 + 26 - 37 = 19$ *and* $\sum_{s \in S} t^s = (1 - t^{21} - t^{27} - t^{16} - t^{26} - t^{22} + t^{30} + t^{29} + t^{34} + t^{40} + t^{35} - t^{56})/((1 - t^7)(1 - t^9)(1 - t^8)(1 - t^{13})) = 1 + t^7 + t^8 + t^9 + t^{13} + t^{14} + t^{15} + t^{16} + t^{17} + t^{18} + t^{20}/(1 - t)$.

Corollary 16 *If* $S = \langle n_1, \ldots, n_4 \rangle$ *is 4-generated pseudosymmetric, then* $PF(S) = \{n_1\alpha_1 + n_2 + n_4 - N, n_1\alpha_1 + n_2\alpha_2 + n_3(\alpha_3 - 1) - N\}$, *where* $N = \sum_{i=1}^{4} n_i$, .

Corollary 17 *If S is 4-generated pseudosymmetric, we always have*

$A = \alpha_1 n_1 + n_2 = \alpha_3 n_3 + \alpha_{21} n_1 = n_2 + n_3 + (\alpha_4 - 1) n_4$

$B = \alpha_1 n_1 + (\alpha_3 - 1) n_3 = \alpha_3 n_3 + (\alpha_4 - 1) n_4 =$

$$(\alpha_1 - \alpha_{21} - 1) n_1 + n_2 + (\alpha_4 - 1) n_4$$

$$C = \alpha_2 n_2 + n_1 + (\alpha_3 - 1) n_3 = \alpha_4 n_4 + n_2 =$$

$$(\alpha_{21} + 1) n_1 + (\alpha_3 - 1) n_3 + n_4$$

$$D = \alpha_1 n_1 + n_4 = (\alpha_1 - \alpha_{21}) n_1 + \alpha_2 n_2 =$$

$$n_1 + (\alpha_2 - 1) n_2 + \alpha_3 n_3 = n_3 + \alpha_4 n_4$$

$$E = \alpha_2 n_2 + (\alpha_4 - 1) n_4 = \alpha_{21} n_1 + \alpha_4 n_4 = ($$

$$\alpha_{21} + 1) n_1 + (\alpha_2 - 1) n_2 + (\alpha_3 - 1) n_3$$

and

$$A + n_4 = B + (\alpha_{21} + 1)n_1 = C + n_3 =$$

$$D + n_2 = E + \alpha_3 n_3$$

and

$$A+(\alpha_2-1)n_2+(\alpha_3-1)n_3 = \alpha_2 n_2+\alpha_3 n_3+(\alpha_4-1)n_4 =$$

$$B + \alpha_2 n_2 = C + (\alpha_1 - 1)n_1 =$$

$$D + \alpha_{21} n_1 + (\alpha_3 - 1)n_3 = E + \alpha_3 n_3.$$

This follows from the different ways to determine the degrees of $H_2(\mathbf{F})_*$ and $H_3(\mathbf{F})_*$ in the resolution $\mathbf{F}_*$.

Example 18 *Let* $\alpha_1 = \alpha_2 = \alpha_4 = 3, \alpha_3 = 2, \alpha_{21} = 1$. *Then* $S = \langle 13, 9, 11, 14\rangle$ *is 4-generated pseudosymmetric. We get* $PF(S) = \{39 + 9 + 14 - 47, 39 + 11 + 27 - 47\} = \{15, 30\}$ *and* $\sum_{s\in S} t^s = (1 - t^{39} - t^{27} - t^{22} - t^{42} - t^{37} + t^{48} + t^{49} + t^{50} + t^{51} + t^{53} + t^{55} - t^{62} - t^{77})/((1-t^{13})(1-t^9)(1-t^{11})(1-t^{14})) = 1 + t^9 + t^{11} + t^{13} + t^{14} + t^{18} + t^{20} + t^{22} + t^{23} + t^{24} + t^{25} + t^{26} + t^{27} + t^{28} + t^{29} + t^{31}/(1-t)$.