Geometrical Structures of Space-Time in General Relativity

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Abstract. Space-Time in general relativity is a dynamical entity because it is subject to the Einstein field equations. The space-time metric provides different geometrical structures: conformal, volume, projective and linear connection. A deep understanding of them has consequences on the dynamical role played by geometry. We present a unified description of those geometrical structures, with a standard criterion of naturalness, and then we establish relationships among them and try to clarify the meaning of associated geometric magnitudes.

Keywords: Volume of space-time, linear connection, Lorentzian conformal structure, projective differential geometry.

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INTRODUCTION

The space-time of general relativity (GR), from the point of view of differential geometry, is a 4-dimensional manifold $M$, with a $C^\infty$ atlas $\mathcal{A}$. The atlas is the differential structure of our space-time.

The principle of general covariance of GR establishes the invariance by diffeomorphisms. This leads us to think that a physical event is not a point, but a geometrical structure on a neighborhood. The fundamental geometrical structures that we can consider defined in the space-time are:

- Volume (4-form)
- Conformal structure (Lorentzian)
- Metric (Lorentzian)
- Linear connection (symmetric)
- Projective structure

They are defined in terms of the most primitive differential structure, via the concept of $G$-structure. Volume, conformal structure and metric are first order $G$-structures. But linear connection and projective structure are second order $G$-structures.

For certain $G$’s, classified in [8], every first order $G$-structure lead to a unique second order structure, named its prolongation. This is the case for the volume, metric and conformal structures.

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FRAME BUNDLES

The $r$-th order frame bundle $\mathcal{F}^r M$ is a quotient space of a subset of $\mathcal{A}$ [13, p. 38]. An $r$-frame, $j^r \varphi \in \mathcal{F}^r M$ is an $r$-jet at 0, where $x = \varphi^{-1}$ is a chart with 0 as a target.

We restrict our interest to first and second order. The first order frame bundle $\mathcal{F}^1 M$ is usually identified with the linear frame bundle $LM$. For a better understanding of the second order frame bundle consider $LLM$, the linear bundle of $LM$. There is a canonical inclusion $\mathcal{F}^2 M \hookrightarrow LLM$. $j^2 \varphi \mapsto j^1 \tilde{\varphi}$, where $\tilde{\varphi}$ is the diffeomorphism induced by $\varphi$, between neighborhoods of $0 \in \mathbb{R}^{n+n^2}$ and $j^1 \varphi \in LM$ [10, p. 139].

Let $J^1 LM$ be the bundle of 1-jets of (local) sections of $LM$ and $s$ be a section of $LM$. Each $j^1_s$ is characterized by the transversal $n$-subspace $H_1 = s_*(T_p M) \subset T_1 LM$ [6]. Then, there is also a canonical inclusion $J^1 LM \hookrightarrow LLM$, $j^1_s \mapsto z$, where $z$ is the basis of $T_1 LM$, whose first $n$ vectors span $H_1$ and correspond to the usual basis of $\mathbb{R}^n$, via the canonical form of $LM$, and the last $n^2$ vectors are the fundamental vectors corresponding to the standard basis of $\mathfrak{gl}(n, \mathbb{R})$ [9].

By the previous canonical maps, it happens that $\mathcal{F}^2 M$ is mapped one to one into the subset of $J^1 LM$, corresponding with the torsion-free transversal $n$-subspaces in $TLM$.

**Theorem 1** We have the canonical embeddings: [13, p. 54]

$$\mathcal{F}^2 M \hookrightarrow J^1 LM \hookrightarrow LLM$$

The $r$-th order frame bundles are principal bundles. They are fundamental because every natural bundle, in the categorial approach, can be described as an associated bundle to some $\mathcal{F}^r M$ [12] and the so-called geometrical objects can be identified with sections of those associated bundles.

STRUCTURAL GROUPS

The structural group of the principal bundle $\mathcal{F}^r M$ is the group $G_r^n$ of $r$-jets at 0 of diffeomorphisms of $\mathbb{R}^n$, $j_0^r \phi$, with $\phi(0) = 0$.

The group $G^n_1$ is identified with $GL(n, \mathbb{R})$. Then there is a canonical inclusion of $G^n_1$ into $G^n_r$, if we take the $r$-jet at 0 of every linear map of $\mathbb{R}^n$. Furthermore, $G^n_r$ is the semidirect product of $G^n_1$ with a nilpotent normal subgroup [16]. Let us see this decomposition for $G^n_2$. We consider the underlying additive group of the vector space $S^n_2$ of symmetric bilinear maps of $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^n$. Then there is a monomorphism $\iota: S^n_2 \hookrightarrow G^n_2$ defined by $\iota(s) = j_0^2 \phi$, with $s = (s_{jk}^i)$ and $\phi(u^j) := (u^i + \frac{1}{2} s_{jk}^i u^j u^k)$.

**Theorem 2** We obtain the split exact sequence of groups:

$$0 \rightarrow S^n_2 \overset{\iota}{\rightarrow} G^n_2 \overset{\varpi}{\rightarrow} G^n_1 \rightarrow 1$$

It makes $G^n_2$ isomorphic to the semidirect product $G^n_1 \rtimes S^n_2$, whose multiplication rule is $(a,s)(b,t) := (ab, b^{-1}s(b,b) + t)$. The isomorphism is given by $j_0^2 \phi \mapsto (D\phi|_0, D\phi|_0^{-1} D^2 \phi|_0)$. 
G-STRUCTURES

We define an $r$-th order $G$-structure on $M$ as a reduction of $\mathcal{F}^rM$ to a subgroup $G \subset G_n^r$ [10]. This idea of geometrical structure on $M$ concerns the classification of charts in $\mathcal{A}$, when the meaningful classes are chosen guided by an structural group.

We exemplify the concept of a $G$-structure studying a volume on a manifold, which rarely is treated this way [3]. Let us define a volume on $M$ as a first order $G$-structure $V$, with $G = \text{SL}^\pm_n := \{ a \in \text{GL}(n, \mathbb{R}) : \det a = \pm 1 \}$. For an orientable $M$, $V$ has two components for two $\text{SL}(n, \mathbb{R})$-structures, for two equal, except sign, volume $n$-forms. For a general $M$, volume corresponds to odd type $n$-form, as in [4, pp. 21-27].

From principal bundle theory [9], $\text{SL}^\pm_n$-structures are the sections of the bundle associated with $LM$ and the left action of $G_n^1$ on $G_n^1/\text{SL}^\pm_n$. This is the volume bundle, $\mathcal{V}M$. Furthermore, the sections of $\mathcal{V}M$ correspond to $G_n^1$-equivariant functions $f$ of $LM$ to $G_n^1/\text{SL}^\pm_n$. The equivariance condition is $f(\la) = \det a \cdot f(l)$, $\forall a \in G_n^1$.

We have the bijections:

\[
\text{Volumes on } M \leftrightarrow \text{Sec } \mathcal{V}M \leftrightarrow C^\infty_{\text{eq}}(LM, G_n^1/\text{SL}^\pm_n)
\]

The isomorphisms $G_n^1/\text{SL}^\pm_n \simeq H_n$, with $H_n := \{ kI_a : k > 0 \}$ and $H_n \simeq \mathbb{R}^+$, the multiplicative group of positive numbers, allow to represent a volume as an (odd) scalar density on $M$.

SECOND ORDER STRUCTURES

We can view a symmetric linear connection (SLC) on $M$ as a $G_n^1$-structure of second order. A SLC is also the image of an injective homomorphism of $LM$ to $\mathcal{F}^2M$ [10].

From the principal bundle theory [9], SLC’s on $M$ are sections of the SLC bundle, $\mathcal{D}M$, associated with $\mathcal{F}^2M$ and the action of $G_n^2$ on $G_n^2/G_n^1 \simeq S_n^2$. Furthermore, each SLC, $\nabla$, corresponds to a $G_n^2$-equivariant function $f^\nabla: \mathcal{F}^2M \rightarrow S_n^2$, verifying $f^\nabla(z(a,s)) = a^{-1}f^\nabla(z)(a,a) + s$.

We have the bijections:

\[
\text{SLC’s on } M \leftrightarrow \text{Sec } \mathcal{D}M \leftrightarrow C^\infty_{\text{eq}}(\mathcal{F}^2M, S_n^2)
\]

Given two SLC’s, $\nabla$ and $\tilde{\nabla}$, the difference function $f^\nabla - f^{\tilde{\nabla}}: \mathcal{F}^2M \rightarrow S_n^2$ verifies $z(a,s) \mapsto a^{-1}(f^\nabla(z) - f^{\tilde{\nabla}}(z))(a,a)$. Then, it is projectable to a function $f: LM \rightarrow S_n^2$ verifying $f(\la) = a^{-1}f(l)(a,a)$, which corresponds to a tensor $\rho = (\rho_{i,j}^l)$ on $M$.

A projective structure (PS) is an equivalence class of SLC’s which have the same family of pregeodesics. This is the cornerstone to understand the freely falling bodies in GR [5]. We can define a PS on $M$ as a second order $G_n^1 \times p$-structure, $Q$, with $p := \{ s \in S_n^2 : \delta^i_j \mu_k + \mu_j \delta^i_k, \mu = (\mu_i) \in \mathbb{R}^{n*} \}$.

Now, for two SLC included in the same PS (i.e. literally $\nabla, \tilde{\nabla} \subset Q$) the tensor $\rho$, expressing their difference, is determined by the contraction $C(\rho) = (\rho_{s,i}^l)$, which is an 1-form on $M$. 
PROLONGATIONS

Let $B$ be a first order $G$-structure. A connection in $B$ is a distribution $H$ of transversal $n$-subspaces, $H_l \subset T_l B$. If the subspaces are free-torsion, these determine a second order $G$-structure, whose $G^1_n$-extension [7, p. 206] is a SLC on $M$. Then, we say that $B$ admits a SLC. Let us give two examples:

- A SLC and a parallel volume is an equiaffine structure on $M$ [11]; hence, it is a second order SL$_n^+$-structure.
- A SLC compatible with a conformal structure is a Weyl structure; hence, it is a second order CO($n$)-structure [2].

For a linear group $G$, let $\mathfrak{g}$ denote the Lie algebra of $G$. The first prolongation of $\mathfrak{g}$ is defined by $\mathfrak{g}^1 := S^2_n \cap L(\mathbb{R}^n, \mathfrak{g})$. We obtain that $G \rtimes \mathfrak{g}^1$ is a subgroup of $G^1_n \rtimes S^2_n$, and hence, a subgroup of $G^2_n$ (see more details in [1]).

**Theorem 3** Let $B \subset LM$ be a $G$-structure, admitting a SLC. Then, the set of 2-frames, corresponding with torsion-free transversal $n$-subspaces which are included in $TB$, is a reduction of $F^2 M$ to $G \rtimes \mathfrak{g}^1$. It is named the prolongation of $B$ and denoted by $B^2$ (for a proof, see [13, pp. 150-155]).

Let us give a well known example: if $B$ is an $O(n)$-structure, $B^2$ is isomorphic to $B$ on account of $O(n)_1 = \{0\}$; this explain the uniqueness of Levi-Civita connection.

There is an important theorem [8] classifying the groups $G$ such that every $G$-structure admits a SLC: the groups of volume, metric and conformal structures, and a class of groups preserving an 1-dimensional distribution, have this property.

CONCLUDING REMARKS

We have done a unified description of the geometrical structures that have been used by GR to define intrinsic properties of the space-time. The unifying criterion, we used for it, not only is natural in the sense that geometric objects are sections of bundles associated with the $F^2 M$ frame bundles [16], but also in the sense that the objects themselves are reductions of $F^2 M$. Therefore, we have not considered a linear connection with torsion because it is a section of an associated bundle of $F^2 M$, but not a reduction.

We have tried to clarify the relationships between the structures involved. Only simple relations, such as intersection, inclusion, reduction and extension, have been used for it, on account of the previous prolongation of $G$-structures admitting SLC. For instance, it follows readily from the last section that the classical equiaffine or Weyl structures can be defined as the intersection of a SLC with the prolongation of a volume or a conformal structure, respectively.

Recently, some of my research [14] have been taken into consideration for one of the lines of thought about quantum gravity [15]. This contribution is a set of my latest reflections and conclusions about geometrical structures with an eye on the applications to physics.
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