Simple and injective comodules and localization in coalgebras

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New techniques in Hopf algebras and graded ring theory
Brussels, September 2006
Most kinds of coalgebras are defined by properties of:
- its category of comodules
- or merely, its injective or simple comodules

For instance,
- pointed coalgebras
- co-semi-simple coalgebras
- serial coalgebras
- hereditary coalgebras
- semi-perfect coalgebras
- quasi-co-frobenius coalgebras
- and others...
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Study the behavior of injective and simple comodules in different situations

In particular, in this talk,

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Study how the functors associated to a localizing subcategory transform these comodules
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Study how the functors associated to a localizing subcategory transform these comodules
A dense subcategory of an abelian category $\mathcal{C}$:

- There is a quotient functor $T : \mathcal{C} \to \mathcal{C}/\mathcal{A}$.
- $\mathcal{A}$ is localizing if $T$ has a right adjoint functor $S : \mathcal{C}/\mathcal{A} \to \mathcal{C}$ (section functor).
- $\mathcal{A}$ is perfect localizing if $S$ is exact.

**Theorem**

- $T$ is an exact functor.
- $S$ is a fully faithful and left exact functor.
- $TS = 1_{\mathcal{C}/\mathcal{A}}$. 
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Dually,

- \mathcal{A} is \textbf{colocalizing} if \( T \) has a left adjoint functor \( H : \mathcal{C} / \mathcal{A} \to \mathcal{C} \) (\textit{colocalizing functor}).
- \( \mathcal{A} \) is \textbf{perfect colocalizing} if \( H \) is exact.

**Theorem**

- \( H \) is a fully faithful and right exact functor.
- \( TH = 1_{\mathcal{C} / \mathcal{A}} \).
Dually,

- \( \mathcal{A} \) is **colocalizing** if \( T \) has a left adjoint functor \( H : C/\mathcal{A} \to C \) (**colocalizing functor**).
- \( \mathcal{A} \) is **perfect colocalizing** if \( H \) is exact.

**Theorem**

- \( H \) is a fully faithful and right exact functor.
- \( TH = 1_{C/\mathcal{A}} \).
$C$ be a coalgebra and $\mathcal{M}^C$ the right $C$-comodules.

**Theorem**

There are one-to-one correspondences between:
- **Localizing subcategories** of $\mathcal{M}^C$.
- Classes of equivalence of **injective** $C$-comodules.
- **Coidempotent subcoalgebras** of $C$ ($A \wedge A = A$).
- Sets of **simple** $C$-comodules.
- Sets of **indecomposable injective** $C$-comodules.
- Classes of equivalence of **idempotents** in $C^*$. 

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\( \mathcal{T}_e \) localizing subcategory associated to \( e \in C^* \).

- There is an equivalence \( \mathcal{M}^C / \mathcal{T}_e \cong \mathcal{M}^{eCe} \)

\[
\Delta_{eCe}(exe) = \sum_{(x)} ex(1)e \otimes ex(2)e
\]

if \( \Delta_C(x) = \sum_{(x)} x(1) \otimes x(2) \)

- The quotient \( T = e(-) = -\square_C eC = \text{Cohom}_C(Ce, -) \)
- The section \( S = -\square_{eCe} Ce \)
- The colocalizing functor \( H = \text{Cohom}^e_{eCe}(eC, -) \)
\( \mathcal{T}_e \) localizing subcategory associated to \( e \in C^* \).

- There is an equivalence \( \mathcal{M}_e^C / \mathcal{T}_e \cong \mathcal{M}^{eC} \)

\[
\Delta_{eC}(exe) = \sum_{(x)} e \times_1 e \otimes e \times_2 e
\]

if \( \Delta_C(x) = \sum_{(x)} x_1 \otimes x_2 \)

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- The colocalizing functor \( H = \text{Cohom}_{eC}(eC, -) \)
Example: localization of path coalgebras

\[ Q = (Q_0, Q_1) \text{ quiver} \]
\[ e \text{ idempotent in } (KQ)^* \]
\[ X \subseteq Q_0 \text{ vertices associated to } e \]

\[ X \leftrightarrow e(p) = \begin{cases} 
1 & \text{if } p \in X \\
0 & \text{otherwise} 
\end{cases} \]

Let \( p \) be a path in \( Q \)

\[
\begin{array}{cccccc}
O & \rightarrow & O & \rightarrow & \cdots & \rightarrow & O & \rightarrow & O \\
X_1 & & X_2 & & \cdots & & X_{n-1} & & X_n
\end{array}
\]

- \( p \) is a **cell** relative to \( X \) if \[
\begin{cases}
x_1, x_n \in X, \\
x_2, x_3, \ldots, x_{n-1} \notin X
\end{cases}
\]
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\begin{array}{c}
\circ \rightarrow \circ \rightarrow \circ \rightarrow \cdots \rightarrow \circ \rightarrow \circ \\
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Theorem

\( e(KQ)e \cong KQ^e \), where \( Q^e = (X, \text{Cell}_X^Q) \).

Example

\( e(\circ) = 1 \) and \( e(\bullet) = 0 \)

\[ A_3 : \circ \rightarrow \bullet \rightarrow \circ \]

\[ (A_3)^e : \circ \rightarrow \circ \]
**Theorem**

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**Example**

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\[ \begin{array}{c}
\xymatrix{
\bigtriangleup^3 : & \circ \ar[r] & \bullet \ar[r] & \circ \\
& & \circ \\
& & \circ \\
& & \circ 
}
\end{array} \]

\[ (\bigtriangleup^3)^e : \begin{array}{c}
\circ \\
\circ 
\end{array} \]
Theorem

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Localization in coalgebras
Let $C$ any (basic) coalgebra (over a field)

- $\{S_x\}_{x \in I_C}$ simple $C$-comodules
- $\{E_x\}_{x \in I_C}$ indecomposable injectives

Let $e \in C^*$ idempotent element

- $K_e = \{S_x\}_{x \in l_e \subseteq I_C}$ simple $eC_ee$-comodules
- $\{\overline{E}_x\}_{x \in l_e}$ indecomposable injectives
Let $C$ any (basic) coalgebra (over a field)

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- $\{\bar{E}_x\}_{x \in l_e}$ indecomposable injectives
Simple and injective comodules

\[ \mathcal{M}^C \xleftarrow{T = e(-) = - \square_C eC} \mathcal{M}^{eCe} \xrightarrow{S = - \square_{eCe} Ce} \mathcal{M}^{eCe} \cdot \]

**Lemma**

\[ T(S_x) = \begin{cases} S_x, & \text{if } S_x \in \mathcal{K}_e \\ 0, & \text{if } S_x \notin \mathcal{K}_e \end{cases} \]

**Lemma**

\[ S(\overline{E}_x) = E_x \text{ for all indecomposable injective comodule} \]
Simple and injective comodules

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$$T(S_x) = \begin{cases} 
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Lemma

$$S(\overline{E}_x) = E_x$$ for all indecomposable injective comodule
Section functor and simple comodules

Problem

Is $S(S_x) = S_x$ for any simple $eC_e$-comodule?

Example

$S(S_o) = S_o \square_{eC_e} C_e \cong C_e \cong < x, \alpha > \neq S_o$

$S$ does not preserve simples

Example

$S(S_o) = S_o \square_{eC_e} C_e \cong C_e \cong < 1, \{\alpha_1 \cdots \alpha_{n-1} \alpha_n\}_{n \geq 1} >$

$S$ does not preserve finite dimension
Section functor and simple comodules

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Is $S(S_x) = S_x$ for any simple $eCe$-comodule?

Example

$\bullet \xrightarrow{\alpha} \circ$

$S(S_o) = S_o \bigotimes_{eCe} Ce \cong Ce \cong < x, \alpha > \neq S_o$

$S$ does not preserve simples

$\cdots \xrightarrow{\alpha_{n+1}} \bullet \xrightarrow{\alpha_n} \bullet \xrightarrow{\alpha_{n-1}} \bullet \cdots \xrightarrow{\alpha_2} \bullet \xrightarrow{\alpha_1} \circ$

$S(S_o) = S_o \bigotimes_{eCe} Ce \cong Ce \cong < 1, \{\alpha_1 \cdots \alpha_{n-1} \alpha_n\}_{n \geq 1} >$

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**Problem**

Is $S(S_x) = S_x$ for any simple $eCe$-comodule?

**Example**

$$S(S_o) = S_o \Box^e_{eCe} Ce \cong Ce \cong \langle x, \alpha \rangle \neq S_o$$

*S does not preserve simples*

**Example**

$$S(S_o) = S_o \Box^e_{eCe} Ce \cong Ce \cong \langle 1, \{\alpha_1 \cdots \alpha_{n-1} \alpha_n\}_{n \geq 1} \rangle$$

*S does not preserve finite dimension*
Geometry of the Ext-quiver

\[ \Gamma_C \text{ Ext-quiver of } C \equiv \left\{ \begin{array}{l}
\text{The vertices are } \{S_x\}_{x \in I_C} \\
S_y \rightarrow S_x \iff \text{Ext}_C^1(S_y, S_x) \neq 0
\end{array} \right\} \]

**Example**

\[
\begin{array}{c}
\circ y \xrightarrow{\alpha} \circ z \xrightarrow{\beta} \circ x
\end{array}
\]

- \(C_1 = KQ\)
- \(C_2 = \langle x, y, z, \alpha, \beta \rangle\)

Then \(\Gamma_{C_1} = \Gamma_{C_2}\) is

\[ S_y \rightarrow S_z \rightarrow S_x \]
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**Example**

\[ \begin{array}{ccc} y & \xrightarrow{\alpha} & z \\ \downarrow & & \downarrow \\ x & \xrightarrow{\beta} & \end{array} \]

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Example

\[
\begin{array}{ccc}
\circ_y & \xrightarrow{\alpha} & \circ_z \\
\beta & & \xrightarrow{} \circ_x \\
\end{array}
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- $C_1 = KQ$
- $C_2 = \langle x, y, z, \alpha, \beta \rangle$

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Geometry of the Ext-quiver

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S_y \to S_z \to S_x
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Lemma

\[ \text{Soc}\left( \frac{E_x}{S_x} \right) = \bigoplus \text{immediate predecessors of } S_x. \]

Consider the Loewy filtration of \( E_x \),

\[ 0 \subset \text{Soc} E_x \subset \text{Soc}^2 E_x \subset \cdots \subset E_x \]

Definition

\[ \text{Soc}\left( \frac{E_x}{\text{Soc}^n(E_x)} \right) = \bigoplus n\text{-predecessors of } S_x. \]

Proposition

If \( S_y \) is an \( n \)-predecessor of \( S_x \) then there is a \( n \)-length path from \( S_y \) to \( S_x \).

The converse holds if \( C \) is hereditary.
Lemma
\[ \text{Soc} \left( \frac{E_x}{S_x} \right) = \bigoplus \text{immediate predecessors of } S_x. \]

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The converse holds if \( C \) is hereditary.
Proposition

TFAE:

- $S_y$ is a $n$-predecessor of $S_x$.
- There exists a morphism $f : E_x \to E_y$ such that
  
  $f(\text{Soc}^i E_x) = 0$ for all $i = 1, \ldots, n$
  
  $f(\text{Soc}^{n+1} E_x) \neq 0$

Corollary

$S_y$ is a predecessor of $S_x$ if and only if $\text{Rad}_C(E_x, E_y) \neq 0$. 
Geometry of the Ext-quiver

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Corollary

$S_y$ is a predecessor of $S_x$ if and only if $\text{Rad}_C(E_x, E_y) \neq 0$. 
Section functor and simple comodules

Problem
Who is $S(S_x)$? At least we know $S_x \subseteq S(S_x) \subseteq E_x$.

Theorem
\[
\frac{S(S_x)}{S_x} \text{ is the torsion subcomodule of } \frac{E_x}{S_x}
\]
Problem

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Corollary

TFAE:

- $S(S_x) = S_x$
- $\frac{E_x}{S_x}$ is torsion-free
- $\not\exists S_y \to S_x$ such that $T(S_y) = 0$
- $\text{Hom}_C(E_x, E_y) = 0$ when $T(S_y) = 0$

Corollary

$S(S_x) = E_x$ if and only if all predecessors of $S_x$ are torsion.
Corollary

TFAE:

- \( S(S_x) = S_x \)
- \( \frac{E_x}{S_x} \) is torsion-free
- \( \nexists \ S_y \to S_x \) such that \( T(S_y) = 0 \)
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Corollary

\( S(S_x) = E_x \) if and only if all predecessors of \( S_x \) are torsion.
Consider the Loewy filtration of $S(S_x)$,

$$0 \subset \text{Soc} S(S_x) \subset \text{Soc}^2 S(S_x) \subset \cdots \subset S(S_x)$$

**Theorem**

If $S_y \subseteq \frac{S(S_x)}{\text{Soc}^n S(S_x)}$ for some $n \geq 1$, then:

- $S_y$ is torsion.
- $S_y$ is a $n$-predecessor of $S_x$.
- There exists a path

$$S_y \rightarrow S_{n-1} \rightarrow \cdots \rightarrow S_2 \rightarrow S_1 \rightarrow S_x$$

such that $S_i$ is torsion for all $i = 1, \ldots, n-1$.

The converse also holds if $C$ is hereditary.
Corollary

Let $Q$ be a quiver and $X \subseteq Q_0$. For each vertex $x \in X$, the $KQ$-comodule $S(S_x)$ is generated by the set of paths

$$\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \bullet \rightarrow \circ \rightarrow x$$
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Who is $T(E_x)$?

Problem

Is $T(E_x) = \begin{cases} \bar{E}_x & \text{if } T(S_x) = S_x, \\ 0 & \text{if } T(S_x) = 0. \end{cases}$?

Example

\begin{equation}
\begin{tikzcd}
\bullet \arrow[r, \alpha] & \circ \\
& \circ \arrow[u, \circlearrowleft, x] & y
\end{tikzcd}
\end{equation}

Then $T(E_y) = e < y, \alpha > \cong S_x \neq 0$. 
Problem

*Who is $T(E_x)$?*

Is $T(E_x) = \begin{cases} \overline{E}_x & \text{if } T(S_x) = S_x, \\ 0 & \text{if } T(S_x) = 0. \end{cases}$?

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Localization in coalgebras
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Then $T(E_y) = e < y, \alpha \cong S_x \neq 0$. 

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Quotient functor and injective comodules

Theorem

For a torsion simple comodule \( S_y \). TFAE:

- \( T(E_y) = 0 \)
- \( \text{Hom}_C(E_y, E_x) = 0 \) when \( T(S_x) = S_x \)
- \( S_y \) has no torsion-free predecessors
- \( \nexists S_x \to S_y \) such that \( T(S_x) = S_x \)
- \( T_e \) is a stable subcategory
Corollary

Let $S_y$ be a torsion simple comodule. If $S_x \subseteq \text{Soc } T(E_y)$ then:

- $S_x$ is torsion-free
- $S_x$ is a predecessor of $S_y$
- There exists a path

$$S_x \rightarrow S_n \rightarrow \cdots \rightarrow S_2 \rightarrow S_1 \rightarrow S_y$$

such that $S_i$ is torsion for all $i = 1, \ldots, n$

If $C$ is hereditary, the converse also holds.
Corollary

Let $Q$ be a quiver and $X \subseteq Q_0$. For each vertex $y \notin X$, $S_x \subseteq T(E_y)$ if and only if there is a path

\[ \text{x} \quad \Rightarrow \quad \text{•} \quad \Rightarrow \quad \text{•} \quad \Rightarrow \quad \text{•} \quad \Rightarrow \quad \text{•} \quad \Rightarrow \quad \text{y} \]
Summary

Theorem

TFAE:

- $T_e$ is a stable subcategory.
- $T(E_x) = \begin{cases} \overline{E}_x & \text{if } T(S_x) = S_x, \\ 0 & \text{otherwise.} \end{cases}$
- $\text{Hom}_C(E_y, E_x) = 0$ when $T(S_x) = S_x$ and $T(S_y)$
- Any torsion vertex has no torsion-free predecessor
- $\mathcal{K} = \{ S \in (\Gamma_C)_0 \mid T(S) = S \}$ is right link-closed
- There is no path from a torsion-free vertex to a torsion vertex
- $e$ is a left semicentral idempotent in $C^*$.

If $T_e$ is a colocalizing

- $H(S_x) = S_x$ for any simple

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Theorem

TFAE:

- $\mathcal{T}_{1-e}$ is a stable subcategory.
- $\text{Hom}_C(E_x, E_y) = 0$ when $T(S_x) = S_x$ and $T(S_y)$
- Any torsion-free vertex has no torsion predecessor.
- There is no path from a torsion vertex to a torsion-free vertex
- $\mathcal{K} = \{ S \in (\Gamma C)_0 \mid T(S) = S \}$ is left link-closed
- $e$ is a right semicentral idempotent in $C^*$.
- $S(S_x) = S_x$ for any simple
Simple and injective comodules and localization in coalgebras

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