Localization and Comodule Types of Coalgebras

Pascual Jara — Luis Merino — Gabriel Navarro

Department of Algebra
University of Granada

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Some well known results on finite dimensional algebras are the following:

**Theorem (Gabriel’s Theorem)**

*Over an algebraically closed field, every basic finite dimensional algebra is the path algebra of a quiver with relations.*

**Corollary**

*There is an equivalence $\mathcal{M}_{\text{fd}}^A \cong \text{rep}_K(\mathcal{Q}, \Omega).$*

**Theorem (Tame-Wild Dichotomy)**

*Any finite dimensional algebra, over an algebraically closed field, is either of tame representation type, or of wild representation type, and these two classes of algebras are disjoint.*
Some well known results on finite dimensional algebras are the following:

**Theorem (Gabriel’s Theorem)**

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**Corollary**

There is an equivalence $\mathcal{M}^{fd}_A \cong \text{rep}_K(\mathcal{Q}, \Omega)$.

**Theorem (Tame-Wild Dichotomy)**

Any finite dimensional algebra, over an algebraically closed field, is either of tame representation type, or of wild representation type, and these two classes of algebras are disjoint.
Daniel Simson defines two disjoint classes of coalgebras:

- **C is tame** if for every $v \in K_0(C)$ there exist $K[t]$-$C$-bimodules $L^{(1)}, \ldots, L^{(r_v)}$, which are finitely generated free $K[t]$-modules, such that all but finitely many indecomposable finite dimensional left $C$-comodules $M$ with $\text{length } M = v$ are of the form $M \cong K^1_\lambda \otimes_{K[t]} L^{(s)}$, where $s \leq r_v$, $K^1_\lambda = K[t]/(t - \lambda)$ and $\lambda \in K$ (algebraically closed field).

\[ \text{length } M = v = (v_i)_{i \in I_C}, \text{ where } v_i \text{ is the number of composition factors isomorphic to the simple comodule } S_i. \]

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\[ ^0 \text{D. Simson, Path coalgebras of quivers with relations and a tame-wild dichotomy problem for coalgebras, Lectures Notes in Pure and Applied Mathematics, 236(2005), pp. 465-492.} \]
Daniel Simson defines two disjoint classes of coalgebras:

- **C is wild** if there exists an exact $K$-linear embedding $T : \text{mod} - KQ \to M^{C}_{fd}$ that respects isomorphism classes and carries indecomposables right $KQ$-modules to indecomposable right $C$-comodules., where $Q$ is the quiver $\circ \rightarrow \rightarrow \rightarrow \circ$.

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Lemma

If $C$ is wild there exists an exact $k$-linear embedding representation functor $\mathcal{M}_H^{\text{fd}} \to \mathcal{M}_C^{\text{fd}}$, for each finite dimensional algebra $H$.

$C$ comprises the representation theory of all finite dimensional (co)algebras and therefore it is not a realistic aim to classify a wild coalgebra.

In a natural way appears

Conjecture (Tame-wild dichotomy for coalgebras)

Any coalgebra is either of tame comodule type, or of wild comodule type, and these types are mutually exclusive.
Lemma

If $C$ is wild there exists an exact $k$-linear embedding representation functor $\mathcal{M}_H^{Hf} \to \mathcal{M}_C^{Cf}$, for each finite dimensional algebra $H$.

$C$ comprises the representation theory of all finite dimensional (co)algebras and therefore it is not a realistic aim to classify a wild coalgebra.

In a natural way appears

Conjecture (Tame-wild dichotomy for coalgebras)

Any coalgebra is either of tame comodule type, or of wild comodule type, and these types are mutually exclusive.
Simson \(^1\) proposes:

**Definition**

For any \((Q, \Omega)\) quiver with relations,

\[ C(Q, \Omega) = \{ a \in KQ \text{ such that } \langle a, \Omega \rangle = 0 \} \]

with \(\langle , \rangle : KQ \times KQ \rightarrow K\) defined by

\[ \langle v, w \rangle = \delta_{v,w} = \begin{cases} 0 & \text{if } v \neq w \\ 1 & \text{if } v = w \end{cases} \]

for all paths \(v, w \in KQ\).

**Corollary**

There is an equivalence \(\mathcal{M}_\text{fd}^C \cong \text{nilrep}_K^{\text{lf}}(Q, \Omega)\).

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\(^1\) D. Simson, Coalgebras, comodules, pseudocompact algebras and tame comodule type, Colloq. Math., 90 (2001), 101-150.
This a situation of a non-degenerate pairing

\[ C^\perp KQ = \{ p \in KQ \mid \langle C, p \rangle = 0 \} \]

\[ \Omega^\perp = C(Q, \Omega) \]
Problem

For any $C \leq KQ$ admissible, there is an admissible ideal $\Omega$ such that $C = C(Q, \Omega)$
Example

Let $Q$ be the quiver

\[
\begin{array}{c}
\alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \\
\end{array} 
\quad 
\begin{array}{c}
\beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \\
\end{array} 
\quad 
\begin{array}{c}
\gamma_i = \beta_i \alpha_i \text{ for all } i \in \mathbb{N}
\end{array}
\]

$H \leq CQ$ generated by $\Sigma = \{\gamma_i - \gamma_{i+1}\}_{i \in \mathbb{N}}$.

$H$ contains the wild algebra $K\Gamma$. 

\[
\begin{array}{c}
\alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\
\end{array} 
\quad 
\begin{array}{c}
\alpha_5 \\
\end{array} 
\quad 
\begin{array}{c}
\alpha_1 \\ \alpha_2 \\
\end{array} 
\quad 
\begin{array}{c}
\alpha_1 \\
\end{array}
\]
Example

Let $Q$ be the quiver

$$
\begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_n \\
\end{array}
\quad
\begin{array}{c}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_n \\
\end{array}
\quad
\begin{array}{c}
\alpha_i \\
\beta_i \\
\vdots \\
\end{array}
\quad
\begin{array}{c}
\gamma_i = \beta_i \alpha_i \text{ for all } i \in \mathbb{N}
\end{array}
$$

$H \leq CQ$ generated by $\Sigma = \{\gamma_i - \gamma_{i+1}\}_{i \in \mathbb{N}}$.

$H$ contains the wild algebra $K\Gamma$.

\[\Gamma \equiv \]

\[\begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_4 \\
\alpha_3 \\
\end{array}
\quad
\begin{array}{c}
\alpha_5 \\
\end{array}\]
Theorem (Criterion)

Let \( C \leq KQ \) be an admissible subcoalgebra. TFAE:

- \( C \) is not the path coalgebra of a quiver with relations.
- There exist an infinite number of different paths \( \{\gamma_i\}_{i \in \mathbb{N}} \) in \( Q \) such that:
  - All of them have common source and common sink.
  - None of them is in \( C \).
  - There exist elements \( a^n_j \in K \) for all \( j, n \in \mathbb{N} \) such that the set
    \[ \{ \gamma_n + \sum_{j > n} a^n_j \gamma_j \}_{n \in \mathbb{N}} \]
    is contained in \( C \).
The problem is reformulated to consider a smaller class of coalgebras.

Problem

Any basic tame coalgebra, over an algebraically closed field, is isomorphic to the path coalgebra of a quiver with relations.

In order to attack this problem a useful technique is the localization and colocalization of (pointed) coalgebras.
Let $\mathcal{A}$ be a dense subcategory of an abelian category $\mathcal{C}$:
- There is a quotient functor $T : \mathcal{C} \to \mathcal{C}/\mathcal{A}$.
- $\mathcal{A}$ is localizing if $T$ has a right adjoint functor $S : \mathcal{C}/\mathcal{A} \to \mathcal{C}$ (section functor).
- $\mathcal{A}$ is perfect localizing if $S$ is exact.

**Theorem**
- $T$ is an exact functor.
- $S$ is a fully faithful and left exact functor.
- $TS = 1_{\mathcal{C}/\mathcal{A}}$. 
Dually

- $\mathcal{A}$ is **colocalizing** if $T$ has a left adjoint functor $H : C / A \to C$ (**colocalizing functor**).
- $\mathcal{A}$ is **perfect colocalizing** if $H$ is exact.

**Theorem**

- $H$ is a fully faithful and right exact functor.
- $TH = 1_{C / A}$. 
Let $C$ be a coalgebra and $\mathcal{M}^C$ the category of right $C$-comodules.

**Theorem**

There are one-to-one correspondences between:

- **Localizing subcategories** of $\mathcal{M}^C$.
- Sets of **simple** $C$-comodules.
- Sets of **indecomposable injective** $C$-comodules.
- Classes of equivalence of **idempotents** elements in $C^*$.

If $C$ is an admissible subcoalgebra of $KQ$ (**pointed**)

- Subsets of **vertices** of $Q$. 

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Representation Theory of Coalgebras
Localization in terms of idempotents

$\mathcal{T}_e$ localizing subcategory associated to $e \in C^*$.

- $eC e$ acquires a coalgebra structure given by

$$\Delta_{eC e}(exe) = \sum_{(x)} ex_1 e \otimes ex_2 e$$

if $\Delta_C(x) = \sum_{(x)} x_1 \otimes x_2$ and $\epsilon_{eC e}(exe) = \epsilon(x)$.

- There is an equivalence $\mathcal{M}_C / \mathcal{T}_e \cong \mathcal{M}^{eC e}$.

- The functor $T = e(-) = - \square_C eC = \text{Cohom}_C(Ce, -)$. 
Localization of pointed coalgebras

Let $p$ be a path in $Q$

$$\begin{array}{ccccccc}
\circ & \rightarrow & \circ & \rightarrow & \circ & \rightarrow & \circ \\
x_1 & , & x_2 & , & \cdots & , & x_{n-1} & , & x_n
\end{array}$$

$p$ is a **cell** relative to $X$ if

$$\begin{cases} 
& x_1, x_n \in X, \\
& x_2, x_3, \ldots, x_{n-1} \notin X
\end{cases}$$

$p$ is a **tail** relative $X$ if

$$\begin{cases} 
& x_1 \in X, \\
& x_2, x_3, \ldots, x_{n-1}, x_n \notin X
\end{cases}$$

$Q$ quiver

$e$ idempotent in $(KQ)^*$

$X \subseteq Q_0$ vertices associated to $e$
Localization of pointed coalgebras

$\mathcal{Q}$ quiver
$e$ idempotent in $(K\mathcal{Q})^*$
$X \subseteq \mathcal{Q}_0$ vertices associated to $e$

Let $p$ be a path in $\mathcal{Q}$

\[
\begin{array}{cccccccc}
& \circ \rightarrow & \circ \rightarrow & \circ \rightarrow & \circ \rightarrow & \circ \rightarrow & \circ \rightarrow & \circ \\
X_1 & X_2 & & & & & X_{n-1} & X_n
\end{array}
\]

- $p$ is a **cell** relative to $X$ if \[ \begin{cases} x_1, x_n \in X, \\ x_2, x_3, \ldots, x_{n-1} \notin X \end{cases} \]
- $p$ is a **tail** relative $X$ if \[ \begin{cases} x_1 \in X, \\ x_2, x_3, \ldots, x_{n-1}, x_n \notin X \end{cases} \]
A Problem for Coalgebras
Localization in (Pointed) Coalgebras
Localization and Comodule Types

Localization of pointed coalgebras

Q quiver
e idempotent in \((KQ)^*\)
\(X \subseteq Q_0\) vertices associated to e

Let \(p\) be a path in Q

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\end{cases}
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Localization of pointed coalgebras

Let $p$ be a path in $Q$

\[ X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_{n-1} \rightarrow X_n \]

- $p$ is a **cell** relative to $X$ if $\{x_1, x_n \in X, x_2, x_3, \ldots, x_{n-1} \notin X\}$

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A Problem for Coalgebras
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\[ \xymatrix{ X \ar[r]^{\alpha_1} & \bullet \ar[r]^{\alpha_2} & \bullet \ar[r]^{\alpha_3} & \bullet \ar[r]^{\alpha_4} & Y \\
\circ \ar[u]^\beta_1 \ar[r]_{\beta_2} & \bullet \ar[u]^\beta_2 \ar[r]_{\beta_3} & \bullet \ar[u]^\beta_3 \ar[r] & \circ \ar[u]^\beta_1 & } \]
Theorem

\[ e(KQ)e \cong KQ^e, \text{ where } Q^e = (X, \text{Cell}_X^Q). \]
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\[ e(KQ)e \cong KQ^e, \text{ where } Q^e = (X, Cell_{X}^{Q}). \]
Theorem

e(KQ)e \cong KQ^e, \text{ where } Q^e = (X, Cell^Q_X).
Theorem

Let \( C \leq KQ \) admissible and \( e \in C^* \) idempotent associated to \( X \subseteq Q_0 \). Then the \( eCe \) is an admissible subcoalgebra of \( KQ^e \), where:

- \((Q^e)_0 = X\).
- \( \text{Card} \{ x \rightarrow y \} = \dim_K \text{Cell}^Q_X(x, y) \cap C \), for all \( x, y \in X \).

\[
C = KQ_0 \oplus KQ_1 \oplus K(\alpha_2\alpha_1 + \beta_2\beta_1)
\]
Theorem

Let $C \subseteq KQ$ admissible and $e \in C^*$ idempotent associated to $X \subseteq Q_0$. Then the $eCe$ is an admissible subcoalgebra of $KQ^e$, where:

- $(Q^e)_0 = X$.
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- $(Q^e)_0 = X$.
- $\text{Card}(\{x \rightarrow y\}) = \dim_K \text{Cell}_{X}^{Q}(x, y) \cap C$, for all $x, y \in X$.

$$eCe = KX \oplus K(\theta) = KQ^e$$
Theorem

\[ e(KQ) = \bigoplus_{x \in X} E_X^{\text{Card} \left( \text{Tail}_X^Q(x) \right) + 1} \text{ as right } e(KQ)e\text{-comodules}. \]

Corollary

- \( T_X \) is colocalizing \( \iff \text{Tail}_X^Q(x) \) is finite for all \( x \in X \).
- \( T_e \) colocalizing \( \iff T_e \) perfect colocalizing.
Colocalization of pointed coalgebras

**Theorem**

\[ e(KQ) = \bigoplus_{x \in X} E_x \text{Card}(\text{Tail}_X^Q(x)) + 1 \] as right \( e(KQ)e \)-comodules.

**Corollary**

- \( T_X \) is colocalizing if and only if \( \text{Tail}_X^Q(x) \) is finite for all \( x \in X \).
- \( T_e \) colocalizing if and only if \( T_e \) perfect colocalizing.
Theorem

Let $Q$ quiver and $C \leq KQ$ be an admissible subcoalgebra. Consider $X \subseteq Q_0$. TFAE:

- $\mathcal{T}_X \subseteq \mathcal{M}^C$ is colocalizing.
- $\dim_K K\text{Tail}_X^Q(x) \cap C$ is finite for all $x \in X$. 
\{ S_x \}_{x \in l_C} \text{ simple } C\text{-comodules.}
\mathcal{K} = \{ S_x \}_{x \in l_e \subseteq l_C} \text{ simple } eC e\text{-comodules.}

**Lemma**

\[ T(S_x) = \begin{cases} 
S_x, & \text{if } S_x \in \mathcal{K} \\
0, & \text{if } S_x \notin \mathcal{K}
\end{cases} \]
Length vector and quotient functor

\{S_x\}_{x \in l_C} \text{ simple } C\text{-comodules.}
\mathcal{K} = \{S_x\}_{x \in l_e \subseteq l_C} \text{ simple } eCe\text{-comodules.}

**Lemma**

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S_x, & \text{if } S_x \in \mathcal{K} \\
0, & \text{if } S_x \notin \mathcal{K}
\end{cases} \]
**Lemma**

Let $L$ be a finite dimensional right $C$-comodule, then $(\text{length } L)_i = (\text{length } eL)_i$ for all $i \in I_e$.

The following diagram is commutative

\[
\begin{array}{ccc}
M^C_f & \xrightarrow{e(-)} & M^{eCe}_f \\
\downarrow \text{length} & & \downarrow \text{length} \\
\mathbb{Z}^{lC} \cong K_0(C) & \xrightarrow{f} & \mathbb{Z}^{l_e} \cong K_0(eCe)
\end{array}
\]

$f$ is the projection.
Length vector and section functor

Restriction: \( S \) does not preserve finite dimension.

Example

Let \( Q \) be the quiver

\[
\cdots \xrightarrow{\alpha_{n+1}} n+1 \xrightarrow{\alpha_n} n \xrightarrow{\alpha_{n-1}} n-1 \cdots \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_1} 1
\]

\( S(S_1) \cong C \mathfrak{e} \cong < 1, \{\alpha_1 \cdots \alpha_{n-1} \alpha_n\}_{n \geq 1} > . \)

Moreover, \( S \) does not preserve simple comodules.
Restriction: $S$ does not preserve finite dimension.

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Let $Q$ be the quiver

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\ldots \xrightarrow{\alpha_{n+1}} n+1 \xrightarrow{\alpha_n} n \xrightarrow{\alpha_{n-1}} n-1 \xrightarrow{\cdots} 3 \xrightarrow{\alpha_2} 2 \xrightarrow{\alpha_1} 1
$$

$$S(S_1) \cong C\epsilon \cong \langle 1, \{\alpha_1 \cdots \alpha_{n-1} \alpha_n\}_{n \geq 1} \rangle .$$

Moreover, $S$ does not preserve simple comodules.
**Lemma**

The following are equivalent:
1. \( S(S_x) \) is finite dimensional for each \( x \in I_e \).
2. \( S \) preserves finite dimensional comodules.

How can we describe the comodule \( S(S_x) \)?

In path coalgebras,
Lemma

The following are equivalent:

- $S(S_x)$ is finite dimensional for each $x \in I_e$.
- $S$ preserves finite dimensional comodules.

How can we describe the comodule $S(S_x)$?

In path coalgebras,
The vertices are $\{S_x\}_{x \in I_C}$

$S_x \to S_y \iff \text{Ext}_C^1(S_x, S_y) \neq 0$

**Proposition**

The following are equivalent:

- $S(S_x) = S_x$
- There is no arrows from a torsion vertex to $S_x$ in $\Gamma_C$. 

$\Gamma_C$ Ext-quiver of $C \equiv \left\{ \begin{align*}
\text{The vertices are } & \{S_x\}_{x \in I_C} \\
S_x \to S_y & \iff \text{Ext}_C^1(S_x, S_y) \neq 0
\end{align*} \right.$
Proposition

Assume $S$ preserves finite dimensional comodules. Then $\Omega_V$ is finite.
Theorem

Assume $S$ preserves finite dimensional comodules. If $C$ is tame then $eCee$ is tame.

Example: left semiperfect coalgebras.
Wildness and the colocalizing functor

It is well known that $H$:

- is a **fully faithful** functor.
- is a **right exact** functor.
- respects **isomorphism classes**.
- preserves **finite dimensional** comodules.
- preserves **finite dimensional indecomposable** comodules.

**Proposition**

Assume that $T_e$ is **perfect colocalizing**. If $eC_e$ is wild $\Rightarrow$ $C$ is wild.
Wildness and the colocalizing functor

It is well known that $H$:

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**Proposition**

Assume that $\mathcal{T}_e$ is perfect colocalizing. If $eCe$ is wild $\Rightarrow$ $C$ is wild.
Wildness and the section functor

It is well known that $S$:

- is a **fully faithful** functor.
- is a **left exact** functor.
- respects **isomorphism classes**.
- preserves **quasi-finite indecomposable** comodules.

**Proposition**

Assume that:

- $T_e$ be **perfect localizing**
- $S$ preserves **finite dimensional** comodules.

If $eCe$ is wild $\Rightarrow$ $C$ is wild.

**Example**: left semiperfect coalgebras
Wildness and the section functor

It is well known that $S$:
- is a **fully faithful** functor.
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**Proposition**

Assume that:
- $T_e$ be **perfect localizing**
- $S$ **preserves finite dimensional** comodules.

If $eC e$ is wild $\Rightarrow$ $C$ is wild.

**Example**: left semiperfect coalgebras
Theorem

Let $Q$ be an **acyclic** quiver and $C$ be an admissible subcoalgebra of $KQ$. If $C$ is **not the path coalgebra of a quiver with relations** then $C$ is of **wild** comodule type.

Corollary (Acyclic Gabriel’s theorem for coalgebras)

Let $Q$ be an acyclic quiver. Then any tame admissible subcoalgebra of $KQ$ is the path coalgebra of a quiver with relations.
Theorem

Let $Q$ be an acyclic quiver and $C$ be an admissible subcoalgebra of $KQ$. If $C$ is not the path coalgebra of a quiver with relations then $C$ is of wild comodule type.

Corollary (Acyclic Gabriel’s theorem for coalgebras)

Let $Q$ be an acyclic quiver. Then any tame admissible subcoalgebra of $KQ$ is the path coalgebra of a quiver with relations.
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Representation Theory of Coalgebras