Probabilistic inverse problem to characterize tissue-equivalent material mechanical properties

Nicolas Bochud\textsuperscript{1} and Guillermo Rus\textsuperscript{1}

\textit{Special Issue: Novel Embedded Systems for Ultrasonic Imaging and Signal Processing - Medical Imaging}

Abstract

The understanding of internal processes that affect the changes of consistency of soft tissue is a challenging problem. An ultrasound monitoring Petri dish has been designed to control in real-time the evolution of relevant mechanical parameters during engineered tissue formation processes. A better understanding of the measured ultrasonic signals required the use of numerical models of the ultrasound-tissue interactions. The extraction of relevant data and its evolution with sufficient sensitivity and accuracy is addressed by applying well-known signal processing techniques to both the experimental and numerically-predicted measurements. In addition, a stochastic model-class selection formulation is used to rank which of the proposed interaction models are more plausible. The sensitivity of the system is verified by monitoring a gelation process.

Index Terms

Tissular mechanics, Inverse Problem, Model-class selection, Ultrasound, Signal processing, Tissue characterization.

I. INTRODUCTION

The rational principles of continuum mechanics are proposed together with a formal signal processing framework to address the problem of characterizing mechanical properties of tissue cultures based on noninvasive and non-ionizing ultrasonic measurements.

\textsuperscript{1}Department of Structural Mechanics, University of Granada, Politécnico de Fuentenueva, 18071 Granada, Spain. Email: \{nbochud,grus\}@ugr.es, www.ugr.es/~grus
In the non-destructive evaluation (NDE) community, signal processing techniques have played an increasing role over the last decades [1]. An essential element in NDE systems is the analysis of the captured signal, by means of a noise-free parameter extraction, in order to obtain relevant information from the tested specimen. Among the methods of noise impact minimization, it is worth to note filtering techniques: Rodríguez et al. [2] selected the frequency range corresponding to the detected echoes location. Thus, by making use of the Wigner-Ville transform, the echoes from the crack can be distinguished from the noisy echoes generated by the scattered ultrasonic waves through the material grains. Bilgutay et al. [3] proposed a transformation based on filter-bank techniques by applying the split spectrum processing (SSP) method to obtain a reconstructed signal less affected by noise.

The first proposals to solve the deconvolution problem in NDE use classical techniques, such as the Wiener filter, the spectral extrapolation, deconvolution of minimal least variance, estimation of the least squares [4], or homomorphic deconvolution by computation of the cepstrum [5]. The cepstrum also has been used as an efficient parametrization tool for ultrasonics signals, since the cepstral coefficients involve deconvolutionated signal information [7]. Higher-order statistics (HOS) enables the developing of blind deconvolution techniques, avoiding any prior information on the signal noise or defect [8]. In some cases, the extracted signal parameters can be processed again to reduce the dimensionality of the feature vectors, by applying classical linear discriminant analysis (LDA) [9] or principal components analysis (PCA) [10].

Recently, signal processing methods have been increasingly used to provide suitable features extraction from biomedical imaging, especially for elastography imaging (B-scan), to diagnose pathologies. Ultrasound data were captured from postmortem coronary arteries to develop radio frequency analysis techniques for the characterisation of atherosclerotic plaque [11]. A system was proposed by Scheipers et al. [12] for prostate diagnostics based on multifeature tissue characterization. Their Neuro-fuzzy inference system combine spectral features and textural features of first and second order with clinical variables and morphological descriptors for diseases classification. Abeyratne et al. [13] proposed the use of a wavelet transform based technique to estimate the inter-scatterer-distribution (ISS) in diagnosing focal diseases of the liver. Maggio et al. [14] employed a multi-feature kernel classification model based on generalized discriminant analysis to support prostate cancer diagnosis. Siebers et al. [15] presented an ultrasound based system for computer aided characterization of biologic tissue and its application...
to differential diagnosis of parotid gland lesions, based on a supervised classification of malignant and benign cases using tissue-describing features derived from ultrasound radio-frequency (RF) echo signals and image data.

However, most of the above-mentioned applications deal with determination of the presence of damages/pathologies in the evaluated specimen. Hence, the final goal of the system is limited to a classification of the states damage/no-damage or malignant/benign, respectively. These techniques require a huge amount of experimental data and an expensive training process, without providing any information at the physical level, nor using knowledge about the physiology to aid the decision.

Several laboratory-scale imaging modalities have been proposed by other authors, that provide limited information about shear elastic properties of tissue. Shear waves can be generated within tissues using focused, impulsive, acoustic radiation force excitations [16]–[18] acoustic remote palpation, sonoelastography or shear wave dispersion ultrasound vibrometry, and the resulting displacement response can be ultrasonically tracked through time. An alternative way to measure shear modulus is elastography [19]–[21], where the displacement is induced by indentation and the strain is computed by a speckle-tracing method. The impulsive force excitation combined with ultrasound vibrometry developed by Sandrin et al. [22] has been commercialized by Hitachi under the name of Fibroscan® as a liver disorder diagnosis device. Only a few works incorporate mechanical models for the identification of clinical processes. Moulton et al. [23] solved an inverse boundary value problem to determine the unknown material parameters for a nonlinear, nonhomogeneous material law, using a p-version finite element model of the heart. Han et al. [24] presented a finite element based nonlinear inverse scheme to reconstruct the elastic properties of soft tissues subjected to an external compression. Recently, Guo et al. [25] developed a novel finite element method-based direct method for the material reconstruction in soft tissue elastography.

Some recent experimental observation may tangentially suggest that nonlinear mechanical properties may be a key signature to quantify tissue consistency changes, even better than just bulk or shear moduli. Barannik et al. [26] revealed the mechanical relaxation processes appearing at higher frequencies than in the quasi-static regime. The appearance of these relaxation dynamics was associated to the presence of inhomogeneities in tissue. In the past years, numerous models have been developed in the NDE community that investigate the possibility of using nonlinear
techniques to detect damaged interface in solids [27], [28] or dynamic nonlinear stress-strain features in micro-inhomogeneous materials [29]–[31]. Recently, some authors have extended some of these experimental techniques (NRUS, NEWS, etc.) for application to damage assessment in cortical bone [32] and to measure acoustic nonlinearity in trabecular bone [33].

The proposed system avoids the need of steps such as cross-correlation of echographic images, by a more direct procedure, hence increasing the precision. Second, it allows the use of a complex propagation model in order to quantify model parameters of interest beyond modulus and attenuation. Nonlinear phenomena have not been considered in this study, but this is a research topic currently under development at our laboratory. This void in the mechanical characterization and interpretation of material defects may be overcome by adopting model-based inverse problem strategies. This is the main goal of the present work, which is successfully applied for tissue characterization.

Many types of uncertainties involved in the modeling of interaction between ultrasonic waves and tissue, such as excitation, material viscosity, and material heterogeneity are responsible for noise in the output. In this paper, several models of ultrasound-tissue interaction are proposed, implemented and contrasted against experimental observations. All assume homogeneous media with varying moduli and energy-dissipation forms that are expressed as attenuation models. High frequency ultrasound is adopted for excitation and measurement in order to analyze and interact with small-scale specimens. The minimization between experimental and numerically predicted measurements is addressed by solving a probabilistic inverse problem which includes advanced signal processing techniques.

This approach allows to obtain not only the optimal parameters in a model class, but also the uncertainty associated with the parameter estimates. Some recent developments and civil engineering applications of Bayesian model class selection have been carefully reviewed by Yuen [34]. The model-class selection is formulated following Beck et al. [35]. Finally, a simple formulation of the joint probability is proposed, from which either the inverse problem or the model-class selection can be derived just by extracting specific marginal probabilities, thus unifying all the approaches.

A model-class selection algorithm is useful to understand unknown propagation models in complex materials such as engineered tissue, which is the specimen this device will be used for. With the purpose of validating the model-class selection algorithm, a gelation process is
monitored and analyzed, of which the behavior is controlled and assumed by the majority of the literature to be viscoelastic [36]–[38].

II. Methodology

The proposed methodology combines four elements. (1) The signal acquisition of the ultrasonic signals obtained from the waves interaction with a sample of tissue, a (2) set of alternative attenuation models that simulate the ultrasound-tissue interaction, which is numerically solved by the transfer matrix formalism, a (3) stochastic model-class selection formulation used to rank which of the models parametrization are more plausible, and a (4) NDE oriented signal processing framework that extract relevant features from both the experimental and numerically predicted signals. The latter is used to reconstruct the evolution of the relevant mechanical parameters during the culture reaction time.

A. Experimental setup

A Petri dish with a specifically designed sub-wavelength thickness ultrasonic transmitter and receiver in angle position was manufactured for real-time measurement of mechanical and geometrical properties of thin layers of tissue culture (of the order of 100 [µm]). The monitored Petri dish is connected to the electronic setup detailed in Fig 1.

The transmitting and receiving transducers are designed to be in angle position (45°) in order to avoid reverberation echoes inside the petri dish plate parts. The transmitted signal is generated as
a 1-cycle burst composed by a 20 [MHz] sine of amplitude that amounts to 5 [V] with a repetition rate of 1 [ms], using an arbitrary wave generator (Agilent 33220). The recording is digitized with a high resolution A/D converter with a base of 10 [V] after 40 [dB] preamplification, during a period of 5 [µs] and a sampling rate of 400 [MHz]. The gelation process is monitored during half an hour at 5 [s] intervals, resulting in a database of 350 measurements. Each measurement corresponds to the average of 300 captures of the signal, providing an effective reduction of noise according to the signal-to-noise ratio (25 dB). Only compressional waves are generated by the transducers and no mode conversion waves are measured in the present case, although the methodology is extensible to shear or other waves.

The materials and the concentration for the gel culture were chosen according to Ortega et al. [39]: 92.5 % of water, 5 % of glycerol and 2.5 % of agar. In order to obtain an homogeneous solution, water has been first heated and then the remaining materials have been added. The material properties of the Petri dish material (PMMA, polymethylmetacrilate) and the gel culture are summarized in Table I.

| TABLE I | MECHANICAL PROPERTIES OF MATERIALS (GOLUMBECK [40]). |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| Material        | Modulus [Pa]    | Poisson ν       | Density ρ [Kg/m³] | Speed cp [m/s] |
| PMMA            | $E = 2960 \cdot 10^6$ | 0.43            | 1180             | 2672            |
| Gel (initial)   | unknown         | 0.5             | 1000             | unknown         |

B. Propagation models

The experimental system is idealized by a mathematical model of the propagation and interaction of the transmitted ultrasonic waves with all the parts of the system until they are received by the sensor. The relevant ultrasonic paths along the Petri dish material (PMMA, polymethylmetacrilate) and the gel culture are illustrated in Fig 1.

Several models are tested to idealize the removal of energy by dissipation or radiation. Three alternative damping models are used: viscous, hysteretic, and proportional to integer time derivatives of the particle movement, based on their fractional time derivatives. The damping is defined...
in terms of the wave modulus $M$, which is modified from the undamped one $M_0$ to generate a dispersive one, which is a frequency-dependent complex modulus $M^*(\omega)$, where $\omega$ is the angular frequency if the modulus dispersion is represented by its frequency domain. The viscous model is defined in terms of the frequency-dependent loss factor $\eta$, obtained as the ratio between loss and storage moduli [41]. In this context, a specific view of hysteretic damping is taken, where it is expressed as a frequency-independent damping [42]. The last model, based on fractional time derivatives, leads to a damping function that may be expressed as a power law, and thus improves curve-fitting properties for relaxation [41], [42]. These models are selected according to their performance demonstrated in a previous study [43]. The viscoelastic $\eta$ and hysteretic $\zeta$ models are defined according to Maia et al. [44],

$$M^*(\omega) = M^0 (1 - i\omega\eta - i\zeta)$$

where $\eta$ and $\zeta$ are the viscoelastic and hysteretic damping coefficients of tissue, respectively. The fractional time derivative damping is defined as,

$$M^*(\omega) = M^0 \frac{1 + b(i\omega)^\beta}{1 + a(i\omega)^\alpha}$$

The three models are summarized hereafter, highlighting the combination of the considered parameters,

**TABLE II**
**Combination of models.**

<table>
<thead>
<tr>
<th>Tag</th>
<th>Size</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>$K_{tissue}$ z\text{ampl} z\text{time}</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>$K_{tissue}$ $\eta$ z\text{ampl} z\text{time}</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>$K_{tissue}$ $a$ $b$ $\alpha = \beta$ z\text{ampl} z\text{time}</td>
</tr>
</tbody>
</table>

where $K_{tissue}$ denotes the Bulk modulus of tissue. The fractional derivative constants are defined as $a$, $b$, and $\alpha = \beta$. Two additional parameters $z_{\text{ampl}}$ and $z_{\text{time}}$ are introduced to control the correction of the amplitude and the time-shift of the input in the culture, which corrects effects of temperature and other phenomena on the sensors, that affect attenuation and delay on the path from the electronics to the arrival of the signal at the culture specimen. The factor
$z_{\text{ampl}}$ corrects variations of the amplitude and phase of the excitation over the reaction process time, that may suffer the influence of several simultaneous factors including the temperature. Nonetheless, we assume that these factors can be summarized as a whole by introducing a phenomenological factor, labeled as $z_{\text{time}}$.

The mathematical model is approximated by a semi-analytical model of the wave interactions within multilayered materials based on the transfer matrix formalism (TMF) [45], describing the ultrasonic waves interactions between the Petri dish and the culture.

C. Probabilistic inverse problem

Following the probabilistic formulation of the model reconstruction inverse problem established by Tarantola et al. [46], the solution is not a single-valued set of model parameters $\mathcal{M}$. On the contrary, the solution is provided by probability density functions (PDF) $p(\mathcal{M})$ of the values of the model parameters $\mathcal{M}$ within the manifold $\mathfrak{M}$ of possible values. The probability density is assigned the sense of plausibility of the model values to be true.

Statistical inference theory is used to incorporate to the a priori information about the measured observations $\mathcal{O}$, the model parameters $\mathcal{M}$ and the model class $\mathcal{C}$, the information of idealized relationship between them $\mathcal{O} = \mathcal{O}(\mathcal{M})$ computed by a numerical model pertaining to a model class $\mathcal{C}$. The former are defined by the probability densities to prior (labeled $^0$) data $p^0(\mathcal{O})$, $p^0(\mathcal{M})$ and $p^0(\mathcal{C})$ respectively, whereas the additional information about relationship (labeled $^m$) between observations and model provided by the model class $\mathcal{C}$ is given by the PDF $p^m(\mathcal{O}, \mathcal{M}|\mathcal{C})$.

The a posteriori probability $p(\mathcal{O}, \mathcal{M}, \mathcal{C})$ of the hypothetical model $\mathcal{M}$ is obtained jointly with the observations $\mathcal{O}$ and class $\mathcal{C}$,

$$p(\mathcal{O}, \mathcal{M}, \mathcal{C}) = k_1 \frac{p^0(\mathcal{O}, \mathcal{M}, \mathcal{C})p^m(\mathcal{O}, \mathcal{M}, \mathcal{C})}{\mu(\mathcal{O}, \mathcal{M}, \mathcal{C})}$$

(3)

where $\mu(\mathcal{O}, \mathcal{M}, \mathcal{C})$ is the noninformative density function and $k_1$ is a normalization constant. Assuming first that $\mathcal{O}$, $\mathcal{M}$ and $\mathcal{C}$ are independent a priori allows to split the joint prior information and the uniform distribution. Secondly, the probabilistic model can be represented by a computation of $\mathcal{O}$ depending on $\mathcal{M}$. Finally, the model is not assumed to provide conditional information between model and class. This simplifies the expression to,

$$p(\mathcal{O}, \mathcal{M}, \mathcal{C}) = k_1 \frac{p^0(\mathcal{O})p^0(\mathcal{M})p^0(\mathcal{C})p^m(\mathcal{O}|\mathcal{M}, \mathcal{C})}{\mu(\mathcal{O})}$$

(4)
The posterior probability of the model $\mathcal{M}$ is obtained from the joint probability by extracting the marginal probability $p(\mathcal{M})|_{C=C_i}$ for all possible observations and provided the model class $C_i \in \mathcal{C}$ is assumed to be true,

$$p(\mathcal{M})|_{C=C_i} = \int_{C=C_i} \int_{\mathcal{D}} p(\mathcal{O}, \mathcal{M}, C)d\mathcal{O}dC = k_2 \int_{\mathcal{D}} \frac{p^0(\mathcal{O})p^0(\mathcal{M})p^n(\mathcal{O}|\mathcal{M}, C)}{\mu(\mathcal{O})} d\mathcal{O}$$  \hspace{1cm} (5)

As a last simplification, we assume to have no prior information about the model $p^0(\mathcal{M})$, which is therefore represented by the noninformative distribution $p^0(\mathcal{M}) = \mu(\mathcal{M})$, which can in turn be dropped in the case of Jeffreys parameters are used,

$$p(\mathcal{M})|_{C=C_i} = k_3 \int_{\mathcal{D}} p^0(\mathcal{O})p^n(\mathcal{O}|\mathcal{M}, C)d\mathcal{O}$$  \hspace{1cm} (6)

where $k_3$ is a normalization constant that replaces the dropped uniform distributions, and is needed for $p(\mathcal{M})|_{C=C_i}$ to fulfill the theorem of total probability.

The observations are assumed to follow a Gaussian distribution $\mathcal{O} \sim \mathcal{N}(E[\mathcal{O}^{\text{exp}}], C^{\text{exp}})$ with mean of the experimental observations $\mathcal{O}^{\text{exp}}$ and covariance matrix $C^{\text{exp}}$ standing for the measurement noise. Additionally, the observations are assumed to be a Gaussian process $\mathcal{O} \sim \mathcal{N}(\mathcal{O}(\mathcal{M}), C^{\text{num}})$ centered at the numerically computed ones $E[\mathcal{O}^{\text{num}}] = \mathcal{O}(\mathcal{M})$ with covariance matrix $C^{\text{num}}$.

The probabilistic observations $\mathcal{O}$ are in our case a vector of functions of time $\mathcal{O} = o_i(t)$ at every measuring time $t \in [0, T]$ and repetition $i \in [1...N_i]$, and the assumptions made above are valid for every instant $t$ and sensor $i$. Considering that the compound probability of the information from all sensors and time instants is the productory of that of each one individually, and that this productory is equivalent to a summation within the exponentiation, the Gaussian distribution allows for an explicit expression of the probability densities,

$$J(\mathcal{M}) = \frac{1}{2} \sum_{i,j=1}^{N_i} \int_{t=0}^{t=T} O_i (c_{ij}^{\exp} + c_{ij}^{\text{num}})^{-1} O_j dt$$  \hspace{1cm} (7)

where $O_k = (o_k(t, \mathcal{M}) - o_k^{\text{exp}}(t))$, $\forall k = i, j$. The term $J(\mathcal{M})$ corresponds to a misfit function between model and observations,

$$p(\mathcal{M})|_{C=C_i} = k_6 e^{-J(\mathcal{M})}$$  \hspace{1cm} (8)
and the constant $k_6$ is derived from the theorem of total probability applied over all possible models $\mathfrak{M}$, which is integrated by Quasi Montecarlo using a Sobol sequence with $2^{18}$ points. The best-fitting model is found by minimizing $J(\mathcal{M})$ instead of maximizing $p(\mathcal{M})$ since,

$$\hat{\mathcal{M}} = \arg\max_{\mathcal{M}} \left\{ p(\mathcal{M}) |_{\mathcal{C} = \mathcal{C}_i} = k_6 e^{-J(\mathcal{M})} \right\} = \arg\min_{\mathcal{M}} \{ J(\mathcal{M}) \}$$

Let model class $\mathcal{C} \in \mathcal{C}$ denotes an idealized mathematical model hypothesized to simulate the experimental system, whereas model $\mathcal{M}$ denotes the set of physical parameters that the model-class depends on. Different model classes can be formulated and hypothesized to idealize the experimental system, and each of them can be used to solve the probabilistic inverse problem, yielding different values of model parameters but also physically different sets of parameters. To select among the infinitely many possible model classes that can be defined, a probabilistic criteria can be defined based on their compatibility between prior information about observations $\mathcal{O}$, model parameters $\mathcal{M}$ and model class $\mathcal{C}$, and probabilistic model information [35].

The goal is to find the probability $p(\mathcal{C})$, understood as a measure of plausibility of a model class $\mathcal{C}$ [47]. It can be derived as the marginal probability of the posterior probability $p(\mathcal{O}, \mathcal{M}, \mathcal{C})$ defined in Equation 4,

$$p(\mathcal{C}) = \int_{\mathcal{D}} \int_{\mathfrak{M}} p(\mathcal{O}, \mathcal{M}, \mathcal{C}) d\mathcal{M} d\mathcal{O}$$

$$= k_7 p(\mathcal{C}) \int_{\mathcal{D}} \int_{\mathfrak{M}} \frac{p(\mathcal{O}) p(\mathcal{M}) p^{\alpha}(\mathcal{O}, \mathcal{M}, \mathcal{C})}{\mu(\mathcal{O})} d\mathcal{M} d\mathcal{O}$$

The plausibility of a residue definition given a measurement domain and model class $\mathcal{C}_i$ in the sense of matching between model and experimental observations can be computed by introducing the residue space $\mathcal{R} \in \mathfrak{R}$, and deriving the corresponding marginal probability as,

$$p(\mathcal{R}) = \int_{\mathcal{D}} \int_{\mathfrak{M}} p(\mathcal{R}, \mathcal{O}, \mathcal{M}, \mathcal{C}) |_{\mathcal{C} = \mathcal{C}_i} d\mathcal{M} d\mathcal{O}$$

$$= \tilde{k}_7 \int_{\mathcal{D}} \int_{\mathfrak{M}} \frac{p(\mathcal{O}) p(\mathcal{M}) p^{\alpha}(\mathcal{O} | \mathcal{R}, \mathcal{M}, \mathcal{C}_i)}{\mu(\mathcal{O})} d\mathcal{M} d\mathcal{O}$$

where the constant $\tilde{k}_7 = k_7 p(\mathcal{C}) p(\mathcal{R})$.

The minimization of $\tilde{p}(\mathcal{M})$ for monitoring the evolution of the culture is carried out by two sequential algorithms: When an initial guess is not available, which is the case at the beginning of the process to be monitored, genetic algorithms are used as a full-range random search technique [48]. Since the change between consecutive measurements of the process is expected to be small, the BFGS-algorithm is employed thereon as a local search based on Hessian update [49], assisted by finite differentiation and line search.
In the case the model parameters $m$ are Jeffreys constants [50], they are replaced by unitary logarithmic parameters $\overline{m}$,

$$m_m = m^0_e \overline{m}^{m^1_m - m^0_m}$$

which maps the dimensional parameters $m_m$ from the preferential range $m_m \in [m^0_m, m^1_m]$ to a nondimensional Jeffreys parameter $\overline{m} \in [0, 1]$. This furthermore stabilizes the search algorithms. In this case, the noninformative distribution $\mu(M)$ can just be replaced by a constant.

D. NDE oriented signal processing

This part aims to provide a suitable representation of the ultrasonic signals, appropriate for tissular reaction process identification. The signals are first preprocessed, by means of a temporal windowing. Then, different parametrization approaches can be applied, and the obtained spectral parameters are usually transformed, to provide a more uncorrelated and dimensionally reduced representation. It results from the applied analysis that each signal can be represented by a feature vector containing the analysis parameters. Finally, some methods are defined to improve the calculation of the discrepancy between the experimental and numerically predicted feature vectors.

1) Preprocessing: After acquisition, the signals have been decimated at a sample frequency of 40 MHz, in order to reduce part of the noise and focus on the frequency band of interest. Secondly, a normalization with respect to the peak amplitude has been applied, considering that the system must be unsensitive to changes in signals amplitude. Finally, the signals have been multiplied by a Hamming window. Since ultrasonic signals are finite by nature, the window is foremost used to weight the signal samples over the time, and thus to show off the signal echoes [51].

2) Feature extraction and spectral-domain: The proposed methodology includes a nonparametric technique approach, that directly estimates the spectrum features from the signal itself, providing a sufficiently accurate representation for many types of signals in many different applications [52], where as in NDE systems, the pursued information is hidden in a complex signal. The spectral analysis is performed by determining the magnitude spectrum of the detected signals, easily obtained by applying the discrete Fourier transform (DFT).
3) Homomorphic transformations: Among the homomorphic transformations, the basic idea of the cepstrum consists of converting a convolution into a sum [53], and thus to obtain a decorrelated representation of the signals. The cepstrum of a discrete signal \( s(n) \), whose corresponding spectrum is denoted by \( S(\omega) \), is defined as the inverse Fourier transform of the logarithmic spectrum:

\[
\hat{c}(n) = \mathcal{F}^{-1}[\log(S(\omega))] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(S(\omega)) d\omega
\]

(13)

Generally, the spectrum \( S(\omega) \) is a complex and even function obtained by applying the Fourier transform. Alternatively, a real cepstrum can be obtained by considering the magnitude spectrum \( |S(\omega)| \) as,

\[
c(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(|S(\omega)|) d\omega
\]

(14)

In practice, the real cepstrum can be easily obtained by applying the fast Fourier transform (FFT) as,

\[
c(n) = \text{IFFT}[\log(|\text{FFT}(s(n))|)]
\]

(15)

where IFFT denotes the inverse fast Fourier transform. In this case, the real cepstrum is usually called cepstrum FFT. In an algebraic sense, the associated complex cepstrum could be obtained similarly. However, computing the complex cepstrum is usually cumbersome due to the unwrapping of the digital phase. In the cepstral-domain, due to the harmonic nature of the ultrasonic signals the wave components appear as equidistant peaks at higher quefrencies, rightly separated by a value that corresponds to the fundamental period of the analyzed signal echoes. Thus, this cepstral representation allows to decompose the spectrum in its two main characteristics: The spectral envelope and the fine spectrum.

4) Dimensionality reduction and deconvolution property: It can be useful to restrict the number of cepstral coefficients, by applying a window to rule out lower and/or higher quefrencies. This process is called liftering, and is applied as,

\[
\bar{c}(n) = c(n)w(n)
\]

(16)
where \( n = 1, ..., L \). Several windowing schemes can be directly derived from the rectangular window in the cepstral-domain. This study focuses on the short-pass liftering \([54]\), that corresponds to a smoothing of the spectrum, preserving its spectral envelope while removing the fine spectrum information. Although it has been less studied, it is interesting to point out that the application of a liftering on the complex cepstrum allows to observe the effects of removing frequencies on the recovered signal in the time-domain. Moreover, applying windows different from the rectangular one allows to weight the cepstral coefficients depending on their discriminative performance.

**E. Discrepancy between the experimental and numerical feature vectors**

The residue definitions used in the probabilistic inverse problem are generalized to the vectors obtained from the feature extraction. Thus, the residue is defined as a likelihood measure between two feature vectors. The classical residue \( r_0 \) has been defined as,

\[
    r_0 = s^x - s(p)
\]

where \( s^x \) and \( s(p) \) denote the vectors obtained from the features extraction, corresponding to the experimental and numerically predicted signals, respectively. Some enhancements are proposed by defining weighted residues. First, a weighted residue is defined, which includes the variance of the measurements \( \sigma^x \) over the temporal evolution of the reaction process. Thus, the goodness of fit of the model predictions to the experimental values is assessed with the weighted least squares criterion \([55]\):

\[
    r_1 = \sigma^x(s^x - s(p))
\]

Additionally, a weighted residue is defined, which includes the variance of the measurements \( \sigma_R^x \) over the reaction time of the process.

\[
    r_2 = \frac{s^x - s(p)}{\sigma_R^x}
\]

Finally, a weighted residue is defined, including both aforementioned variances:

\[
    r_3 = \frac{(s^x - s(p))}{\sigma^x} \frac{\sigma^x}{\sigma_R^x}
\]

It is worth to note that the variance \( \sigma \) of the coefficients from the experimental signals amounts to a value close to zero when the corresponding signal intensity is low. Thus, a slightly biased
variance value $\tilde{\sigma}$ is used in the above-mentioned residue definitions by making use of the root mean square of the variance $\sigma$, in order to avoid that the residue tends to infinity:

$$\tilde{\sigma}^x = \sigma^x + 0.1 \sqrt{\frac{1}{L} \sum_{n=0}^{L} (\sigma^x_n)^2} \tag{21}$$

Equations 18-20 allows to (i) reduce the uncertainties due to the measurements noise, by assuming statistical independence of the errors, and (ii) enhance parts of the signals that may contain information of the reaction process. In a probabilistic sense, residue definitions that take into account variance information can be understand as a prior knowledge on the measurement quality and/or evolution of the reaction process. Thus, they may give some enhancements on the interpretation of the gelation process.

III. Results

The recorded signals by the ultrasound-monitored Petri dish every 250 seconds are shown in Fig 2, without and with specimen, respectively. No clear evolution is detectable by bare visual inspection of the signals. The recorded signals are mainly composed of three different waveforms (simplified paths of Fig. 1), namely (1) the wave front that propagates only through the PMMA layer (labeled as $u_1$), (2) a wave that crosses both the PMMA layer and the specimen (labeled as $u_2$), and (3) a wave echo produced by the former wave after crossing twice the specimen (labeled as $u_3$).

It is noteworthy that when the specimen is on place, the majority of the excitation signal (registered without specimen for calibration) is transmitted instead of reflected. Since the wavelength in gel is compatible with the layer thickness, the individual echoes generated by the multiple reflections inside the gel layer can be analyzed separately by signal processing.

A. Signal simulation

The transfer matrix formalism is used to generate sample signals, after calibrating the estimated parameters using the inverse problem, for the first signal (initial evolution time). Time-domain signals and magnitude spectra are shown in Fig. 3-6 for the viscous model (case 2) at the initial time of the reaction process, respectively. In the lower figures, an analysis window (Hamming)
has been applied to the signals, and classical and weighted residue definitions are considered, denoted as $r_0$ and $r_3$, respectively.

A significative ability to simulate the system can be observed visually. The influence of the signal windowing yields to the following observations for the magnitude spectrum (Fig. 5-6): The envelope, that corresponds to the redundant character of the signal, remains almost unchanged. In contrast, the fine spectrum presents accentuated peakiness due to the enhanced echoes of the time-domain signals. In the time-domain, the classical residue $r_0$ is strongly correlated with the signals themselves (Fig. 3), highlighting higher amplitude where the signals energy is higher. However, the weighted residue $r_3$ allows to remove some variability due to the measurements uncertainties (Fig. 4). Additionally, it enhances the signal parts containing information of the reaction process (here mainly the wave front), while allowing to remove the parts which are invariant to the process evolution (scattering parts). In the frequency-domain, the weighted residue allows to remove the frequency range being unsensitive to the reaction process, or being erroneous due to measurements noise. Thus, the resulting magnitude spectra of Fig. 6 show concentrated process information at certain frequencies, in contrast to the magnitude spectrum of Fig. 5.

B. Posterior probability of the model

The probability density function is computed for the viscous attenuation model, and some relevant samples issued from the results obtained in the previous section are shown in Figs. 7-10. Since the PDF is a multidimensional function, without loss of generality, only a slice along two parameters is represented, namely the Bulk modulus of the tissue and the viscous damping coefficient.

The inspection of these plots reveals several local minima, valleys in the probability density function and variations of several orders of magnitude from good to bad model parameters. This implies a bad conditioning of the reconstruction inverse problem and justifies the use of advanced search algorithms such as genetic algorithms. Nonetheless, the use of a weighted residue definition enhance the slope of these local minima, and thus speed up the convergence of the search algorithm. It is noteworthy that the other attenuation models present similar trends. Additionally, some irrelevant samples issued from the cepstral analysis are illustrated in Figs. 11-12.
The inspection of the plots obtained from the cepstral analysis reveals many local minima that approximately have the same values, leading to an ill-conditioned solution space.

C. Model class plausibility

The posterior probability \( p(C) \) of every proposed model class \( C \in \mathcal{C} \) is computed by Quasi-Montecarlo integration using \( 2^{18} \) Sobol sampling points. Additionally, the estimation of Occam’s factor, as well as the certainty metric \( \sigma \) are summarized in Tables III-V for time-domain signals, magnitude spectra and real cepstra, respectively.

**TABLE III**

**PLAUSIBILITY OF MODEL CLASSES, TIME-DOMAIN.**

<table>
<thead>
<tr>
<th>Windowing</th>
<th>Residue</th>
<th>Model class</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( p(C) ) [%]</td>
<td>31.55</td>
<td>32.39</td>
<td>36.06</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Occam ( -\log_{10} )</td>
<td>2.24</td>
<td>4.10</td>
<td>1.90</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Certainty ( \log_{10} )</td>
<td>0.11</td>
<td>0.75</td>
<td>0.37</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Windowing</th>
<th>Residue</th>
<th>Model class</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( p(C) ) [%]</td>
<td>33.33</td>
<td>33.33</td>
<td>33.33</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Occam ( -\log_{10} )</td>
<td>8.11</td>
<td>7.93</td>
<td>11.93</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Certainty ( \log_{10} )</td>
<td>1.69</td>
<td>1.66</td>
<td>1.97</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Windowing</th>
<th>Residue</th>
<th>Model class</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( p(C) ) [%]</td>
<td>15.53</td>
<td>63.10</td>
<td>21.37</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Occam ( -\log_{10} )</td>
<td>7.34</td>
<td>9.95</td>
<td>14.78</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Certainty ( \log_{10} )</td>
<td>2.23</td>
<td>2.74</td>
<td>2.97</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Windowing</th>
<th>Residue</th>
<th>Model class</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( p(C) ) [%]</td>
<td>32.45</td>
<td>32.99</td>
<td>34.57</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Occam ( -\log_{10} )</td>
<td>7.14</td>
<td>6.50</td>
<td>7.44</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Certainty ( \log_{10} )</td>
<td>1.40</td>
<td>1.16</td>
<td>1.02</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Windowing</th>
<th>Residue</th>
<th>Model class</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( p(C) ) [%]</td>
<td>33.33</td>
<td>33.33</td>
<td>33.33</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Occam ( -\log_{10} )</td>
<td>7.93</td>
<td>7.90</td>
<td>13.08</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Certainty ( \log_{10} )</td>
<td>1.83</td>
<td>1.95</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Windowing</th>
<th>Residue</th>
<th>Model class</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( p(C) ) [%]</td>
<td>34.81</td>
<td>31.57</td>
<td>33.62</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Occam ( -\log_{10} )</td>
<td>4.25</td>
<td>6.17</td>
<td>8.72</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Certainty ( \log_{10} )</td>
<td>1.39</td>
<td>1.79</td>
<td>1.45</td>
</tr>
</tbody>
</table>
The most plausible model class is shown to be 2, involving $K^{*}$, viscoelastic damping, and temperature and amplitude corrections. It is closely followed by class 1 (hysteretic), whereas class 3 does not provide results for all proposed signal processing techniques. The magnitude spectrum computed with weighted residue definitions $r_1$ and $r_3$ show significantly higher posterior probability $p(C)$ than the other domains of representation. The real cepstrum provides bad results as well for the posterior probability, which is consistent with the observations in the previous section. This evidence further supports the validity of the probabilistic formulation. Hence, the obtained equiprobable values demonstrate its insensitiveness with respect to the selected mode

<table>
<thead>
<tr>
<th>Windowing</th>
<th>Residue</th>
<th>Model class</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_0$</td>
<td>$r_0$</td>
<td>$p(C)$ [%]</td>
<td>29.79</td>
<td>29.39</td>
<td>40.82</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Occam [-log$_{10}$]</td>
<td>0.09</td>
<td>-0.42</td>
<td>-5.17</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Certainty [log$_{10}$]</td>
<td>0.40</td>
<td>0.38</td>
<td>0.71</td>
</tr>
<tr>
<td></td>
<td>$r_1$</td>
<td>$p(C)$ [%]</td>
<td>49.94</td>
<td>50.06</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Occam [-log$_{10}$]</td>
<td>7.71</td>
<td>8.82</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Certainty [log$_{10}$]</td>
<td>1.52</td>
<td>1.63</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>$r_3$</td>
<td>$p(C)$ [%]</td>
<td>0.02</td>
<td>56.47</td>
<td>43.51</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Occam [-log$_{10}$]</td>
<td>4.84</td>
<td>5.88</td>
<td>22.37</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Certainty [log$_{10}$]</td>
<td>1.93</td>
<td>1.89</td>
<td>4.10</td>
</tr>
<tr>
<td>$w_1$</td>
<td>$r_0$</td>
<td>$p(C)$ [%]</td>
<td>26.90</td>
<td>26.11</td>
<td>46.99</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Occam [-log$_{10}$]</td>
<td>2.47</td>
<td>1.82</td>
<td>-8.55</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Certainty [log$_{10}$]</td>
<td>0.08</td>
<td>0.08</td>
<td>-1.57</td>
</tr>
<tr>
<td></td>
<td>$r_1$</td>
<td>$p(C)$ [%]</td>
<td>49.98</td>
<td>50.02</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Occam [-log$_{10}$]</td>
<td>8.68</td>
<td>7.81</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Certainty [log$_{10}$]</td>
<td>1.80</td>
<td>1.49</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>$r_3$</td>
<td>$p(C)$ [%]</td>
<td>31.98</td>
<td>44.95</td>
<td>23.07</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Occam [-log$_{10}$]</td>
<td>5.11</td>
<td>4.77</td>
<td>16.15</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Certainty [log$_{10}$]</td>
<td>1.61</td>
<td>1.54</td>
<td>2.85</td>
</tr>
</tbody>
</table>
The posterior probability \( p(R) \) of every consistent residue definition \( R \in \mathcal{R} \) is computed accordingly to the posterior probability \( p(C) \). Tables VI-VII summarized the obtained values together with the Occam’s factor estimation and certainty metric, for the hysteretic and viscous damping, respectively.

The most plausible residue definition appears to be \( r_1 \), which involves the inclusion of some prior information on the variance of the measurements over the temporal evolution of the reaction process, when the inverse problem is achieved in the frequency-domain. This ranking remains consistent independently of the model class. Signal windowing has a feeble influence on the results improvement.

### D. Monitoring of evolution

The evolution of the relevant reconstructed mechanical parameters during the reaction process is shown in Fig 13-14 for the most relevant model class and residue definitions, respectively. The value of the reconstructed Bulk modulus at the beginning of the process approximately amounts
to 2.385 [GPa]. By making use of the following formula [42],

$$c_p = \frac{(K + \frac{4\mu}{3}G)}{\rho}$$

(22)

and under the hypothesis that the shear modulus $G$ and the density $\rho$ of the material amount to 1000 [Pa] and 1000 [kg/m$^3$], the wave velocity is found to be 1544 [m/s]. The latter is close to the value depicted by other authors for similar materials, among them Norisuye et al. [37].

Some parameter evolutions reconstructed using residues and models with low plausibility (not shown in this paper for space constraints) show larger scattering and instabilities, consistently with the results in the two previous subsections. Arriving at the same conclusion than crossed observations further supports the validity of the formulation and conclusions.

IV. CONCLUSION

A numerical method to determine the elastic and dynamic energy dissipation properties during a gelation process has been developed by combining the solution of a probabilistic inverse
TABLE VII

P \text{LAUSIBILITY OF RESIDUE DEFINITIONS. MAGNITUDE SPECTRUM.}

<table>
<thead>
<tr>
<th>Model</th>
<th>Window</th>
<th>Residue</th>
<th>( r_0 )</th>
<th>( r_1 )</th>
<th>( r_2 )</th>
<th>( r_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( w_0 )</td>
<td>( p(\mathcal{R}) ) [%]</td>
<td>5.87</td>
<td>38.87</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Occam</td>
<td>0.09</td>
<td>7.71</td>
<td>-14.49</td>
<td>-4.84</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Certainty</td>
<td>-0.40</td>
<td>1.52</td>
<td>-3.98</td>
<td>-1.93</td>
</tr>
<tr>
<td>1</td>
<td>( w_1 )</td>
<td>( p(\mathcal{R}) ) [%]</td>
<td>15.64</td>
<td>39.61</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Occam</td>
<td>2.47</td>
<td>8.68</td>
<td>-11.81</td>
<td>-5.11</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Certainty</td>
<td>0.08</td>
<td>1.80</td>
<td>-3.40</td>
<td>-1.61</td>
</tr>
<tr>
<td>2</td>
<td>( w_0 )</td>
<td>( p(\mathcal{C}) ) [%]</td>
<td>5.82</td>
<td>39.12</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Occam</td>
<td>-0.42</td>
<td>8.82</td>
<td>-14.87</td>
<td>-5.88</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Certainty</td>
<td>-0.38</td>
<td>1.63</td>
<td>-4.00</td>
<td>-1.89</td>
</tr>
<tr>
<td>2</td>
<td>( w_1 )</td>
<td>( p(\mathcal{C}) ) [%]</td>
<td>15.25</td>
<td>39.81</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Occam</td>
<td>1.82</td>
<td>7.81</td>
<td>-12.02</td>
<td>-4.77</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Certainty</td>
<td>0.08</td>
<td>1.49</td>
<td>-3.35</td>
<td>-1.55</td>
</tr>
</tbody>
</table>

problem with signal processing techniques, applying genetic algorithms to minimize a cost functional, and using a semi-analytical model of the interaction between ultrasonic waves and tissue.

The proposed model-class and residue selection and their subjacent class plausibility have enabled to rank both the models and the suitable residue definitions according to their compatibility with the observations. The resulting trade-off between model simplicity and fitting to observations have demonstrated that the viscous damping models, combined with some prior information on the measurements variance over the reaction process evolution, are feasible to characterize the complex evolution of the process.

The reconstructed model parameters highlight the following statements. For the viscoelastic models, the Bulk modulus consistently decreases while increasing the damping coefficient. Therefore, both parameters may be associated to the same phenomena, but a careful interpretation has not been carried out at that time. The evolution of the model parameters has a stronger slope during the first 200-300 seconds of the reaction process, and remain almost constant afterwards.
This trend validates the observations done in situ where the gelation occurred during the first 3–5 minutes. Consequently, the proposed methodology demonstrates capability to discriminate the process during its early solidification phase. For a better understanding of the ultrasonic tissue monitoring, in vitro studies on real tissue combined with histological studies may be conducted.

ACKNOWLEDGMENT

The authors would like to thank the Ministry of Education of Spain through Grant No. PI2010-17065 (MICINN), the Servicio Andaluz de Salud, Junta de Andalucía, through Grant No. PI-0308 (SAS), and the Junta de Andalucía through Grant No. P08-TIC-03911.

REFERENCES


Nicolas Bochud holds a Master degree in Mechanical Engineering from the Swiss Federal Institute of Technology (ETH), Zurich, 2008 and a Master degree in Multimedia System at the University of Granada, 2010. He is currently doing his PhD at the Department of Structural Mechanics of the University of Granada, focusing on the identification of pathologies in advanced and bio-materials using ultrasonics. His research field includes non-destructive evaluation, signal processing and nonlinear ultrasound.

Guillermo Rus started his research on computational mechanics at the University of Granada (UGR, 1995), where he disputed the PhD thesis on Numerical Methods for Nondestructive Identification of Defects (2001), providing defect search algorithms and sensitivity computation with Boundary Elements. He applied these experimentally at the NDE Lab at MIT (USA) as a Fulbright Postdoctoral Fellow, rendering novel robust quantitative approaches to ultrasonics and impact testing. He started up the NDE Lab at the UGR (www.ugr.es/~grus) in 2003 to focus on bioengineering applications like bone implant debonding in collaboration with University College London, bone quality diagnosis with ultrasound with Université Paris VI or biomaterials characterization with the Nanomaterials Technology Laboratory and Institute of Bioengineering, Alicante. He is also transferring this diagnosis technology to civil engineering for monitoring structural health of advanced materials like CFRP, in collaboration with Andong National University, Korea. Rus tenured as associate professor in 2009 at UGR, is the author of 20 SCI papers and 7 books, in addition to 40 international conference contributions and 11 invited presentations. His research career has been awarded by the Juan Carlos Simo prize for young researchers (Spain, 2007), the Honorary Fellowship of the Wessex Institute of Technology (UK, 2005), Fulbright Fellowship (USA, 2002) and the Excellence PhD award (Granada, 2001).
Fig. 2. Signals sample: sequence of signals without specimen (above); signal with specimen registered every 250 seconds (below).
Fig. 3. Example of fitting of experimental and simulated observations. Viscous model. Time-domain. Residue $r_0$. 
Fig. 4. Example of fitting of experimental and simulated observations. Viscous model. Time-domain. Residue $r_3$. 
Fig. 5. Example of fitting of experimental and simulated observations. Viscous model. Magnitude-spectrum. Residue $r_0$. 
Fig. 6. Example of fitting of experimental and simulated observations. Viscous model. Magnitude-spectrum. Residue $r_3$. 
Fig. 7. Posterior probability of the model. Slice along two parameters. Viscous model. Time-domain. Residue $r_0$. 

Probability Density, $P(m)$ [log$_{10}$-scale]

Viscous damping

Bulk modulus, $K$ [Pa]

100e-09
1.00e-09
7.42e-09
2.73e-09
3.69e-10
1.36e-10
5e-11
1.33e+09
1.63e+09
1.99e+09
2.43e+09
2.97e+09
3.63e+09
Fig. 8. Posterior probability of the model. Slice along two parameters. Viscous model. Time-domain. Residue $r_3$. 

![Probability Density, $P(m) [\log_{10} \text{-scale}]$](image)

- Bulk modulus, $K [\text{Pa}]$
- Viscous damping
- Probability Density, $P(m) [\log_{10} \text{-scale}]$
Fig. 9. Posterior probability of the model. Slice along two parameters. Viscous model. Magnitude-spectrum. Residue $r_0$. 

Probability Density, $P(m)$ [log$_{10}$-scale]

Bulk modulus, $K$ [Pa]

Viscous damping
Fig. 10. Posterior probability of the model. Slice along two parameters. Viscous model. Magnitude-spectrum. Residue $r_3$. 
Fig. 11. Posterior probability of the model. Slice along two parameters. Viscous model. Real cepstrum. Residue $r_0$. 
Fig. 12. Posterior probability of the model. Slice along two parameters. Viscous model. Real cepstrum. Residue $r_3$. 