Scalar Conformal Invariants of Weight Zero

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Objectives

The objective of this work is to expose a collection of scalar conformal invariants which, to my knowledge, up to now has been no discussed. We take the following steps:

1. Describe the concept of conformal invariant that we are going to consider.
2. Emphasize the existence of a distinguishing metric in the class of a generic conformal structure. This was noticed by Einstein [1] in a lesser-known paper of 1921.
3. Calculate the simplest invariant of this collection for some well-known spacetimes, comparing the results with usual scalar metric invariants.

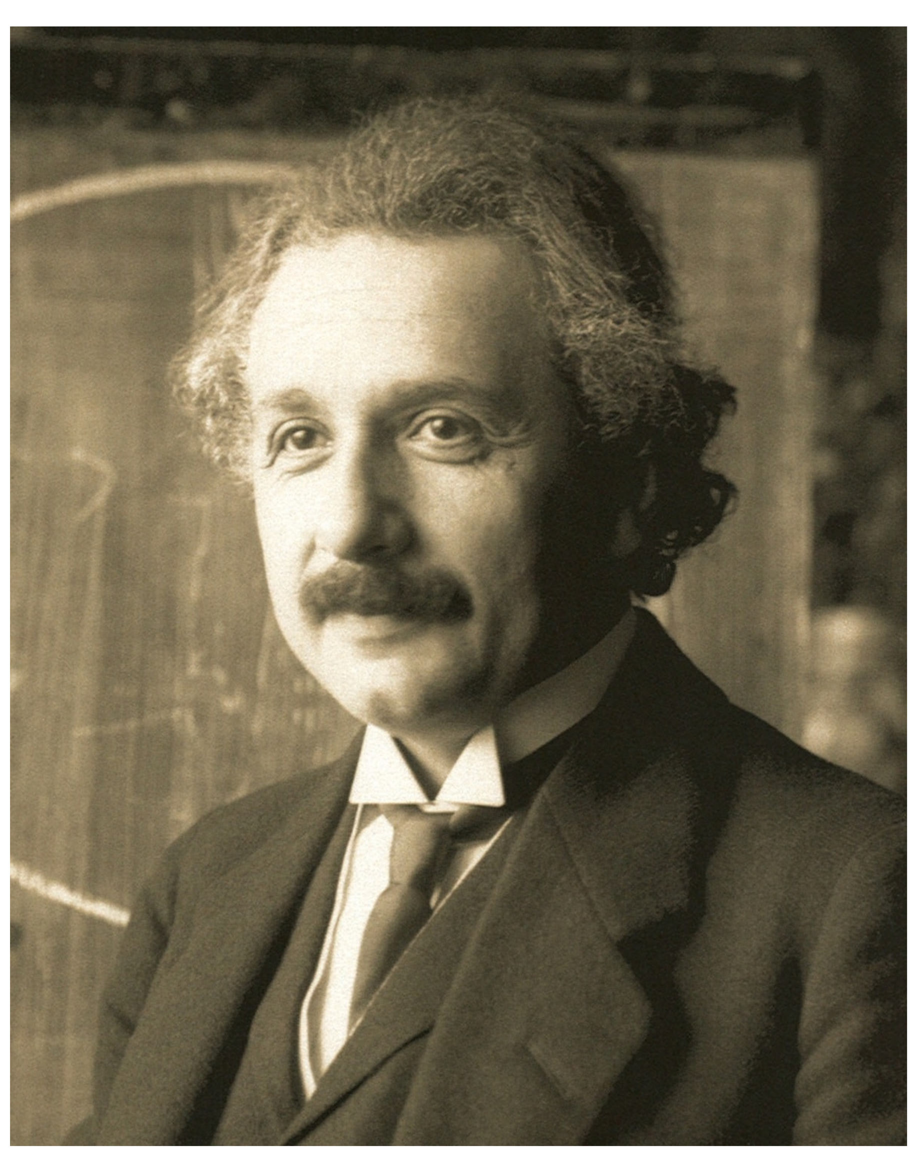
Introduction

We are accustomed to think that in a conformal pseudo-Riemannian class of metrics there is no preferred metric. **This is not true** for reasonably generic conformal structures in four or more dimensions (neither in three dimensions [4]).

In March 1921 Einstein wrote [1] (my notes in brackets):

> *If we put $g'_{\mu\nu} = J g_{\mu\nu}$ [J is given next] then $d\sigma^2 = g'_{\mu\nu} dx_\mu dx_\nu$ is an invariant that depends only upon the ratios of the $g_{\mu\nu}$ [the conformal structure $[g]$]. All Riemann tensors formed as fundamental invariants from $d\sigma$ in the customary manner are —when seen as functions of the $g_{\mu\nu}$— Weyl tensors of weight 0. [...] Therefore, to every law of nature $T(g) = 0$ of the general theory of relativity, there corresponds a law $T(g') = 0$, which contains only the ratios of the $g_{\mu\nu}$.*

We are going to develop this idea of Einstein and we will use it to obtain conformal invariants. In particular, here I show that the Ricci scalar of g' is a scalar conformal invariant.



Albert Einstein during a lecture in Vienna in 1921.

https://commons.wikimedia.org/wiki/File:Einstein1921_by_F_Schmutzer_2.jpg

1. Bundle of G-structures

Given a subgroup G of $Gl_n \equiv GL(n,\mathbb{R})$, we get the *bundle of G-structures* M_G that is the associated bundle to the linear frame bundle LM which fiber is Gl_n/G.

The bundle M^r_G of r-jets of local sections of M_G (r-jets of local G-structures) is associated to the frame bundle $F^{r+1}M$, with its fiber being the space $(\mathbb{R}^n_G)^r_0 \equiv J^r_0(\mathbb{R}^n, Gl_n/G)$ of r-jets at 0 of maps of $\mathbb{R}^n$ to Gl_n/G.

The **bundle of metrics** is obtained taking $G = O_n$, the orthogonal group of given signature (p,q). There is an bijective correspondence between Gl_n/O_n and the set of symmetric invertible matrices of signature (p,q). Hence, we recognize a pseudo-Riemannian metric g of M as a section of M_{O_n} given by the matrix (g_{ij}), over the domain of a chart.

The **bundle of conformal structures** arises when $G = C_n := \mathbb{R}^+ \cdot O_n$. There is an bijective correspondence between Gl_n/C_n and the set of symmetric invertible matrices of signature (p,q) and absolute value of its determinant one. Now, a conformal structure $[g]$ is a section of M_{C_n} which is given by the matrix $(c_{ij}) = |\det(g_{ij})|^{-1/n}(g_{ij})$, over the domain of a chart.

1. Invariants of G-structures

A paradigm of scalar invariant for a differential geometric structure is the scalar curvature, or Ricci scalar, $R_g\colon M \to \mathbb{R}$ of a Riemannian manifold (M,g). For $m \in M$, $R_g(m)$ is made from the partial derivatives up to second order of the metric g at m and does not depend on which chart has been used to perform the derivatives, in other words, $R_g(m)$ is a function of the 2-jet of g at m.

Therefore, the Ricci scalar defines the function $R\colon M^2_{O_n} \to \mathbb{R}$, $R(j^2_m\sigma) := R_g(m)$, with σ the section of M_{O_n} equivalent to g. Ricci scalar is said *invariant by diffeomorphisms* because, for all diffeomorphism φ of M, it is verified $R_{\varphi^*g} = R_g \circ \varphi$, $\forall g$. That is to say that $R \circ \widehat{\varphi}^2 = R$, $\forall \varphi$, with $\widehat{\varphi}^2$ the case $r = 2$ of the typical action of a diffeomorphism φ on M^r_G defined by:

$$\widehat{\varphi}^r\colon M^r_G \to M^r_G, \quad j^r_p\sigma \mapsto j^r_{\varphi(p)}(\bar{\varphi} \circ \sigma \circ \varphi^{-1});$$

$\bar{\varphi}$ being the standard lifting of φ to M_G.

Definition: A *scalar invariant of r-order of G-structures on M* is a function $f\colon M^r_G \to \mathbb{R}$ verifying $f \circ \widehat{\varphi}^r = f$ over the domain of φ, for all local diffeomorphism φ of M.

2. Important Results

For a generic conformal structure $[g]$ on M, **there exists a preferred metric** $g' = Jg \in [g]$, $\forall g \in [g]$, being $J = |H|^{1/2}$ with $H = C_{ijkl}C^{ijkl}$ and $C^i{}_{jkl}$ the conformal curvature tensor (Einstein, 1921 [1]). By generic we mean that $H \neq 0$ for some (and then for all) $g \in [g]$.

Main result: To each scalar metric invariant of r-order corresponds a scalar conformal invariant of $(r+2)$-order.

In fact, an invariant of the preferred metric $g' \in [g]$ of r-order is a conformal invariant of $[g]$ of $(r+2)$-order. In particular, the scalar curvature of the metric g' is a conformal invariant of fourth order that we call the *conformal scalar curvature* of $[g]$.

2. Results

The action of the jet group G^{r+1}_n on $(\mathbb{R}^n_G)^r_0$ is defined [5] by

$$j^{r+1}_0\xi \cdot j^r_0\mu := j^r_0\left((D\xi \cdot \mu) \circ \xi^{-1}\right)$$

being ξ a local diffeomorphism of $\mathbb{R}^n$ with $\xi(0) = 0$, μ a smooth map from a neighborhood of 0 to Gl_n/G (for G closed), and the last dot for the action of Gl_n on Gl_n/G. The orbit space is $(\mathbb{R}^n_G)^r_0/G^{r+1}_n$, consisting of classes of r-jets at 0 of G-structures over $\mathbb{R}^n$ "modulo diffeomorphisms".

It is proved [5] that r-order scalar invariants of G-structures on M are in a natural bijection with functions $\psi\colon (\mathbb{R}^n_G)^r_0/G^{r+1}_n \to \mathbb{R}$ such that $\psi \circ \pi$ is smooth. Such ψ can be called a *scalar G-invariant of r-order* (**it does not depend on M!**). With this simplified concept of scalar invariant it is easier to prove its properties.

Properties of invariants:

- An invariant of r-order is invariant of $(r+s)$-order, $\forall s$.
- If $U \subset G$ is a subgroup, a G-invariant of r-order is a U-invariant of r-order (hence, conformal invariants are metric invariants of the same order).
- A metric invariant of r-order is a conformal invariant of $(r+2)$-order.

3. Conformal scalar curvature

I have calculated the "conformal scalar curvature" of the conformal structure of some well known spacetime metrics. To do this, I take a metric g, calculate J and calculate the Ricci scalar of Jg; I used the package *xAct`xCoba`* for Mathematica. In the table below, I give some results comparing the Ricci scalar R_g, the Kretschmann scalar K_g and the conformal scalar curvature denoted by $S_{[g]}$.

Let me highlight that replacing the metric g by a conformally related metric αg, Ricci and Kretschmann scalars get complicated with terms including up to second order derivatives of α. In contrast, the conformal scalar curvature keeps unalterable, meaning that its weight is zero, in the Weyl's sense [3].

Metrics	R_g	K_g	$S_{[g]}$
pp–wave	0	0	0
Schwarzschild	0	$\frac{48M^2}{r^6}$	$\frac{9\sqrt{3}}{4} - \frac{3\sqrt{3}\,r}{8M}$
Gödel	$-\frac{1}{a^2}$	$\frac{3}{a^4}$	$-\frac{\sqrt{3}}{2}$
Barriola-Vilenkin	$\frac{2-k^2}{k^2r^2}$	$\frac{4(k+1)^2(k-1)^2}{k^4r^4}$	$-\sqrt{3}$

Scalar invariants of some spacetimes

Conclusion

Einstein proposed the natural addition of the differential equation $J = J_0$ (constant) to the field equations of General Relativity.

In dimension four, the existence of a distinguishing metric in the class of conformal structures with $H \neq 0$ arises also from the theory of Weyl gravity (or conformal gravity). This theory is governed by the Lagrangian $H\,\Omega_g$, with Ω_g being the metric volume form. This Lagrangian in 4 dimensions is invariant by conformal transformations (in dimension greater than 4, what is conformally invariant is $|H|^{n/4}\Omega_g$ which is the former Lagrangian, except sign, when $n = 4$). Then $H\Omega_g$ is a distinctive volume form provided only by the conformal structure $[g]$. Therefore, it can be concluded that there exists a unique metric $g' \in [g]$ such that $\Omega_{g'} = H\Omega_g$.

These previous facts suggest new perspectives of research using that preferred metric of the conformal class, which is surprisingly unknown in the research bibliography (I only found one reference to this paper of Einstein pointing at this distinguishing metric [2]).

References

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