Elasticity in Numerical Semigroup Rings and Power Series Rings

Paul Baginski

Fairfield University

INdAM International Meeting on Numerical Semigroups
Cortona, Italy
September 11, 2014

Work building off undergraduate research by Chris Crutchfield, K. Grace Kennedy, Matthew Wright in the 2005 Trinity NSF Research Experience for Undergraduates.

Let $S = \langle n_1, \ldots, n_b \rangle$ be a numerical semigroup, K a field.

The **semigroup ring** of S over K is the subring of $K[x]$ given by

$$K[S] := \left\{ f \in K[x] \;\middle|\; f = \sum_{i=0}^{n} a_i x^i, \text{ where } a_i \neq 0 \Rightarrow i \in S \right\}$$

The **semigroup power series ring** of S over K is the subring of $K[\![x]\!]$ given by

$$K[\![S]\!] := \left\{ f \in K[\![x]\!] \;\middle|\; f = \sum_{i=0}^{\infty} a_i x^i, \text{ where } a_i \neq 0 \Rightarrow i \in S \right\}$$

An element $a \in K[S]^{\bullet} \backslash K[S]^{\times}$ is **irreducible** if $a \neq bc$ for any $b, c \in K[S] \backslash K[S]^{\times}$.
Similarly, $a \in K[\![S]\!]^{\bullet} \backslash K[\![S]\!]^{\times}$ is **irreducible** if $a \neq bc$ for any $b, c \in K[\![S]\!] \backslash K[\![S]\!]^{\times}$.

We want to see how nonzero nonunits of $K[S]$ (or $K[\![S]\!]$) factor as a product of irreducibles (up to associates).

Examples

1. Work in $K[x^2, x^3]$.

$$x^6 = (x^2)^3 = (x^3)^2$$

S is isomorphic to the submonoid $X = \{x^n \mid n \in S\} \subset K[S]$. Up to associates, this submonoid is *saturated* in $K[S]$, meaning all factorizations in $K[S]$ already occur in X. So the factorization theory of S is present in $K[S]$.

Examples

2. Work in $K[x^2, x^5]$ with char $K \neq 2$.

$$\begin{aligned} x^{10}(1-x^2)^2 &= [x^2]^5[1-x^2][1-x^2] \\ &= [x^5]^2[1-x^2][1-x^2] \end{aligned}$$

Examples

2. Work in $K[x^2, x^5]$ with char $K \neq 2$.

$$\begin{aligned} x^{10}(1-x^2)^2 &= [x^2]^5[1-x^2][1-x^2] \\ &= [x^5]^2[1-x^2][1-x^2] \\ &= [x^5(1+x)][x^5(1-x)][1-x^2] \\ &= [x^2][x^4(1+x)][x^4(1-x)][1-x^2] \end{aligned}$$

Examples

2. Work in $K[x^2, x^5]$ with char $K \neq 2$.

$$\begin{aligned} x^{10}(1-x^2)^2 &= [x^2]^5[1-x^2][1-x^2] \\ &= [x^5]^2[1-x^2][1-x^2] \\ &= [x^5(1+x)][x^5(1-x)][1-x^2] \\ &= [x^2][x^4(1+x)][x^4(1-x)][1-x^2] \\ &= [x^5(1+x)(1+x)][x^5(1-x)(1-x)] \\ &= [x^2][x^4(1+x)(1+x)][x^4(1-x)(1-x)] \end{aligned}$$

and variants of the above. So $x^{10}(1-x^2)^2$ factors as a product of $7, 4, 3,$ and 2 irreducibles.

(in char 2 the final two expressions have reducible terms)

We need:

- language to quantify "non-uniqueness" of factorization
- criteria to test for irreducibility
- description of the factorization behavior in $K[S]$, globally and, if lucky, locally.

Factorization language

H a commutative, cancellative, atomic monoid, such as $K[S]^\bullet$ or $K[\![S]\!]^\bullet$ under multiplication.

$H^\times = \{f \in H \mid f \text{ is a unit } \}$

$\mathscr{A}(H) = \{f \in H \backslash H^\times \mid f \text{ is irreducible } \}$ (the **atoms** of H)

Factorization language

The **set of lengths** of f is

$$\mathscr{L}(f) = \{n \in \mathbb{N}_0 | \exists a_1, \ldots, a_n \in \mathscr{A}(H)\, f = a_1 \cdots a_n\}$$

with $\ell(f) = \min \mathscr{L}(f)$ and $L(f) = \max \mathscr{L}(f)$.

The **elasticity** of f is

$$\rho(f) = \frac{L(f)}{\ell(f)}$$

and the **elasticity** of H is $\rho(H) = \sup\{\rho(f) \mid f \in H \backslash H^\times\}$.

Notes:

Factorization language

The **set of lengths** of f is

$$\mathscr{L}(f) = \{n \in \mathbb{N}_0 | \exists a_1, \ldots, a_n \in \mathscr{A}(H)\, f = a_1 \cdots a_n\}$$

with $\ell(f) = \min \mathscr{L}(f)$ and $L(f) = \max \mathscr{L}(f)$.

The **elasticity** of f is

$$\rho(f) = \frac{L(f)}{\ell(f)}$$

and the **elasticity** of H is $\rho(H) = \sup\{\rho(f) \mid f \in H \backslash H^{\times}\}$.

Notes: (1) $\rho(f) \leq \rho(f^n)$ for all $n \in \mathbb{N}$.

Factorization language

The **set of lengths** of f is

$$\mathscr{L}(f) = \{n \in \mathbb{N}_0 | \exists a_1, \ldots, a_n \in \mathscr{A}(H)\, f = a_1 \cdots a_n\}$$

with $\ell(f) = \min \mathscr{L}(f)$ and $L(f) = \max \mathscr{L}(f)$.

The **elasticity** of f is

$$\rho(f) = \frac{L(f)}{\ell(f)}$$

and the **elasticity** of H is $\rho(H) = \sup\{\rho(f) \,|\, f \in H \backslash H^\times\}$.

Notes: (1) $\rho(f) \le \rho(f^n)$ for all $n \in \mathbb{N}$. (2) While $\rho(f) \in \mathbb{Q}$, we could have $\rho(H)$ be irrational or ∞.

Questions:

1. Can we write out $\mathscr{L}(x)$ for any x? or $L(x)$? or $\ell(x)$
2. Can we compute $\rho(x)$? Or $\rho(H)$?
3. Which rationals between 1 and $\rho(H)$ can occur as $\rho(x)$ of some $x \in H$?
4. If $\rho(H) \in \mathbb{Q}$, is there an $x \in H$ such that $\rho(x) = \rho(H)$?

Questions:

1. Can we write out $\mathscr{L}(x)$ for any x? or $L(x)$? or $\ell(x)$
2. Can we compute $\rho(x)$? Or $\rho(H)$?
3. Which rationals between 1 and $\rho(H)$ can occur as $\rho(x)$ of some $x \in H$?
4. If $\rho(H) \in \mathbb{Q}$, is there an $x \in H$ such that $\rho(x) = \rho(H)$?

H is **fully elastic** if for all $q \in \mathbb{Q} \cap [1, \rho(H))$, there is $x \in H$ with $\rho(x) = q$. H has **accepted elasticity** if there is $x \in H$ such that $\rho(x) = \rho(H) \in \mathbb{Q}$.

Irreducibles

We have submonoids

$$K_0[x] = \{f \in K[x] \mid f(0) = 1\} \cong (K^\times + xK[x])/K^\times$$
$$K_0[S] = \{f \in K[S] \mid f(0) = 1\} = K_0[x] \cap K[S]$$

If $f \in K[S]$, then we can write f uniquely in $K[x]$ as

$$f = ux^n g_1 \cdots g_k,$$

where $u \in K^\times$ and $g_1, \ldots, g_k \in \mathscr{A}(K_0[x])$.

We can immediately conclude $n \in S$.

Suppose $n = 0$. What is special about $g_1, \ldots, g_k$?

Associated group

Consider

$$K_0[\![x]\!] = \{f \in K[\![x]\!] \mid f(0) = 1\} \cong (K^\times + xK[\![x]\!])/K^\times$$
$$K_0[\![S]\!] = \{f \in K[\![S]\!] \mid f(0) = 1\} = K_0[\![x]\!] \cap K[\![S]\!]$$

Clear that $K[\![x]\!]^\times = K^\times K_0[\![x]\!]$.

Associated group

Consider

$$K_0[\![x]\!] = \{f \in K[\![x]\!] \mid f(0) = 1\} \cong (K^\times + xK[\![x]\!])/K^\times$$
$$K_0[\![S]\!] = \{f \in K[\![S]\!] \mid f(0) = 1\} = K_0[\![x]\!] \cap K[\![S]\!]$$

Clear that $K[\![x]\!]^\times = K^\times K_0[\![x]\!]$. Less obvious: $K_0[\![S]\!]$ is a group and $K[\![S]\!]^\times = K^\times K_0[\![S]\!]$. Now set

$$G_S(K) = K_0[\![x]\!]/K_0[\![S]\!]$$

Associated group

Consider

$$K_0[\![x]\!] = \{f \in K[\![x]\!] \mid f(0) = 1\} \cong (K^\times + xK[\![x]\!])/K^\times$$
$$K_0[\![S]\!] = \{f \in K[\![S]\!] \mid f(0) = 1\} = K_0[\![x]\!] \cap K[\![S]\!]$$

Clear that $K[\![x]\!]^\times = K^\times K_0[\![x]\!]$. Less obvious: $K_0[\![S]\!]$ is a group and $K[\![S]\!]^\times = K^\times K_0[\![S]\!]$. Now set

$$G_S(K) = K_0[\![x]\!]/K_0[\![S]\!]$$

We have a homomorphism $K_0[x] \to G_S(K)$ where $g \in K_0[x] \mapsto \overline{g} \in G_S(K)$.

Take $g_1, \ldots, g_k \in \mathscr{A}(K_0[x])$. If $g_1 \cdots g_k \in K_0[S]$ then

$$\overline{g_1} \cdots \overline{g_k} = \overline{g_1 \cdots g_k} = 1$$

Lemma For $g_1, \dots, g_k \in \mathscr{A}(K_0[x])$, the product $g_1 \cdots g_k \in K_0[S]$ iff $\overline{g_1} \cdots \overline{g_k} = 1$.

Connects $K_0[S]$ to a classic object in factorization theory, the block monoid.

If G is an abelian group and $G_0 \subseteq G$, we construct $\mathscr{F}(G_0)$, the free abelian monoid over G_0 and have an evaluation map $\sigma : \mathscr{F}(G_0) \to G$. The **block monoid** is the kernel of this map, namely

$$\mathscr{B}(G, G_0) = \{a_1 \cdot \dots \cdot a_n \in \mathscr{F}(G_0) \,|\, a_1 \cdots a_n = 1 \text{ in } G\}$$

Theorem: $K_0[S]$ is a saturated submonoid of $K[S]$ and there is a transfer homomorphism* $\theta : K_0[S] \to \mathscr{B}(G_S(K), G_A)$, where

$$G_A = \{\, \overline{g} \in G_S(K) \,|\, g \in \mathscr{A}(K[x])\,\}$$

So if we ask any of our factorization questions about $K_0[S]$, we can work in $\mathscr{B}(G_S(K), G_A)$ instead. There we "factor" a sequence over G_A that multiplies to 1 into subsequences which multiply to 1.

Irreducibles $g_1 \cdots g_k \in \mathscr{A}(K_0[S])$ correspond to irreducible blocks $\overline{g_1} \cdots \overline{g_k} \in \mathscr{A}(\mathscr{B}(G_S(K), G_A))$.

* Caveat: Technically we must first throw away the prime elements from $K_0[S]$ to obtain a trivial kernel for θ.

We have completely characterized irreducibles of the form x^n and of the form $g_1 \cdots g_k$ for $g_i \in \mathscr{A}(K_0[x])$.

Still one more possibility for irreducibles, namely $x^n g_1 \cdots g_k$, where $n, k \geq 1$.

We have completely characterized irreducibles of the form x^n and of the form $g_1 \cdots g_k$ for $g_i \in \mathscr{A}(K_0[x])$.

Still one more possibility for irreducibles, namely $x^n g_1 \cdots g_k$, where $n, k \geq 1$.

Here we do not have a complete characterization. We know:

- $n \in S$, but n does not have to be a generator. However $n \leq \mathscr{F}(S) + n_1$.
- there can be no $I \subseteq \{1, \ldots, k\}$ such that $\prod_{i \in I} \overline{g_i} \in \mathscr{B}(G_S(K), G_A)$.

Summary

Up to associates, irreducibles in $K[S]$ take one of three forms:

- x^n, where n a generator of S
- $g_1 \cdots g_k$, $g_i \in \mathscr{A}(K_0[x])$, where $\overline{g_1} \cdots \overline{g_k} \in \mathscr{A}(\mathscr{B}(G_S(K), G_A))$
- $x^n g_1 \cdots g_k$, where $n \in S$, $n \leq \mathscr{F}(S) + n_1$, and $\overline{g_1} \cdots \overline{g_k}$ does not have a block as a subsequence.

At this point we have:

- $X = \{x^n \mid n \in S\}$ is saturated in $K[S]$ and behaves like the numerical semigroup S.
- $K_0[S]$ is saturated in $K[S]$ and behaves like the block monoid $\mathscr{B}(G_S(K), G_A)$.
- There are other poorly understood irreducibles $x^n g_1 \cdots g_k$ which link the two.

Questions:

1. Can we write out $\mathscr{L}(x)$ for any x? or $L(x)$? or $\ell(x)$
2. Can we compute $\rho(x)$? Or $\rho(K[S])$?
3. Which rationals between 1 and $\rho(K[S])$ can occur as $\rho(x)$ of some $x \in K[S]$?
4. If $\rho(K[S]) \in \mathbb{Q}$, is there an $x \in K[S]$ such that $\rho(x) = \rho(K[S])$?

Assume K finite with $|K| = p^q$. Let t be minimal such that $p^t \in S$.

Then $|G_S(K)| = p^{qg(S)}$, where $g(S)$ is the genus of S and $\exp(G_S(K)) = p^t$.

In particular, if $p \in S$, then $G_S(K) \cong (\mathbb{Z}/p\mathbb{Z})^{qg(S)}$.

If $p \notin S$, we have not determined the isomorphism type of $G_S(K)$, but at least we know it is a finite abelian p-group.

By the Dirichlet-Kornblum Theorem, since K is finite, $K_0[x]$ has infinitely many irreducibles of the form

$$1 + a_1 x + a_2 x^2 + \ldots + a_{\mathscr{F}(S)} x^{\mathscr{F}(S)} + x^{\mathscr{F}(S)+1} h(x)$$

for any choice of $a_1, \ldots, a_{\mathscr{F}(S)} \in K$.

Two consequences:

(1) $\mathscr{A}(K_0[x]) \cap K_0[S]$ is infinite, so $K_0[S]$ has infinitely many prime elements.

(2)

By the Dirichlet-Kornblum Theorem, since K is finite, $K_0[x]$ has infinitely many irreducibles of the form

$$1 + a_1x + a_2x^2 + \ldots + a_{\mathscr{F}(S)}x^{\mathscr{F}(S)} + x^{\mathscr{F}(S)+1}h(x)$$

for any choice of $a_1, \ldots, a_{\mathscr{F}(S)} \in K$.

Two consequences:

(1) $\mathscr{A}(K_0[x]) \cap K_0[S]$ is infinite, so $K_0[S]$ has infinitely many prime elements.

(2) For all $h \in G_S(K)$, there exists $g \in \mathscr{A}(K_0[x])$ such that $\overline{g} = h$. So $G_A = G_S(K)$.

(2) For all $h \in G_S(K)$, there exists $g \in \mathscr{A}(K_0[x])$ such that $\overline{g} = h$. So $G_A = G_S(K)$.

So $K_0[S]$ behaves like $\mathscr{B}(G_S(K), G_S(K))$, which is very well understood. In particular,

$$\rho(K_0[S]) = \rho(\mathscr{B}(G_S(K), G_S(K))) = D/2$$

for a constant $D = D(G_S(K))$ known as the **Davenport constant** of the group. An explicit formula is known for $D(G)$ for p-groups G.

(2) For all $h \in G_S(K)$, there exists $g \in \mathscr{A}(K_0[x])$ such that $\overline{g} = h$. So $G_A = G_S(K)$.

So $K_0[S]$ behaves like $\mathscr{B}(G_S(K), G_S(K))$, which is very well understood. In particular,

$$\rho(K_0[S]) = \rho(\mathscr{B}(G_S(K), G_S(K))) = D/2$$

for a constant $D = D(G_S(K))$ known as the **Davenport constant** of the group. An explicit formula is known for $D(G)$ for p-groups G.

But this is $\rho(K_0[S])$. What is $\rho(K[S])$? Could $\rho(K[S]) = \infty$?

No, $\rho(K[S])$ is finite. Because irreducibles have one of the forms:

- x^n, where n a generator of S
- $g_1 \cdots g_k$, where $\overline{g_1} \cdots \overline{g_k} \in \mathscr{A}(\mathscr{B}(G_S(K), G_A))$
- $x^n g_1 \cdots g_k$, where $n \in S$, $n \leq \mathscr{F}(S) + n_1$, and $\overline{g_1} \cdots \overline{g_k}$ does not have a block as a subsequence.

Irreducibles of the first kind involve at least n_1 copies of x and at most n_r copies of x.
Irreducibles of the second kind involve at least 2 of the g_i and at most D of the g_i.

Irreducibles of the third kind involve at least (n_1 of the x and 1 of g_i) and at most ($\mathscr{F}(S) + n_1$ of the x and $D - 1$ of the g_i).

So if $f = x^n g_1 \cdots g_k \in K[S]$, then rough heuristics give

$$\rho(f) \leq \frac{n_1 D + 2\mathscr{F}(S) + n_1}{2n_1}$$

a bound derived by Anderson and Scherpenisse (1997).

They showed that you get equality when $S = \langle n_1, n_1 + 1, \ldots, 2n_1 - 1 \rangle$, in which case

$$\rho(f) = \frac{n_1 D + 2(2n_1 - 1) + n_1}{2n_1} = \frac{D}{2} + \frac{5}{2} - \frac{1}{n_1}$$

So if $f = x^n g_1 \cdots g_k \in K[S]$, then rough heuristics give

$$\rho(f) \leq \frac{n_1 D + 2\mathscr{F}(S) + n_1}{2n_1}$$

a bound derived by Anderson and Scherpenisse (1997).

They showed that you get equality when $S = \langle n_1, n_1 + 1, \ldots, 2n_1 - 1\rangle$, in which case

$$\rho(f) = \frac{n_1 D + 2(2n_1 - 1) + n_1}{2n_1} = \frac{D}{2} + \frac{5}{2} - \frac{1}{n_1}$$

Compare this to naive lower bounds

$$\rho(f) \geq \max\{\rho(X), \rho(K_0[S])\} = \max\{\rho(S), \rho(\mathscr{B}(G_S(K), G_S(K)))\}$$
$$= \max\left\{\frac{n_r}{n_1}, \frac{D}{2}\right\} = \max\left\{2 - \frac{1}{n_1}, \frac{D}{2}\right\}$$

We have obtained a lot of information about $L(f)$ for general $f \in K[S]$, including cases where it can be explicitly computed (using S and $\mathscr{B}(G_S(K), G_S(K))$).

For instance, if $f = x^n g_1 \cdots g_k$ and $\overline{g_1} \cdots \overline{g_k} \in \mathscr{B}(G_S(K), G_A)$, then

$$L(f) = L_S(n) + L_{\mathscr{B}}(\overline{g_1} \cdots \overline{g_k})$$

These should be useful to get better upper bounds for $\rho(K[S])$ in general for K finite.

Dirichlet-Kornblum also gave us:

(1) $\mathscr{A}(K_0[x]) \cap K_0[S]$ is infinite, so $K_0[S]$ has infinitely many prime elements.

Lemma: If H has prime elements and for all $x \in H$ there exists $N \in \mathbb{N}$ such that $\rho(x^{Nm}) = \rho(x^N)$ for all $m \in \mathbb{N}$, then H is fully elastic (i.e. for all $q \in [1, \rho(H)) \cap \mathbb{Q}$, there is $x \in H$ with $\rho(x) = q$).

Both conditions are true when K is finite, so $K[S]$ is fully elastic.

If K is infinite, much less is known.

$G_S(K)$ is infinite. If K has char 0, then $G_S(K)$ is torsion free. If K has char $p > 0$, then $G_S(K)$ has exponent p^t, the least power of p in S.

Regardless of characteristic, one can show

$$\rho(K_0[S]) = \rho(\mathscr{B}(G_S(K), G_A)) = \infty$$

and since $\rho(K_0[S]) \leq \rho(K[S])$, we have $\rho(K[S]) = \infty$.

The structure of G_A is not well understood.

If K is algebraically closed, $G_A = \{\overline{1+ax} \mid a \in K\} \subsetneq G_S(K)$. If K is real closed, G_A is a little bigger but also a strict subset of $G_S(K)$ (unless $S = \langle 2, 3 \rangle$).

If $K = \mathbb{Q}$, you can use Eisenstein's Criterion to get $G_A = G_S(K)$.

Generally, you need to know about the existence of irreducibles in $K[S]$ with given initial terms.

Our more precise results about $L(f)$ for arbitrary $f \in K[S]$ hold as well in the case where K is infinite case.

For instance, if $f = x^n g_1 \cdots g_k$ and $\overline{g_1} \cdots \overline{g_k} \in \mathscr{B}(G_S(K), G_A)$, then

$$L(f) = L_S(n) + L_{\mathscr{B}}(\overline{g_1} \cdots \overline{g_k})$$

As for full elasticity, we had previously used

Lemma: If H has prime elements and for all $x \in H$ there exists $N \in \mathbb{N}$ such that $\rho(x^{Nm}) = \rho(x^N)$ for all $m \in \mathbb{N}$, then H is fully elastic (i.e. for all $q \in [1, \rho(H)) \cap \mathbb{Q}$, there is $x \in H$ with $\rho(x) = q$).

For some K infinite, we can construct prime elements in $K[S]$. What about the second property (it is called **tautness**)?

As for full elasticity, we had previously used

Lemma: If H has prime elements and for all $x \in H$ there exists $N \in \mathbb{N}$ such that $\rho(x^{Nm}) = \rho(x^N)$ for all $m \in \mathbb{N}$, then H is fully elastic (i.e. for all $q \in [1, \rho(H)) \cap \mathbb{Q}$, there is $x \in H$ with $\rho(x) = q$).

For some K infinite, we can construct prime elements in $K[S]$. What about the second property (it is called **tautness**)?

When K has char $p > 0$, we can show $K[S]$ is taut.

As for full elasticity, we had previously used

Lemma: If H has prime elements and for all $x \in H$ there exists $N \in \mathbb{N}$ such that $\rho(x^{Nm}) = \rho(x^N)$ for all $m \in \mathbb{N}$, then H is fully elastic (i.e. for all $q \in [1, \rho(H)) \cap \mathbb{Q}$, there is $x \in H$ with $\rho(x) = q$).

For some K infinite, we can construct prime elements in $K[S]$. What about the second property (it is called **tautness**)?

When K has char $p > 0$, we can show $K[S]$ is taut.

When K has char 0, we can only show $K_0[S]$ is taut, but have not been able to determine if $K[S]$ is taut. Nonetheless, tautness of $K_0[S]$ should suffice.

What to do when no primes exist in $K[S]$? Then you can't use the lemma. This includes cases where K is algebraically closed or real closed.

We have to try explicit constructions of x such that $\rho(x) = a/b$. This has run into many, many obstacles due to structural considerations about S.

Thank you.

References

- D.F. Anderson, C. Scherpenisse. *Factorization in $K[S]$*. Commutative ring theory (Fès, 1995), 4556, Lecture Notes in Pure and Appl. Math., 185, Dekker, New York, 1997.
- P. Baginski, C. Chapman, C. Crutchfield, K.G. Kennedy, M. Wright. *Elastic properties and prime elements*. Results Math. 49 (2006), no. 3-4, 187200.