An Approach to Differential Invariants of G-Structures

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Presentation

- The paradigm of scalar differential invariant of a G-structure is the scalar curvature $\mathbf{S_g} \colon M \to \mathbb{R}$ of a Riemannian manifold $(M, \mathbf{g})$. On each $m \in M$ it is defined $\mathbf{S_g}(m)$ from the values of $\mathbf{g}$ and its partial derivatives in m up to second order, that is to say, it depends of the 2-jet of the metric at m. It is said that the scalar curvature is invariant by diffeomorphisms because if $\varphi \colon M \to M$ is a diffeomorphism then the scalar curvature of the metric $\varphi^{-1*}\mathbf{g}$ verifies $\mathbf{S}_{\varphi^{-1*}\mathbf{g}} = \mathbf{S_g} \circ \varphi^{-1}$.
- The scalar differential invariants of the metrics are well studied and it is known the number of functionally independent metric invariants – depending on the dimension of M and the maximal order of derivatives of $\mathbf{g}$ involved – and how they are obtained from the Riemann curvature.
- This essay is a preparatory study in order to access the scalar differential invariants of other G-structures types. Particularly, we are interested in the differential invariants of conformal structures – do not confuse these with the *conformal invariants with weight*; in any case, the invariants of our work are of *weight zero*.

Introduction

- P, principal bundle over M with group G; right action $p \cdot g$; $P/G = M$.
- W, manifold; left action $g \cdot w$; $G \backslash W$ – topological space, in general not a manifold; notation: $[w]$.
- $P \times W$; right action $(p, w) \cdot g := (p \cdot g, g^{-1} \cdot w)$; we obtain the associated bundle $P(W) \equiv (P \times W)/G$; notation: $[p, w]$.
- LM, linear frame bundle of M with group Gl_n.
- $\mathrm{Gl}_n/\mathrm{O}_n$, quotient manifold; left action of Gl_n on $\mathrm{Gl}_n/\mathrm{O}_n$.
- $LM(\mathrm{Gl}_n/\mathrm{O}_n)$, the bundle of metrics: Giving a metric $\mathbf{g}$ is equivalent to give a section $\sigma_{\mathbf{g}} \colon M \to LM(\mathrm{Gl}_n/\mathrm{O}_n)$, $\sigma_{\mathbf{g}}(m) = [l, \mathrm{O}_n]$, with $l \in LM$ being some basis $\mathbf{g}$-orthonormal. Reciprocally, a section σ defines a metric establishing which are the orthonormal bases.
- The bundle $J^r LM(\mathrm{Gl}_n/\mathrm{O}_n)$ of r-jets of sections of $LM(\mathrm{Gl}_n/\mathrm{O}_n)$ has, for each element $j^r_m\sigma$, the information of the r-jet at m of the corresponding metric $\mathbf{g}$.
- We can define the scalar curvature function $\mathbf{S} \colon J^2 LM(\mathrm{Gl}_n/\mathrm{O}_n) \to \mathbb{R}$ so that $\mathbf{S} \circ j^2\sigma_{\mathbf{g}} = \mathbf{S}_{\mathbf{g}}$ – that is $\mathbf{S}(j^2_m\sigma_{\mathbf{g}}) := \mathbf{S}_{\mathbf{g}}(m)$.

Introduction

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- The bundle $J^r LM(\mathrm{Gl}_n/\mathrm{O}_n)$ of r-jets of sections of $LM(\mathrm{Gl}_n/\mathrm{O}_n)$ has, for each element $j^r_m\sigma$, the information of the r-jet at m of the corresponding metric $\mathbf{g}$.
- We can define the scalar curvature function $\mathbf{S}\colon J^2 LM(\mathrm{Gl}_n/\mathrm{O}_n) \to \mathbb{R}$ so that $\mathbf{S} \circ j^2\boldsymbol{\sigma}_{\mathbf{g}} = \mathbf{S}_{\mathbf{g}}$ – that is $\mathbf{S}(j^2_m\boldsymbol{\sigma}_{\mathbf{g}}) := \mathbf{S}_{\mathbf{g}}(m)$.
- Action of a diffeomorphism $\varphi\colon M \to M$
  - on LM: $l \mapsto \varphi_* \circ l$.
  - on $LM(W)$: $\bar{\varphi}\colon [l, w] \mapsto [\varphi_* \circ l, w]$.
  - on sections $\boldsymbol{\sigma}\colon M \to LM(W)$: $\boldsymbol{\sigma} \mapsto \bar{\varphi} \circ \boldsymbol{\sigma} \circ \varphi^{-1}$.
  - on jets of sections: $\widehat{\varphi}^r\colon j^r_m\boldsymbol{\sigma} \mapsto j^r_{\varphi(m)}(\bar{\varphi} \circ \boldsymbol{\sigma} \circ \varphi^{-1})$.
- In the case $W = \mathrm{Gl}_n/\mathrm{O}_n$, it is easy to prove that $\boldsymbol{\sigma}_{\varphi^{-1*}\mathbf{g}} = \bar{\varphi} \circ \boldsymbol{\sigma}_{\mathbf{g}} \circ \varphi^{-1}$ and, using this fact, to prove that

$$\mathbf{S}_{\varphi^{-1*}\mathbf{g}} = \mathbf{S}_{\mathbf{g}} \circ \varphi^{-1},\ \forall\, \mathbf{g},\ \forall\, \varphi \quad \Longleftrightarrow \quad \mathbf{S} = \mathbf{S} \circ \widehat{\varphi}^2,\ \forall\, \varphi$$

The action of diffeomorphisms on frame bundles and associated bundles

- Let F^rM be the r-th order frame bundle over M, with $\dim M = n$. This is a principal bundle with group G_n^r, the r-th jet group. The right action of $j_0^r\xi \in \mathrm{G}_n^r$ on $j_0^r\psi \in F^rM$ is defined by $j_0^r(\psi \circ \xi) \in F^rM$.

  ξ is a diffeomorphism between neighborhoods of $0 \in \mathbb{R}^n$, with $\xi(0) = 0$,
  ψ is a diffeomorphism between a neighborhood of 0 and an open set of M.

  The pseudogroup $\mathcal{D}M$ of diffeomorphisms between open sets of M acts on F^rM: The left action of $\varphi \in \mathcal{D}M$ on $j_0^r\psi \in F^rM$ is defined by $j_0^r(\varphi \circ \psi) \in F^rM$, – when $\psi(0) \in \operatorname{dom}\varphi$.

  The action of diffeomorphisms is transitive:

  $$\forall\, j_0^r\psi, j_0^r\psi' \in F^rM,\ \exists\, \varphi \in \mathcal{D}M \text{ such that } j_0^r(\varphi \circ \psi) = j_0^r\psi' .$$

- Let $F^rM(W)$ be an associated bundle to F^rM; the left action of $\varphi \in \mathcal{D}M$ defined there is given by – over $\operatorname{dom}\varphi$:

  $$\bar{\varphi}^r : F^rM(W) \to F^rM(W), \quad [j_0^r\psi, w] \mapsto [j_0^r(\varphi \circ \psi), w].$$

Scalar differential invariants

Definition (Invariants of associated bundles)

A function $f\colon F^rM(W) \to \mathbb{R}$ is a *scalar differential invariant* of $F^rM(W)$ if, $\forall\, w \in W$, $\forall\, \varphi \in \mathcal{D}M$ and $\forall\, j^r_0\psi \in F^rM$ with $\psi 0 \in \operatorname{dom}\varphi$, it is verified:

$$f[j^r_0\psi, w] = f[j^r_0(\varphi \circ \psi), w].$$

Equivalently, $\forall\, \varphi \in \mathcal{D}M$, $f \circ \bar{\varphi}^r = f$ over $\operatorname{dom}\varphi$.

We can reduce the problem of finding the scalar differential invariants of $F^rM(W)$ to a problem that is independent from M and its diffeomorphisms.

Theorem (Independence from M of the invariants)

The set of scalar differential invariants of $F^rM(W)$ is in bijective correspondence with the set of functions $h\colon \mathrm{G}^r_n \backslash W \to \mathbb{R}$ such that $h \circ \Pi$ is differentiable – being $\Pi\colon W \to \mathrm{G}^r_n\backslash W,\ w \mapsto [w]$.

Scalar differential invariants

Demostration

Given an invariant $f\colon F^rM(W) \to \mathbb{R}$, the function $h[w] := f[j^r_0\psi, w]$ is well defined because:

(*i*) for another $j^r_0\psi'$, we obtain a diffeomorphism $\varphi = \psi' \circ \psi^{-1}$ between neighbohoods of $\psi 0$ y $\psi' 0$, and then $f[j^r_0\psi', w] = f[j^r_0(\varphi \circ \psi), w] = f[j^r_0\psi, w]$;

(*ii*) if $w' \in [w]$, there exists $j^r_0\xi \in \mathrm{G}^r_n$ such that $[j^r_0\psi, w'] = [j^r_0\psi, j^r_0\xi \cdot w] = [j^r_0(\psi \circ \xi), w]$, then $f[j^r_0\psi, w'] = f[j^r_0(\psi \circ \xi), w]$ and, from (*i*), $f[j^r_0\psi, w'] = f[j^r_0\psi, w]$.

The differentiability follows from the identity $h \circ \Pi = f \circ \pi \circ \iota_z$, where $\pi\colon F^rM \times W \to F^rM(W)$ is the natural projection and $\iota_z\colon W \to F^rM \times W$, $\iota_z(w) := (z, w)$, with $z \in F^rM$.

Reciprocally, given $h\colon \mathrm{G}^r_n\backslash W \to \mathbb{R}$, the function $f[j^r_0\psi, w] := h[w]$, $\forall\, j^r_0\psi \in F^rM$, which obviously is invariant, is well defined because

$f[j^r_0(\psi \circ \xi), j^r_0\xi^{-1} \cdot w] = h[j^r_0\xi^{-1} \cdot w] = h[w], \quad \forall\, j^r_0\xi \in \mathrm{G}^r_n$

It is known that f is differentiable if and only if $f \circ \pi$ is differentiable; as $f \circ \pi = h \circ \Pi \circ \pi_2$, with $\pi_2\colon F^rM \times W \to W$ being the projection $\pi_2(z, w) = w$, the differentiability of f follows of the differentiability of $h \circ \Pi$.

Bundles of r-jets of sections

- Given a bundle E over M with fiber W, it is obtained the bundle of r-jets of local sections of E, $J^r E$, whose basis is M and fiber is the space $J^r_0(\mathbb{R}^n, W)$ of r-jets at 0 of applications of $\mathbb{R}^n$ to W.
- Let us see how work in the case $E = F^k M(W)$. We use the natural section of $F^k M$ induced from a chart (x, U):

$$\widehat{x}^k \colon U \to F^k M, \quad p \mapsto \widehat{x}^k p := j^k_0(x^{-1} \circ \tau_{x(p)}),$$

with $\tau_{x(p)}$ being the translation by $x(p)$ on $\mathbb{R}^n$. A local trivialization of E is:

$$\Psi^x \colon E|_U \to U \times W, \quad [\widehat{x}^k p, w] \mapsto (p, w).$$

Now, a local section $\sigma \colon U \to E$ is characterized by the application $\sigma^x \colon U \to W$ such that $\Psi^x(\sigma(p)) = (p, \sigma^x(p))$. We can write $\sigma = [\widehat{x}^k, \sigma^x]$. Then, a local trivialization of $J^r E$ is

$$J^r E|_U \to U \times J^r_0(\mathbb{R}^n, W), \quad j^r_p \sigma \mapsto \left(p, j^r_0(\sigma^x \circ x^{-1} \circ \tau_{xp})\right).$$

The bundle of jets of sections is an associated bundle

Proposition (1)

If $E = F^k M(W)$ is an associated bundle to $F^k M$ with typical fiber W then $J^r E$ is an associated bundle to $F^{r+k} M$ with typical fiber $J^r_0(\mathbb{R}^n, W)$.

Demostration

We consider the following action of $j_0^{r+k}\xi \in \mathrm{G}_n^{r+k}$ on $j_0^r \mu \in J_0^r(\mathbb{R}^n, W)$ defined by

$$j_0^r\left((j^k\xi \cdot \mu) \circ \xi^{-1}\right) \in J_0^r(\mathbb{R}^n, W), \qquad (1)$$

in which occurs the W-valued function given by

$$(j^k\xi \cdot \mu)(v) := j_0^k(\tau_{-\xi(v)} \circ \xi \circ \tau_v) \cdot \mu(v), \quad v \in \mathbb{R}^n;$$

the last dot here refers to the action of G_n^k on W.

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Demostration

(cont.) It is proved that (1) defines a left action of G^{r+k}_n on $J^r_0(\mathbb{R}^n, W)$ and then it produces an associated bundle $F^{r+k}M(J^r_0(\mathbb{R}^n, W))$. We can define a fibered application – in the domain of the chart x – by

$$\begin{aligned} \Lambda\colon J^rE &\longrightarrow F^{r+k}M(J^r_0(\mathbb{R}^n, W)) \\ j^r_p\sigma &\longmapsto \left[\widehat{x}^{r+k}p,\, j^r_0(\sigma^x \circ x^{-1} \circ \tau_{x(p)})\right], \end{aligned}$$

which is an isomorphism of bundles. We proved that this definition is independent of the chart and, thus, that it is a bundle isomorphism globally defined.

Action of diffeomorphisms on jets of sections

In a bundle $E = F^k M(W)$ it is defined the action of a diffeomorphism φ by:

$$\bar{\varphi}^k \colon E \to E, \ [j_0^k \psi, w] \mapsto [j_0^k(\varphi \circ \psi), w].$$

If σ is a section of E in a neighborhood of $p \in M$ then $\bar{\varphi}^k \circ \sigma \circ \varphi^{-1}$ is a section of E in a neighborhood of $\varphi(p)$. Thus, it is defined the action of φ on $J^r E$ by:

$$\widehat{\varphi}^{r,k} \colon J^r E \to J^r E, \quad j_p^r \sigma \mapsto j_{\varphi(p)}^r (\bar{\varphi}^k \circ \sigma \circ \varphi^{-1}).$$

Definition (Invariants of bundles of jets of sections)

A differentiable function $f \colon J^r(F^k M(W)) \to \mathbb{R}$ is a *scalar differential invariant (SDI) of r-th order over* $F^k M(W)$ if, $\forall \varphi \in \mathcal{D}M$, $f \circ \widehat{\varphi}^{r,k} = f$ on $\operatorname{dom} \varphi$.

The two definitions of SDI of r-th order over $F^k M(W)$ and of SDI of $F^{r+k} M(J_0^r(\mathbb{R}^n, W))$ are equivalent because of the commutative diagram:

$$\begin{array}{ccc} J^r(F^k M(W)) & \xrightarrow{\Lambda} & F^{r+k} M(J_0^r(\mathbb{R}^n, W)) \\ \downarrow \widehat{\varphi}^{r,k} & & \downarrow \bar{\varphi}^{r+k} \\ J^r(F^k M(W)) & \xrightarrow{\Lambda} & F^{r+k} M(J_0^r(\mathbb{R}^n, W)) \end{array}$$

Bundle of G-structures

Let G be a closed subgroup of Gl_n. The left action of Gl_n on Gl_n/G gives the associated bundle $M_G \equiv LM(\mathrm{Gl}_n/G)$ which is called the *bundle of G-structures*. Its sections, $\sigma\colon M \to M_G$, are in correspondence with the principal subbundles, $P = \{l \in LM\colon [l, G] \in \sigma(M)\}$, with group G which are the so-called G-structures on M.
Let $J^r M_G$ be the bundle of r-jets of local sections of M_G, whose fiber is $J^r_0(\mathbb{R}^n, \mathrm{Gl}_n/G)$. The action of $\varphi \in \mathcal{D}M$ on $J^r M_G$ is defined by:

$$\widehat{\varphi}^{r,1}\colon J^r M_G \to J^r M_G, \quad j^r_p\sigma \mapsto j^r_{\varphi(p)}(\bar{\varphi} \circ \sigma \circ \varphi^{-1}).$$

Definition (Scalar differential invariants of G-structures)

A *scalar differential invariant of r-th order of G-structures on M* is a differentiable function $f\colon J^r M_G \to \mathbb{R}$ which verifies $f \circ \widehat{\varphi}^{r,1} = f$ over $\mathrm{dom}\,\varphi$, $\forall \varphi \in \mathcal{D}M$.

[Addded in traslation: A paradigmatic example is the scalar curvature of (pseudo-)Riemannian geometry $\mathbf{S}\colon J^2 M_{\mathrm{O}_n} \to \mathbb{R}$, being $M_{\mathrm{O}_n} \equiv LM(\mathrm{Gl}_n/\mathrm{O}_n)$ (see the Introduction)]

Scalar differential invariants of G-structures

The isomorphism Λ of Proposition (1) allows to identify the bundle of jets of sections $J^r M_G$ and the associated bundle $F^{r+1}M\big(J^r_0(\mathbb{R}^n, \mathrm{Gl}_n/G)\big)$ with respect to the action of $j^{r+1}_0\xi \in \mathrm{G}^{r+1}_n$ on $j^r_0\mu \in J^r_0(\mathbb{R}^n, \mathrm{Gl}_n/G)$ defined by

$$j^r_0\big((D\xi \cdot \mu) \circ \xi^{-1}\big) \in J^r_0(\mathbb{R}^n, \mathrm{Gl}_n/G),$$

being $(D\xi \cdot \mu)(v) := D\xi|_v \cdot \mu(v)$, with $v \in \mathbb{R}^n$ and the last dot for the action of Gl_n on Gl_n/G.

Therefore, a scalar differential invariant of G-structures can be seen as a differentiable function $f\colon F^{r+1}M\big(J^r_0(\mathbb{R}^n, \mathrm{Gl}_n/G)\big) \to \mathbb{R}$ which verifies $f \circ \bar{\varphi}^{r+1} = f$, $\forall \varphi \in \mathcal{D}M$. Now, as one application to G-structures of the theorem of independence from M of the invariants, we obtain:

Theorem (Scalar differential G-invariants of r-th order)

The set of scalar differential invariants of r-th order of G-structures on a manifold M is in natural bijective correspondence with the functions $h\colon \mathrm{G}^{r+1}_n \backslash J^r_0(\mathbb{R}^n, \mathrm{Gl}_n/G) \to \mathbb{R}$ such that $h \circ \Pi$ is differentiable. We say that h is a scalar differential G-invariant of r-th order.

Minimum number of scalar differential G-invariants

It can be proved that the subspace of $\mathrm{G}_n^{r+1}\backslash J_0^r(\mathbb{R}^n, \mathrm{Gl}_n/G)$, denoted with G^r, which is the union of the orbits of maximal dimension – in $J_0^r(\mathbb{R}^n, \mathrm{Gl}_n/G)$ – is an open dense subset which – I conjecture – is a differentiable manifold whose dimension, m_r, will be the number of functionally independent scalar differential G-invariants of r-th order.

Theorem (Minimum number of scalar differential G-invariants)

Let G be a closed subset of Gl_n and $m = \dim G$. The number m_r of functionally independent scalar differential G-invariants of r-th order verifies

$$m_r \geq (n^2 - m)\binom{n+r}{n} - n\binom{n+r+1}{n} + n.$$

Demostration

Taking into account that $\dim J_0^r(\mathbb{R}^n, \mathrm{Gl}_n/G) = (n^2 - m)\binom{n+r}{n}$ and $\dim \mathrm{G}_n^{r+1} = n\left(\binom{n+r+1}{n} - 1\right)$, the result follows of $\dim(\mathrm{G}_n^{r+1}\backslash G^r) \geq \dim G^r - \dim \mathrm{G}_n^{r+1}$.

Minimum number of metric and conformal invariants

In the case of parallelizations of M – $G = \{I_n\}$ – or in the case of fields of projective frames – $G = \{kI_n : k \neq 0\}$ – the minimum number of the theorem coincides with the exact number of invariants.

In the metric case the minimum coincides with the exact number of O_n-invariants, except for[†] $n = 2$ and $r = 2$ in which there is an invariant.

In the conformal case, the exact number of CO_n-invariants is an open problem [see (*) in the references].

Min. number of O_n-invariants

$n \backslash r$	1	2	3	4	5
1	0	0	0	0	0
2	-	**0**[†]	2	5	9
3	-	3	18	45	87
4	-	14	74	200	424
5	-	40	215	635	1475
6	-	90	510	1644	4164

Min. number of CO_n-invariants

$n \backslash r$	1	2	3	4	5
1	-	-	-	-	-
2	-	-	-	-	-
3	-	-	-	10	31
4	-	-	39	130	298
5	-	19	159	509	1223
6	-	62	426	1434	3702

References

Pedro L. García, J. Muñoz Masqué,
Differential invariants on the bundles of G-structures,
Lect. Notes in Math. **1410** (1989), 177–201.

J. Muñoz Masqué, Antonio Valdés,
The number of functionally independent invariants of a pseudo-Riemannian metric,
J. Phys. A: Math. Gen. **27** (1994), 7843–7855.

R. A. Sarkisyan,
On differential invariants of geometric structures,
Izv. Ross. Akad. Nauk Ser. Mat. 70:2 (2006), 99–158

B. Kruglikov,*
Conformal differential invariants,
J. Geom. Phys. **113** (2017), 170–175.

*[Added in translation. This author gives the exact numbers of SDI for conformal structures. These coincide with the minima given by us, except for $n = 3$ & $r = 3$ with 1 SDI, for $n = 4$ & $r = 2$ with 3 SDI and for $n \geq 5$ & $r = 2$ with n SDI more that the minimum given.]