# Characterization of curvature forms in dimension four

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# Presentation

- It is well known (DeTurck, Goldschmidt, Talvacchia) that in dimension 3 every 2-form with values in a semi-simple Lie algebra is generically and locally the curvature of a connection form.
- In dimension 4, as shown below, a 2-form with values in a Lie algebra for which certain condition holds (in particular, for a semi-simple Lie algebra) is generically and locally a curvature form of a connection if and only if it is a solution to a second-order partial differential system.

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# Introduction

- Let $P \xrightarrow{\pi} M$ be a principal bundle over an $n$-dimensional manifold $M$ with structure Lie group $\mathcal{G}$. If $\mathfrak{g}$ is the Lie algebra of $\mathcal{G}$, a connection on $P$ can be given by a $\mathfrak{g}$-valued 1-form on $P$, $\omega \in \Omega^1(P) \otimes \mathfrak{g}$, verifying

  1) $\omega(A^*) = A, \qquad \forall A \in \mathfrak{g},$
  2) $R_a^* \omega = Ad_{a^{-1}} \cdot \omega, \quad \forall a \in \mathcal{G}.$

  Once we give a local section $\sigma \colon U \subset M \to P$ we obtain a trivialization $U \times \mathcal{G} \cong \pi^{-1}(U) \subset P$. Since we will work on a local problem, we will take, for simplicity, $P = M \times \mathcal{G}$.

- Hence, a connection will be described for a $\mathfrak{g}$-valued 1-form on $M$, $w \in \Omega^1(M) \otimes \mathfrak{g}$. Specifically, the connection form $\omega$ on $P = M \times \mathcal{G}$ determines and is determined by $w$ by the equality

$$\omega_{(x,e)}(X_x, 0_e) = w_x(X_x), \qquad \forall X_x \in T_x M, \quad \forall x \in M,$$

  with $e$ being the unit of $\mathcal{G}$. For other tangent vectors or in other points of $P$, the form $\omega$ is fixed by 1) and 2).
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$$F = dw + \frac{1}{2} w \wedge w,$$

where the exterior product is taken with respect to the Lie bracket of $\mathfrak{g}$; in this case $(w \wedge w)(X, Y) := [w(X), w(Y)] - [w(Y), w(X)]$. Sometimes it is denoted $w \wedge w \equiv [w, w]$.

- If the group of structure is abelian, the problem of *"When is a $\mathfrak{g}$-valued 2-form a curvature?"* is already solved and well known by all:

  As the Lie bracket is always naught, $F = dw$ and the components of $F$ (which are as many 2-forms on $M$ as the dimension of $\mathfrak{g}$) must be exact forms to be $F$ the curvature of a connection.

  Hence, the Poincaré lemma assures that $F$ is (locally) a curvature form if it satisfies the equation $dF = 0$; which is no other than the Bianchi identity: $dF + w \wedge F = 0$, in the abelian case.

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- In nonabelian case, the Bianchi identity $dF + w \wedge F = 0$ remains a necessary condition for $F$ to be a curvature. But the equation includes the unknown $w$, and even if the identity is solved for some existent $w$, that doesn't assure that $F$ is a curvature of $w$ or of someone else connection.

  Funnily, when the base manifold is of dimension 4 and thanks to the Bianchi identity, there is a simple trick that works on a large class of Lie algebras including semisimple algebras, and that solves the problem generically. From now on $\dim M = 4$.

- We consider the following assumption:

($\alpha$) For every $x \in M$ there exists $F_x \in \wedge^2 T_x^* M \otimes \mathfrak{g}$ such that the linear map

$$\phi(F_x)\colon T_x^* M \otimes \mathfrak{g} \to \wedge^3 T_x^* M \otimes \mathfrak{g}, \qquad \phi(F_x)(w_x) = F_x \wedge w_x,$$

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*(i) If* $(\alpha)$ *holds for* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$*, then it also holds for* $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

*(ii) If* $(\alpha)$ *holds, then the center of* $\mathfrak{g}$ *is zero and* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

*(iii) The assumption* $(\alpha)$ *holds if and only if the following holds:*

*($\beta$) There exist* $A_{ij} \in \mathfrak{g}$, $1 \leq i < j \leq 4$, *such that for* $A_i \in \mathfrak{g}$ *the equations*

$$\begin{cases} [A_{12}, A_3] + [A_{23}, A_1] - [A_{13}, A_2] = 0 \\ [A_{12}, A_4] + [A_{24}, A_1] - [A_{14}, A_2] = 0 \\ [A_{13}, A_4] + [A_{34}, A_1] - [A_{14}, A_3] = 0 \\ [A_{23}, A_4] + [A_{34}, A_2] - [A_{24}, A_3] = 0 \end{cases} \qquad (1)$$

*imply* $A_1 = \ldots = A_4 = 0$.

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# A class of Lie algebras

**Proof.**

*(i) If $(\alpha)$ holds for $\mathfrak{g}_1$ and $\mathfrak{g}_2$, then it also holds for $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$.*

It is immediate.

# A class of Lie algebras

## Proof.

*(ii) If $(\alpha)$ holds, then the center of $\mathfrak{g}$ is zero and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.*

It is necessary for the injectivity and surjectivity of $\phi(F_x)$:

1) if $A$ is a vector in the center of $\mathfrak{g}$, then $\phi(F_x)(\mu_x \otimes A) = 0$, $\forall \mu_x \in T_x^* M$, $\forall F_x \in \wedge^2 T_x^* M \otimes \mathfrak{g}$, then injectivity implies $\mu_x \otimes A = 0$, then $A = 0$;
2) $\mathrm{im}\phi(F_x) \subseteq \wedge^3 T_x^* M \otimes [\mathfrak{g}, \mathfrak{g}] \subseteq \wedge^3 T_x^* M \otimes \mathfrak{g}$, for all $F_x \in \wedge^2 T_x^* M \otimes \mathfrak{g}$, then surjectivity implies $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

# A class of Lie algebras

## Proof.

*(iii)* Suppose ($\beta$) holds. Let $(x^1, \ldots, x^4)$ be coordinates on $U \subseteq M$ and let $(B_1, \ldots, B_m)$ be a basis of $\mathfrak{g}$. Define $F_x = \sum_{h<i} (dx^h \wedge dx^i)_x \otimes A_{hi}$, $x \in U$. As $\dim T_x^* M \otimes \mathfrak{g} = \dim \wedge^3 T_x^* M \otimes \mathfrak{g}$, $\phi(F_x)$ is an isomorphism iff $\ker \phi(F_x) = 0$. Assume $w_x = \mu_j^k (dx^j)_x \otimes B_k \in \ker \phi(F_x)$. We set $A_j = \mu_j^k B_k$. Then, we have

$$\begin{aligned}
\phi(F_x)(w_x) &= \phi(F_x)\left((dx^j)_x \otimes A_j\right) \\
&= (dx^1 \wedge dx^2 \wedge dx^3)_x \otimes ([A_{12}, A_3] + [A_{23}, A_1] - [A_{13}, A_2]) \\
&+ (dx^1 \wedge dx^2 \wedge dx^4)_x \otimes ([A_{12}, A_4] + [A_{24}, A_1] - [A_{14}, A_2]) \\
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\end{aligned}$$

and, from $\phi(F_x)(w_x) = 0$ and ($\beta$), we obtain $A_1 = \ldots = A_4 = 0$, which implies $w_x = 0$. Thus, the assumption ($\alpha$) holds.

# A class of Lie algebras

## Proof.

*(iii)* Conversely, assume that $(\alpha)$ holds and let $F_x = \sum_{h<i}(dx^h \wedge dx^i)_x \otimes A_{hi}$ be a 2-form in $\wedge^2 T_x^* M \otimes \mathfrak{g}$, $x \in U$, such that $\phi(F_x)$ is an isomorphism. Let $A_1, \dots, A_4$ be four arbitrary vectors in $\mathfrak{g}$. According to the equation

$$\begin{aligned}
\phi(F_x)(w_x) &= \phi(F_x)\left((dx^j)_x \otimes A_j\right) \\
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\end{aligned}$$

the 1-form $w_x = (dx^j)_x \otimes A_j$ belongs to the kernel of $\phi(F_x)$ (and hence $w_x = 0$) if and only if the equations (1) in $(\alpha)$ hold. Thus, the assumption $(\beta)$ follows.

# A class of Lie algebras

## Proof.

*(iv) If $(\alpha)$ holds, $\mathcal{O}$ is an $\mathrm{Ad}\mathcal{G}$-invariant dense open subbundle in $\wedge^2 T^*M \otimes \mathfrak{g}$.*
We order conveniently the bases $dx^h \otimes B_k$; $dx^h \wedge dx^i \otimes B_k$, $h < i$;
$dx^h \wedge dx^i \wedge dx^j \otimes B_k$, $h < i < j$, of $T^*U \otimes \mathfrak{g}$, $\wedge^2 T^*U \otimes \mathfrak{g}$, $\wedge^3 T^*U \otimes \mathfrak{g}$.
If $F = \sum_{h<i} F^j_{hi} dx^h \wedge dx^i \otimes B_j$, the $4m \times 4m$ matrix of $\phi(F)$ is:

$$\Lambda(F) = \begin{pmatrix} \Lambda_{11}(F) & \dots & \Lambda_{1m}(F) \\ \vdots & \ddots & \vdots \\ \Lambda_{m1}(F) & \dots & \Lambda_{mm}(F) \end{pmatrix},$$

$$\text{with} \quad \Lambda_{hj}(F) = \begin{pmatrix} c^h_{ij} F^i_{23} & -c^h_{ij} F^i_{13} & c^h_{ij} F^i_{12} & 0 \\ c^h_{ij} F^i_{24} & -c^h_{ij} F^i_{14} & 0 & c^h_{ij} F^i_{12} \\ c^h_{ij} F^i_{34} & 0 & -c^h_{ij} F^i_{14} & c^h_{ij} F^i_{13} \\ 0 & c^h_{ij} F^i_{34} & -c^h_{ij} F^i_{24} & c^h_{ij} F^i_{23} \end{pmatrix},$$

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## Proof.

*(iv) If $(\alpha)$ holds, $\mathcal{O}$ is an $\mathrm{Ad}\mathcal{G}$-invariant dense open subbundle in $\wedge^2 T^*M \otimes \mathfrak{g}$.*

where $(c_{kl}^j)_{k<l}$ denote the structural constants, $[B_k, B_l] = c_{kl}^j B_j$.
Hence $\det \Lambda(F)$ is a homogeneous polynomial of degree $4m$ in the functions $F_{hi}^j$ whose coefficients are homogeneous polynomials of degree $4m$ in $\mathbb{Z}[c_{kl}^j]$, and $\det \Lambda(F)$ does not vanish identically by virtue of the hypothesis that $(\alpha)$ holds; hence $\mathcal{O}_x$ is a dense open subset for every $x \in M$. This argument proves (iv) taking into account that, as a simple calculation shows, we have

$$\phi(\mathrm{Ad}_g \circ F) = \mathrm{Ad}_g \circ \phi(F) \circ \mathrm{Ad}_{g^{-1}}, \quad \forall g \in \mathcal{G}.$$

# A class of Lie algebras

## Proof.

*(v) The assumption $(\beta)$ holds for $\mathfrak{g}$ if and only if it holds for $\mathfrak{g}^c = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.*

If $\phi(F_x)\colon T_x^*M \otimes \mathfrak{g} \to \wedge^3 T_x^*M \otimes \mathfrak{g}$ is an isomorphism, the same holds for $\phi(F_x \otimes 1_{\mathbb{C}})\colon T_x^*M \otimes \mathfrak{g}^c \to \wedge^3 T_x^*M \otimes \mathfrak{g}^c$. If $(\alpha)$ does not hold for $\mathfrak{g}$, then there exists a point $x \in M$ such that the matrix $\Lambda(F_x)$ is singular for every $F_x$ in $\wedge^2 T_x^*M \otimes \mathfrak{g}$, or equivalently $\det \Lambda(F_x) = 0$. This means that certain polynomials in $\mathbb{Z}[c_{kl}^j]$ vanish. Hence $(\alpha)$ does not hold for $\mathfrak{g}^c$ either, as $\mathfrak{g}$ and $\mathfrak{g}^c$ have the same structural constants. This proves (v). □

## Proposition (Mostow, Shnider)

*Every semisimple finite-dimensional Lie algebra satisfies the assumption $(\beta)$.*

However, the class of algebras satisfying the condition $(\beta)$ is strictly larger than that of semisimple Lie algebras.

# Curvature forms in dimension 4

Let us consider the system

$$dF = F \wedge G, \tag{2}$$

$$F = dG + \tfrac{1}{2} G \wedge G, \tag{3}$$

on a manifold $M$ of arbitrary dimension, where $G \in \Omega^1(M) \otimes \mathfrak{g}$, $F \in \Omega^2(M) \otimes \mathfrak{g}$.

**Lemma**

*The system* (2)–(3) *is formally integrable.*

The result is obtained using the standard definitions and results of the theory of formal integrability, such as are developed, for instance, in the book R. L. Bryant et al..

In summary, the Lemma is proved verifying that a formal solution of order $k$ can be extended to a formal solution of order $k + 1$.

# Curvature forms in dimension 4

## Theorem

*Assume* $(\beta)$ *holds for the Lie algebra* $\mathfrak{g}$ *of the Lie group* $\mathcal{G}$. *A* $\mathfrak{g}$*-valued 2-form* $F$ *on* $M$ *taking values in* $\mathcal{O}$, *is the curvature form of a connection on the principal bundle* $M \times \mathcal{G} \to M$ *if and only if the following equation holds:*

$$F = d\left(\phi(F)^{-1}(dF)\right) + \tfrac{1}{2}\left(\phi(F)^{-1}(dF)\right) \wedge \left(\phi(F)^{-1}(dF)\right). \tag{4}$$

*Furthermore, if* $M$ *is of class* $C^\omega$, *given a point* $x_0 \in M$ *and a 2-jet* $j^2_{x_0}F_0$ *at* $x_0$ *in* $\wedge^2 T^*M \otimes \mathfrak{g}$ *such that,* $F_0(x_0) \in \mathcal{O}$ *and*

$$F_0(x_0) = d\left(\phi(F_0)^{-1}(dF_0)\right)_{x_0} + \tfrac{1}{2}\left(\phi(F_0)^{-1}(dF_0)\right)_{x_0} \wedge \left(\phi(F_0)^{-1}(dF_0)\right)_{x_0},$$

*then there exist an open neighbourhood* $U$ *of* $x_0$ *and a solution* $F \in \Gamma(U, \mathcal{O})$ *to the equation (4) such that* $j^2_{x_0}F = j^2_{x_0}F_0$.

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## Proof.

Assume that a connection form $G$ exists on $M \times \mathcal{G}$ whose curvature form is $F$. Then, from the structure equation $F = dG + \frac{1}{2} G \wedge G$ we obtain $dF = F \wedge G = \phi(F)(G)$. Hence $G = \phi(F)^{-1}(dF)$ and the equation (4) holds. Conversely, if $F$ satisfies the condition (4), then the 1-form $\phi(F)^{-1}(dF)$ defines a connection form whose curvature is $F$.

Finally, if $(\beta)$ holds for $\mathfrak{g}$, then the equation (4) is equivalent to the system (2)–(3) as follows by setting $G = \phi(F)^{-1}(dF)$. Hence the second part of the statement is an immediate consequence of the previous Lemma taking account of the fact that under the assumptions of the statement, the formal integrability of (4) implies the existence of solutions with prescribed initial condition.

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# Curvature forms in dimension 4

## Remarks

- *The equation* (4) *is quasi-linear but strongly non-linear, as the coefficients of the second-order terms* $F^{\alpha}_{hi,x^j x^k}$ *are involved expressions of* $F$ *due to the inverse matrix* $\phi(F)^{-1}$.
- *Moreover, it is worth mentioning that this equation is invariant under gauge transformations; namely, if* $F$ *is a solution to* (4), *then the* 2*-form*

$$(\gamma \cdot F)(x) = \mathrm{Ad}_{\gamma(x)} \circ F(x), \quad \forall x \in M,$$

  *is also a solution to* (4) *for every* $\gamma \in C^{\infty}(M, \mathcal{G})$.
- *If* $F$ *is a global section in* $\mathcal{O}$, *an open covering* $\{U_i\}$ *of* $M$ *exists such that* $F|_{U_i}$ *is a curvature form for every* $i$, *then there exists a globally defined unique* 1*-form* $G$ *the curvature of which is* $F$. *In fact, assume* $G_i$ *is a connection form on* $U_i$ *whose curvature form is* $F|_{U_i}$. *On* $U_i \cap U_j$ *we have* $dF = F \wedge G_i$ *and* $dF = F \wedge G_j$. *Subtracting both equations and recalling* $\phi(F)$ *is an isomorphism by hypothesis, we conclude* $G_i|_{U_i \cap U_j} = G_j|_{U_i \cap U_j}$.

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# Example with $\mathfrak{su}(2)$

Let $(B_1, B_2, B_3)$ be a basis for $\mathfrak{su}(2)$ such that $[B_1, B_2] = B_3$, $[B_2, B_3] = B_1$, $[B_3, B_1] = B_2$. Assume $F = \sum_{h<i} F^{\alpha}_{hi} dx^h \wedge dx^i \otimes B_\alpha \in \Omega^2(M) \otimes \mathfrak{su}(2)$ is a section of the open subbundle $\mathcal{O}$ in the Proposition-(iv).
In the bases $\{dx^h \otimes B_\alpha\}$ and $\{dx^h \wedge dx^i \wedge dx^j \otimes B_\alpha\}$, the matrix of $\phi(F)$ is a $12 \times 12$ matrix:

$$\Lambda(F) = \begin{pmatrix} 0 & \Lambda_{12} & -\Lambda_{13} \\ -\Lambda_{12} & 0 & \Lambda_{23} \\ \Lambda_{13} & -\Lambda_{23} & 0 \end{pmatrix}, \quad \text{with}$$

$$\Lambda_{ij} = \begin{pmatrix} -F^k_{23} & F^k_{13} & -F^k_{12} & 0 \\ -F^k_{24} & F^k_{14} & 0 & -F^k_{12} \\ -F^k_{34} & 0 & F^k_{14} & -F^k_{13} \\ 0 & -F^k_{34} & F^k_{24} & -F^k_{23} \end{pmatrix}, \; (i,j,k) \in \{(1,2,3),(1,3,2),(2,3,1)\}.$$

To compute the inverse matrix if there exists we can proceed efficiently operating by blocks as follows:

# Example with $\mathfrak{su}(2)$

We set

$$\Lambda^{-1}(F) = \begin{pmatrix} \Lambda^{11} & \Lambda^{12} & \Lambda^{13} \\ \Lambda^{21} & \Lambda^{22} & \Lambda^{23} \\ \Lambda^{31} & \Lambda^{32} & \Lambda^{33} \end{pmatrix},$$

All of $\Lambda^{ij}$ being $4 \times 4$ matrix blocks. By imposing

$$\Lambda(F) \cdot \Lambda^{-1}(F) = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix},$$

we obtain a system of $4 \times 4$ matrix equations, with the blocks $\Lambda_{ij}$, $\Lambda^{ij}$ involved. We make the additional hypothesis that $\det \Lambda_{12}$, $\det \Lambda_{13}$, and $\det \Lambda_{23}$ do not vanish, which does not affect the generic character of $F$.

Then, after some easy calculations we can obtain the formulas which give the blocks $\Lambda^{ij}$ of the inverse matrix $\Lambda^{-1}(F)$, in function of the blocks $\Lambda_{ij}$ of $\Lambda(F)$ and its inverses $\Lambda_{ij}^{-1}$, allowing one to compute them efficiently.

# Example with $\mathfrak{su}(2)$

Moreover, we have

$$
\begin{aligned}
dF &= \sum_{h<i<j} F^{\alpha}_{hij} dx^h \wedge dx^i \wedge dx^j \otimes B_\alpha \\
&= \left( F^{\alpha}_{23,x^1} - F^{\alpha}_{13,x^2} + F^{\alpha}_{12,x^3} \right) dx^1 \wedge dx^2 \wedge dx^3 \otimes B_\alpha \\
&\quad + \left( F^{\alpha}_{24,x^1} - F^{\alpha}_{14,x^2} + F^{\alpha}_{12,x^4} \right) dx^1 \wedge dx^2 \wedge dx^4 \otimes B_\alpha \\
&\quad + \left( F^{\alpha}_{34,x^1} - F^{\alpha}_{14,x^3} + F^{\alpha}_{13,x^4} \right) dx^1 \wedge dx^3 \wedge dx^4 \otimes B_\alpha \\
&\quad + \left( F^{\alpha}_{34,x^2} - F^{\alpha}_{24,x^3} + F^{\alpha}_{23,x^4} \right) dx^2 \wedge dx^3 \wedge dx^4 \otimes B_\alpha .
\end{aligned}
$$

Then, $\phi(F)^{-1}(dF)$ is computed by making the following matrix product:

# Example with $\mathfrak{su}(2)$

$$\begin{pmatrix} \Lambda^{11} & \Lambda^{12} & \Lambda^{13} \\ \Lambda^{21} & \Lambda^{22} & \Lambda^{23} \\ \Lambda^{31} & \Lambda^{32} & \Lambda^{33} \end{pmatrix} \begin{pmatrix} F^1 \\ F^2 \\ F^3 \end{pmatrix}, \quad \text{with} \quad F^\beta = \begin{pmatrix} F^\beta_{123} \\ F^\beta_{124} \\ F^\beta_{134} \\ F^\beta_{234} \end{pmatrix}, \quad \beta = 1, 2, 3.$$

Namely,

$$\begin{aligned} \phi(F)^{-1}(dF) &= \sum_{\alpha,\beta} \Lambda^{\alpha\beta}(F) F^\beta \otimes B_\alpha \\ &= \sum_{\alpha,\beta} \Big[ \left(\Lambda^{\alpha\beta}\right)_{i1}(F) F^\beta_{123} + \left(\Lambda^{\alpha\beta}\right)_{i2}(F) F^\beta_{124} \\ &\quad + \left(\Lambda^{\alpha\beta}\right)_{i3}(F) F^\beta_{134} + \left(\Lambda^{\alpha\beta}\right)_{i4}(F) F^\beta_{234} \Big] dx^i \otimes B_\alpha. \end{aligned}$$

If the equation (4) is satisfied, this will be the unique connection form with curvature $F$.

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