Serial coalgebras and their valued Gabriel quivers

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Noncommutative Rings and Geometry

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Joint work with José Gómez Torrecillas
Serial coalgebras and their valued Gabriel quivers
arXiv:0707.0132v1 [math.RT]

Continuing the papers:

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Main aim

*Study serial coalgebras by means of their valued Gabriel quiver*

over an arbitrary field!!!

Looking for analogies with f. d. algebras:

- shape of the quiver
- description of the comodules (A-R quiver)
- are all comodules a direct sum of uniserials?
- Eisenbud-Griffith theorem?
- does localization preserve seriality?
What are we trying to do?

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Definitions: serial coalgebra

Socle filtration: \( \text{Soc } M \subset \text{Soc}^2 M \subset \text{Soc}^3 M \subset \cdots \subset M \)

- \( \text{Soc } M \) is the socle of \( M \)
- \( \frac{\text{Soc}^n M}{\text{Soc}^{n-1} M} = \text{Soc} \left( \frac{M}{\text{Soc}^{n-1} M} \right) \).

**Definition**

\( M \) is uniserial if \( \text{Soc} M \subset \text{Soc}^2 M \subset \cdots \subset M \) is a composition series.

**Definition**

\( C \) is (right, left) serial if (right, left) indecomposable injectives are uniserial.
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C basic coalgebra over an arbitrary field
\{S_i\}_{i \in I_C} set of simple C-comodules

**Definition**

The (right) valued Gabriel quiver of C is:
- Vertices are simple comodules
- There exists \( S_1 \overset{(a,b)}{\longrightarrow} S_2 \) if and only if
  - \( \text{Ext}^1_C(S_1, S_2) \neq 0 \)
  - \( a = \dim \text{End}_C(S_1) \text{Ext}^1_C(S_1, S_2) \)
  - \( b = \dim \text{End}_C(S_2) \text{Ext}^1_C(S_1, S_2) \)
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A basic coalgebra over an arbitrary field $C$ has a set of simple $C$-comodules $\{S_i\}_{i \in I}$.

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Proposition

The immediate predecessors of $S_i$ are given by $\frac{\text{Soc}^2 E_i}{\text{Soc} E_i}$.

Corollary

$C$ is right serial if and only if:

- Each vertex is the target of at most one arrow.
- The arrows are labeled by $(1, d)$.

Proposition

The left valued Gabriel quiver is the opposite to the right valued Gabriel quiver.
Properties of the valued Gabriel quiver

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Theorem

C is **serial** if and only if its valued Gabriel quiver is

(a) $\infty A_{\infty}$: ....... $\circ \rightarrow \circ \rightarrow \circ \rightarrow \circ \rightarrow \circ \rightarrow \circ$ .......

(b) $A_{\infty}$: $\circ \rightarrow \circ \rightarrow \circ \rightarrow \circ \rightarrow \circ \rightarrow \circ$ ........

(c) $\infty A$ : ....... $\circ \rightarrow \circ \rightarrow \circ \rightarrow \circ \rightarrow \circ \rightarrow \circ$

(d) $A_n$: $\circ \rightarrow \circ \rightarrow \circ$ ....... $\circ \rightarrow \circ \rightarrow \circ$ $n \geq 1$

(e) $\tilde{A}_n$: $\circ$ $\circ \leftarrow \circ \circ \rightarrow \circ$ $n \geq 1$

and the labels are (1, 1).
Characterization by means of quivers

**Theorem**

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(a) \( \infty \mathbb{A}_\infty : \quad \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \)

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(d) \( \mathbb{A}_n : \quad \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \quad n \geq 1 \)

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*and the labels are* \((1, 1)\).
Proposition

TFAE:
- \( M \) is uniserial.
- Every f.d. subcomodule of \( M \) is uniserial.

Theorem

\( C \) serial coalgebra, \( M \) indecomposable f.d. comodule
- \( M \cong \text{Soc}^n E_i = S_i^n \), \( E_i \) indecomposable injective.
- \( C \) has almost split sequence, in particular,

\[
\begin{align*}
\text{Soc}^n E & \xrightarrow{(i)} \text{Soc}^{n+1} E \oplus \frac{\text{Soc}^n E}{\text{Soc} E} \xrightarrow{(q-j)} \frac{\text{Soc}^{n+1} E}{\text{Soc} E}
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Finite dimensional comodules

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Proposition

TFAE:

- \( M \) is uniserial.
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Theorem

Let \( C \) be a serial coalgebra, \( M \) an indecomposable finite-dimensional comodule

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Serial coalgebras
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Theorem

A serial finite dimensional algebra, then each module is a direct sum of uniserial modules

And for coalgebras?? Not really!!

Proposition

C serial. Each comodule is direct sum of uniserial comudules if and only if C is pure-semisimple.

Counterexample

Consider the path coalgebra of ◦
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Consider the path coalgebra of ◦
A dense subcategory of an abelian category $\mathcal{C}$:

- There is a quotient functor $T : \mathcal{C} \to \mathcal{C}/\mathcal{A}$.
- $\mathcal{A}$ is localizing if $T$ has a right adjoint functor $S : \mathcal{C}/\mathcal{A} \to \mathcal{C}$ (section functor).
- $\mathcal{A}$ is perfect localizing if $S$ is exact.

\[ \mathcal{C} \xleftrightarrow{T} \mathcal{C}/\mathcal{A} \xleftrightarrow{S} \]

Proposition

- $T$ is an exact functor.
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C be a coalgebra and $\mathcal{M}^C$ the right $C$-comodules.

**Theorem**

There are one-to-one correspondences between:

- **Localizing subcategories** of $\mathcal{M}^C$.
- Classes of equivalence of *injective* $C$-comodules.
- **Coidempotent subcoalgebras** of $C$ ($A \wedge A = A$).
- Sets of *indecomposable injective* $C$-comodules.
- Sets of *simple* $C$-comodules.
- Classes of equivalence of *idempotents* in $C^*$.

**Corollary**

$\mathcal{M}^C / T_e \simeq \mathcal{M}^{eC_0}$
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$\mathcal{M}^C / \mathcal{T}_e \simeq \mathcal{M}^{eCe}$
Example: localization of path coalgebras

\[ Q = (Q_0, Q_1) \text{ quiver} \]
\[ e \text{ idempotent in } (KQ)^* \]
\[ X \subseteq Q_0 \text{ vertices associated to } e \]

\[ X \leftrightarrow e(p) = \begin{cases} 
1 & \text{if } p \in X \\
0 & \text{otherwise}
\end{cases} \]

Let \( p \) be a path in \( Q \)

\[ x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_{n-1} \rightarrow x_n \]

\( p \) is a \textbf{cell} relative to \( X \) if

\[ \begin{cases} 
x_1, x_n \in X, \\
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\[ \xymatrix{ 
& \bullet 
\ar[r] & \bullet 
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\circ \\
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\bullet \\
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\[ e(KQ)e \cong KQ^e, \text{ where } Q^e = (X, Cell_{X}^Q). \]

**Example**

\[ e(\circ) = 1 \text{ and } e(\bullet) = 0 \]

\[ A_3 : \circ \rightarrow \bullet \rightarrow \circ \]

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Some consequences

**Proposition**

If $M$ is a uniserial $C$-comodule, then $T(M) = eM$ is a uniserial $eCe$-comodule.

**Corollary**

- If $C$ is serial, then $eCe$ is serial.
- If any socle-finite $eCe$ is serial then $C$ is serial.

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$KQ$ is not serial.
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Theorem (Eisenbud-Griffith Theorem)

Every proper quotient of a hereditary noetherian prime ring is serial

“Coalgebraic” notions:

- Subcoalgebra
- hereditary (global dimension 0 or 1)
-Strictly quasi-finite (quotients are quasi-finite comodules)
- prime ($A \land B = C$, then $A = C$ or $B = C$).
Eisenbud-Griffith Theorem for coalgebras

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Theorem

Every subcoalgebra of a prime, hereditary and strictly quasi-finite coalgebra is serial.
STEP 1: Reduction to socle-finite coalgebras

\[ eCe \text{ socle-finite “localized” coalgebra} \]

\[ eCe \text{ is} \begin{cases} \text{hereditary} \\ \text{strictly quasi-finite} \\ \text{prime} \end{cases} \]

E-G theorem

\[ eCe \text{ is serial} \]

\[ C \text{ is serial} \]
STEP 2: The colocal case

Example

If \( (2,1) \) then \( C \) is NOT strictly quasi-finite

Therefore,

Lemma

The quiver of a colocal “localized” coalgebra is:

- a single point, or
- \( (1,1) \)
STEP 2: The colocal case

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Eisenbud-Griffith Theorem for coalgebras

STEP 3: Deduction of the quiver

Theorem

- If $C$ is prime, then each vertex is in a cycle.
- If $C$ is hereditary and

\[
\begin{array}{c}
  x \\
  \rightarrow (d_1, d_2) \\
  \rightarrow \bullet \\
  \rightarrow (c_1, c_2) \\
  \rightarrow y
\end{array}
\]

then

\[
\begin{array}{c}
  x \\
  \rightarrow (c_1 d_1, c_2 d_2) \\
  \rightarrow y
\end{array}
\]

Corollary

The quiver of $C$ is

Gabriel Navarro  Serial coalgebras
Eisenbud-Griffith Theorem for coalgebras

STEP 3: Deduction of the quiver

**Theorem**

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- If $C$ is hereditary and

$$
\begin{array}{c}
\circ \arrow[d, (d_1, d_2)] \circ \\
\downarrow \downarrow \\
\bullet \arrow[d, (c_1, c_2)] \circ \\
\downarrow \\
n \end{array} 
\begin{array}{c}
\circ \arrow[d, (c_1 d_1, c_2 d_2)] \circ \\
\downarrow \downarrow \\
\bullet \arrow[d, (c_1, c_2)] \circ \\
\downarrow \\
y 
\end{array}
$$

then

**Corollary**

The quiver of $C$ is

$$
\begin{array}{c}
1 \leftrightarrow 2 \leftrightarrow 3 \\
\downarrow \downarrow \downarrow \\
7 \leftrightarrow 6 \leftrightarrow 5 \\
\downarrow \downarrow \downarrow \\
4 \leftrightarrow 3 \leftrightarrow 2 \\
\downarrow \downarrow \downarrow \\
n \leftrightarrow 1 \leftrightarrow 2 \\
\downarrow \downarrow \downarrow \\
7 \leftrightarrow 6 \leftrightarrow 5 \\
\downarrow \downarrow \downarrow \\
4 \leftrightarrow 3 \leftrightarrow 2 \\
\downarrow \downarrow \downarrow \\
n \leftrightarrow 1 \leftrightarrow 2 \\
\end{array}
$$