G-Structures defined on pseudo-Riemannian manifolds

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Abstract: Concepts and techniques from the theory of G-structures of higher order are applied to the study of certain structures (volume forms, conformal structures, linear connections and projective structures) defined on a pseudo-Riemannian manifold. Several relationships between the structures involved have been investigated. The operations allowed on G-structures, such as intersection, inclusion, reduction, extension and prolongation, were used for it.

1 Introduction

A manifold M is a set of points with the property that we can cover it with an atlas. A *differential structure* $\mathcal{A}$ of M is a C^∞ n-dimensional maximal atlas. The charts in the atlas form the *primordial* structure.

The idea of a *geometrical structure* can be realized by the concept of G-structure when choosing the *allowable meaningful classes* of charts in $\mathcal{A}$. In a pseudo-Riemannian manifold there are defined unambiguously the following structures: *volume n-form, conformal structure, pseudo-Riemannian metric, symmetric linear connection and projective structure.*

Volume, conformal and metric structures are G-structures of first order, but each of them lead to a *prolonged* second order structure. Symmetric linear connection and projective structure are *inherently* G-structures of second order.

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2 Frame bundles

The *r-th order frame bundle* $\mathcal{F}^rM$ is a quotient set of $\mathcal{A}$ (a subset of $\mathcal{A}$ properly; see [7]). An *r-frame*, $j^r_0\varphi \in \mathcal{F}^rM$, is an r-jet at 0, where $x = \varphi^{-1}$ is a chart with 0 as a target. The first order bundle $\mathcal{F}^1M$ is usually identified with the linear frame bundle LM.

Let LLM be the linear bundle of LM. There is a *canonical inclusion* $\mathcal{F}^2M \hookrightarrow LLM$, $j^2_0\varphi \mapsto j^1_0\tilde{\varphi}$, where $\tilde{\varphi}$ is the diffeomorphism induced by φ, between neighborhoods of $0 \in \mathbb{R}^{n+n^2}$ and $j^1_0\varphi \in LM$.

Let J^1LM be the bundle of 1-*jets of (local) sections* of LM and s be a section of LM. Each j^1_ps is characterized by the *transversal n-subspace* $H_l = s_*(T_pM) \subset T_lLM$. Then, there is also a *canonical inclusion* $J^1LM \hookrightarrow LLM$, $j^1_ps \mapsto z$, where z is a basis of T_lLM, whose first n vectors span H_l and are applied by the *canonical form of LM* to the usual basis of $\mathbb{R}^n$.

By the previous canonical maps, it happens that $\mathcal{F}^2M$ is mapped one to one into the subset of J^1LM, corresponding with the *torsion-free transversal n-subspaces* in TLM.

Theorem 1 *We have the canonical embeddings:*

$$\mathcal{F}^2M \hookrightarrow J^1LM \hookrightarrow LLM$$

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3 Structural groups

Each $\mathcal{F}^rM$ is a principal bundle with respect to the group G^r_n of *r-jets at 0 of diffeomorphisms of* $\mathbb{R}^n$, $j^r_0\phi$, with $\phi(0) = 0$. The group G^1_n is identified with $\mathrm{GL}(n,\mathbb{R})$. There is a canonical inclusion of G^1_n into G^r_n, taking the r-jet at 0 of every linear map of $\mathbb{R}^n$; furthermore, G^r_n is the *semidirect product* of G^1_n with a nilpotent normal subgroup (see [10]).

Let's see the case of G^2_n: let S^2_n denote the additive group of symmetric bilinear maps of $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^n$; there is a monomorphism $\imath\colon \mathrm{S}^2_n \to \mathrm{G}^2_n$ defined by $\imath(s) = j^2_0\phi$, with $s = (s^i_{jk})$ and $\phi(u^i) := (u^i + \frac{1}{2}s^i_{jk}u^ju^k)$.

Theorem 2 *We obtain the split exact sequence of groups:*

$$0 \to \mathrm{S}^2_n \xrightarrow{\imath} \mathrm{G}^2_n \rightleftarrows \mathrm{G}^1_n \to 1$$

It makes G^2_n *isomorphic to the semidirect product* $\mathrm{G}^1_n \rtimes \mathrm{S}^2_n$, whose multiplication rule is $(a,s)(b,t) := (ab, b^{-1}s(b,b) + t)$. The isomorphism is given by $j^2_0\phi \mapsto (D\phi|_0\,,\, D\phi|_0^{-1}D^2\phi|_0)$.

For a linear group G, let $\mathfrak{g}$ denote the Lie algebra of G. The *first prolongation of* $\mathfrak{g}$ is defined by $\mathfrak{g}_1 := \mathrm{S}^2_n \cap L(\mathbb{R}^n, \mathfrak{g})$. We obtain that $G \rtimes \mathfrak{g}_1$ *is a subgroup of* $\mathrm{G}^1_n \rtimes \mathrm{S}^2_n$, and hence, a subgroup of G^2_n.

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4 1^{st} order G-structures

We define an *r-th order G-structure* of M as a *reduction* of $\mathcal{F}^rM$ to a subgroup $G \subset \mathrm{G}^r_n$, [6]. We exemplify the concept of a first order G-structure studying a volume on a manifold, which rarely is treated this way (see [1]).

Let's define a *volume on M* as a first order G-structure, with $G = \mathrm{SL}^\pm_n := \{a \in \mathrm{G}^1_n\colon |\det(a)| = 1\}$. If M is orientable, a volume on M has two components: two $\mathrm{SL}(n,\mathbb{R})$-structures for two equal, except sign, *volume n-forms*. For a general M, a volume corresponds to an *odd type n-form*, as in [2, pp. 21-27].

From *principal bundle theory* [5], $\mathrm{SL}^\pm_n$-structures are the sections of the *associated bundle* to LM and the left action of G^1_n on $\mathrm{G}^1_n/\mathrm{SL}^\pm_n$. This is the *volume bundle,* $\mathcal{V}M$. Furthermore, the sections of $\mathcal{V}M$ correspond to G^1_n*-equivariant functions* f of LM to $\mathrm{G}^1_n/\mathrm{SL}^\pm_n$. The isomorphisms $\mathrm{G}^1_n/\mathrm{SL}^\pm_n \simeq \mathrm{H}_n := \{k\,I_n : k > 0\} \simeq \mathbb{R}^+$, allow to write $f\colon LM \to \mathbb{R}^+$, verifying the equivariance condition $f(la) = |\det a|^{\frac{-1}{n}} f(l)$, $\forall a \in \mathrm{G}^1_n$.

Theorem 3 *We have the bijections:*

$$\textit{Volumes on } M \longleftrightarrow \mathrm{Sec}\,\mathcal{V}M \longleftrightarrow C^\infty_{equi}(LM, \mathbb{R}^+)$$

The Lie algebra of $\mathrm{SL}^\pm_n$ is $\mathfrak{sl}(n,\mathbb{R})$; its first prolongation is $\mathfrak{sl}(n,\mathbb{R})_1 = \{s \in \mathrm{S}^2_n\colon s^k_{ik} = 0\}$, it's a Lie algebra of infinite type.

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We define a *pseudo-Riemannian metric* as an $\mathrm{O}_{q,n\text{-}q}$-structure, with $\mathrm{O}_{q,n\text{-}q} := \left\{a \in \mathrm{G}^1_n\colon a^t\eta a = \eta := \begin{pmatrix} -I_q & 0 \\ 0 & I_{n\text{-}q} \end{pmatrix}\right\}$.

As above, we obtain bijections between the metrics and the sections of the associated bundle with typical fiber $\mathrm{G}^1_n/\mathrm{O}_{q,n\text{-}q}$, and also with the equivariant functions of LM in $\mathrm{G}^1_n/\mathrm{O}_{q,n\text{-}q}$. The first prolongation of $\mathfrak{o}_{q,n\text{-}q}$ is $\mathfrak{o}_{q,n\text{-}q\ 1} = 0$; a consequence of this fact is the uniqueness of the *Levi-Civita connection.*

A *pseudo-Riemannian conformal structure* is a $\mathrm{CO}_{q,n\text{-}q}$-structure, with $\mathrm{CO}_{q,n\text{-}q} := \mathrm{O}_{q,n\text{-}q} \cdot \mathrm{H}_n$ (direct product).

The first prolongation of its Lie algebra is $\mathfrak{co}_{q,n\text{-}q\ 1} = \{s \in \mathrm{S}^2_n\colon s^i_{jk} = \delta^i_j\mu_k + \delta^i_k\mu_j - \eta^{il}\eta_{jk}\mu_l,\ \mu = (\mu_i) \in \mathbb{R}^{n\,*}\} \simeq \mathbb{R}^{n\,*}$. The named second prolongation $\mathfrak{co}_{q,n\text{-}q\ 2}$ is equal to 0; this implies the uniqueness of the normal Cartan connection but we do not deal with this here (see [7]).

It is well known that conformal structures and volumes on M are the *extension* (see [3, p. 202]) of pseudo-Riemannian metrics with the groups $\mathrm{SL}^\pm_n$ and $\mathrm{CO}_{q,n\text{-}q}$, respectively.

Also it is noteworthy the following result:

Theorem 4 *A pseudo-Riemannian metric on M is the intersection of a pseudo-Riemannian conformal structure and a volume on M.*

This is proved in [8] as a consequence of the fact that $\mathrm{G}^1_n = \mathrm{SL}^\pm_n \cdot \mathrm{CO}_{q,n\text{-}q}$.

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5 2^{nd} order G-structures

We can define a *symmetric linear connection* (SLC) on M as a G^1_n-structure of second order. An SLC also can be seen as the image of an *injective homomorphism* of LM to $\mathcal{F}^2M$ [6].

From the *principal bundle theory*, SLCs on M are sections of the *associated bundle*, $\mathcal{D}M$, to $\mathcal{F}^2M$ and the action of G^2_n on $\mathrm{G}^2_n/\mathrm{G}^1_n \simeq \mathrm{S}^2_n$. Furthermore, each SLC, ∇, corresponds to a G^2_n-*equivariant function* $f^\nabla\colon \mathcal{F}^2M \to \mathrm{S}^2_n$, verifying $f^\nabla(z(a,s)) = a^{-1}f^\nabla(z)(a,a) + s$. We have the bijections:

$$\text{SLC's on } M \longleftrightarrow \mathrm{Sec}\,\mathcal{D}M \longleftrightarrow C^\infty_{\mathrm{equi}}(LM, \mathrm{S}^2_n)$$

Given two SLCs, ∇ and $\widehat{\nabla}$, *the difference* $f^\nabla - f^{\widehat{\nabla}}$ verifies $z(a,s) \mapsto a^{-1}(f^\nabla(z) - f^{\widehat{\nabla}}(z))(a,a)$. Then, it is projectable to a function $f\colon LM \to \mathrm{S}^2_n$ verifying $f(la) = a^{-1}f(l)(a,a)$, which corresponds to *a symmetric* $\binom{1}{2}$*-tensor* $\rho = (\rho^i_{jk})$ on M.

A *differential projective structure* (DPS) is a set of SLC's which have the same family of *pregeodesics*. We also can define a DPS on M as a $\mathrm{G}^1_n \rtimes \mathfrak{p}$*-structure*, Q, with $\mathfrak{p} := \{s \in \mathrm{S}^2_n\colon s^i_{jk} = \delta^i_j\mu_k + \mu_j\delta^i_k,\ \mu = (\mu_i) \in \mathbb{R}^{n\,*}\} \simeq \mathbb{R}^{n\,*}$.

Now, for two SLC *included* in the same DPS, i.e. $\nabla, \widehat{\nabla} \subset Q$, the tensor ρ, expressing *their difference*, is determined by the contraction $C(\rho) = (\rho^k_{ki})$, which is *an 1-form* on M.

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6 Prolongations

If B is a *first order G-structure* on M, a connection ∇ on B is a distribution of transversal n-subspaces, $H_l \subset T_lB \subset T_lLM$, $\forall l \in B$. If all the subspaces are *torsion-free*, they determine a *second order G-structure*, whose G^1_n-extension is a SLC on M. In this case we say that *B admits an* SLC or that *B* is *1-flat* and also that ∇ is a SLC on B.

Theorem 5 *If B is 1-flat, the set B^2 of 2-frames, corresponding with torsion-free transversal n-subspaces included in TB, is a $G \rtimes \mathfrak{g}_1$-structure.*

We name B^2 the holonomic prolongation of B (for a proof, see [7, p.150-155]). Reciprocally, it is verified that every $G \rtimes \mathfrak{g}_1$-structure is the prolongation of a first order G-structure.

The following result is an important theorem, arisen from the Weyl's *'Raumproblem'*, studied by Cartan and others. The theorem is in [4], with a correction revealed in [9].

Theorem 6 *If a group $G \subsetneq G^1_n$, with $n \geq 3$, satisfies that any given first order G-structure is 1-flat, then its Lie algebra is one of these: $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{o}_{q,n\text{-}q}$, $\mathfrak{co}_{q,n\text{-}q}$, $\mathfrak{gl}_{n,W}$ (algebra of endomorphisms with an invariant 1-dimensional subspace W) $\mathfrak{gl}_{n,W,c}$ (for every $c \in \mathbb{R}$, certain subalgebra of the last one) or, for $n = 4$, $\mathfrak{csp}(2,\mathbb{R})$.*

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7 Concluding remarks

A given second order G-structure, for $G \subset G^1_n$, determines by projection a G-structure $B \subset LM$ and by extension a G^1_n-structure of second order, i. e., a SLC admitted by B.

Typical examples of this are:

- A pseudo-Riemannian metric and its *Levi-Civita connection* are given by a second order $\mathrm{O}_{q,n\text{-}q}$-structure.
- An *equiaffine structure* on M is a SLC with a parallel volume; hence, it is a second order $\mathrm{SL}^\pm_n$-structure.
- A *Weyl structure* is a conformal structure with a SLC compatible; hence, it is a second order $\mathrm{CO}_{q,n\text{-}q}$-structure.

As we have seen, an 1-flat G-structure determines and is determined by a $G \rtimes \mathfrak{g}_1$-structure. Therefore, considering the second order we can analyze the intersections of geometrical structures like in the Th.4 with the first order.

- Always intersect a DPS on M with the prolongation of a volume on M; and the intersection gives a SLC doing the volume parallel. But not all SLC are obtained in this form.
- Not always intersect a DPS on M with the prolongation of a conformal structure. In case of they intersect, the intersection is the Levi-Civita connection of a metric of the conformal class.

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We have tried to clarify the relationships between the structures involved. Only simple relations, such as intersection, inclusion, reduction and extension, have been used for it, on account of the previous prolongation of 1-flat of G-structures.

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