# Representation Theory of Coalgebras. Localization in coalgebras.

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# Contents

# Introduction

Interest on studying the structure of coalgebra comes from different sources. Among them, historically, the best known is perhaps their relation with **Hopf algebras** and **quantum groups**, as one of the structures endowed to these objects. Nevertheless, recently, two new connections have been considered. On the one hand, in the framework of **Noncommutative Geometry**, one can generalize the notion of principal bundles by means of the so-called **entwining structures** $(A, C)_\varphi$, where $A$ is an algebra, $C$ a coalgebra and $\varphi : A \otimes C \to C \otimes A$ a map verifying some compatibility conditions; introduced by Brzeziński and Majid in [BM98], where the rôle of the fibre of the principal bundle is played by the coalgebra $C$. Later, this structures were used in order to unify different kinds of modules as Doi-Koppinen Hopf modules, Hopf modules or Yetter-Drinfeld modules, see for instance [CMZ02] and references therein. Entwining structures were also generalized by the notion of **coring** (cf. [BR03]). In the recent paper [KSa], Kontsevich and Soibelman associated to each **noncommutative thin scheme** (or a **noncommutative formal manifold**) $X$, a huge coalgebra $C_X$, which is called the **coalgebra of distributions** on $X$, and provide an equivalence between the category of coalgebras and the one of noncommutative thin schemes. This coalgebra $C_X$ is Morita-Takeuchi equivalent to the **path coalgebra** of an infinite quiver associated to the coordinate ring $A$ of the noncommutative manifold. Concretely, the quiver which has as its vertices the isomorphism classes of finite dimensional simple representations, and for which the number of directed arrows between the vertices corresponding to the simples $S$ and $T$ is the dimension of $\mathrm{Ext}^1_A(S, T)$.

On the other hand, due to the work of Gabriel [Gab62] dealing with the class of **pseudo-compact modules**, the idea of extending the classical **Representation Theory of Algebras** (finite dimensional over an algebraically closed field) to an infinite dimensional context has motivated to consider the structure of coalgebra from this point of view, since, because of the Fundamental Coalgebra Structure Theorem, it seems plausible to use techniques and get similar results to the ones developed for algebras, see [Chi04] and

[Sim01]. Indeed, **Representation Theory of Coalgebras** has been applied successfully studying properties of a whole Hopf algebra or quantum group which depends uniquely on the coalgebra structure (cf. [AD03] for an example linked with co-Frobenius Hopf algebras, and [Chi] and [Hai01] concerning the representations of the quantum group $SL(2)$ and the quantum groups of type $A_{0|0}$, respectively). Albeit the present memory also treats with arbitrary coalgebras over an arbitrary field, it should be seen inside this second case. Hence, from this point of view, we are aimed to find a **description and classification of coalgebras** extending the one existent for algebras of finite dimension.

The **quiver**-theoretical ideas developed by Gabriel and his school during the seventies have been the origin of many advances in Representation Theory of Algebras for years (cf. [ASS05], [ARS95] and [GR92]). Moreover, many of the present developments of the theory use, up to some extent, these techniques and results. Among these tools, it is mostly accepted that the main one is the famous **Gabriel theorem** which relates any finite dimensional algebra (over an algebraically closed field) to a nice quotient of a path algebra (see for instance [ASS05] or [ARS95]). The key-point of such method lies in the fact that it gives a description not only of the algebras, but also of all finitely generated modules by means of finite dimensional **representations** of the quiver. Nevertheless, due to the ambitious primary aim of the theory, that is, to describe, as a category, the modules over any artinian algebra, this theorem and many of the other techniques **strongly require finite dimensionality** over the field and it does not seem possible to generalize them to an arbitrary algebra.

Recently, some authors have tried to get rid of the imposed finiteness conditions by taking advantages of coalgebras and their category of comodules, see [Chi02], [JMN05], [JMN07], [KS05], [Nav08], [Sim01], [Sim05] or [Woo97]. Main reasons for this are, on the one hand, that coalgebras may be realized, because of the freedom on choosing their dimension, as an intermediate step between finite dimensional and infinite dimensional algebras. More concretely, in [Sim01], it is proven that the category of comodules over a coalgebra is equivalent to the category of pseudo-compact modules, in the sense of [Gab72], over the dual algebra. And, on the other hand, because of their locally finite nature, coalgebras are a good candidate for extending many techniques and results stated for finite dimensional algebras. Therefore it is rather natural to ask oneself about the development of the following points in coalgebra theory:

1. Obtain some **quiver techniques** similar to the classical ones stated for algebras. For instance, a description of coalgebras and comodules

by means of quivers and linear representation of quivers (cf. [JMN05], [JMN07] and [Sim05]).

2. Define the **representation types of comodules** (finite, tame and wild) and find some criterions in order to determine the representation type of a coalgebra. Prove the **tame-wild dichotomy** for coalgebras (cf. [Sim01] and [Sim05]).

3. Describe completely the representation theory of some particular classes of coalgebras. For example, co-semisimple, pure semisimple [KS05], semiperfect [Lin75], **serial** [CGT04], biserial [Chi], **hereditary** [JLMS05], or others.

This thesis is dedicated to develop the above questions. Concretely, our primary aim will be to **find a version for coalgebras of the aforementioned Gabriel Theorem**. The innovative technique consists in to apply the theory of **localization** in coalgebras in order solve these problems. Here, by localization we mean the approach given by Gabriel in [Gab62]. This is a categorical generalization of the well-known theory of localization in non-commutative rings and the construction of the ring of fractions, see the books [GW89] and [MR87]. From this point of view, to any coalgebra, we may attach a set of **"localized" coalgebras** and some functors relating the comodule categories. It is worth noting that the behavior of these functors strongly depends on its action with **simple and injective comodules**. Therefore the underlying idea is to deduce properties of a coalgebra by means of its local structure, that is, by means of its "localized" coalgebras. Although a whole chapter of the memory is devoted to Localization (this is the secondary topic of the thesis!), all along the memory some sections are also concerned about this theory in a more specific context.

The work is structured as follows. In Chapter 1 we give some of the background material that will be used all along the work. In particular, we recall that one may endow the path algebra $KQ$ with a structure of graded $K$-coalgebra with comultiplication induced by the decomposition of paths, that is, if $p = \alpha_m \cdots \alpha_1$ is a non-trivial path from a vertex $i$ to a vertex $j$, then

$$\Delta(p) = e_j \otimes p + p \otimes e_i + \sum_{i=1}^{m-1} \alpha_m \cdots \alpha_{i+1} \otimes \alpha_i \cdots \alpha_1 = \sum_{\eta\tau=p} \eta \otimes \tau$$

and $\Delta(e_i) = e_i \otimes e_i$ for a trivial path $e_i$. The counit of $KQ$ is defined by the formula

$$\epsilon(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is a vertex,} \\ 0 & \text{if } \alpha \text{ is a non-trivial path.} \end{cases}$$

This coalgebra is called the **path coalgebra** of the quiver $Q$. In [Woo97], the author proves that every pointed coalgebra is isomorphic to a subcoalgebra of a path coalgebra. The next step is given by Simson in [Sim01]. There, the author tries to get a better approximation by means of the notion of **path coalgebra of a quiver with relations** $(Q,\Omega)$ as the subspace of $KQ$ given by

$$C(Q,\Omega) = \{a \in KQ \mid \langle a, \Omega\rangle = 0\},$$

where $\langle -,-\rangle : KQ \times KQ \longrightarrow K$ is the bilinear map defined by $\langle v,w\rangle = \delta_{v,w}$ (the Kronecker delta) for any two paths $v,w$ in $Q$.

One of the main motivations given in [Sim01] and [Sim05] for presenting a basic coalgebra $C$ in the form $C(Q,\Omega)$ is the fact that, in this case, there is a linear equivalence of the category $\mathcal{M}_f^C$ of finite dimensional right $C$-comodules with the category $\mathrm{nilrep}_K^{\mathrm{lf}}(Q,\Omega)$ of **nilpotent linear representations of finite length** of the quiver with relations $(Q,\Omega)$ (see [Sim01, p. 135] and [Sim05, Theorem 3.14]). Then that definition is consistent with the classical theory and reduces the study of the category $\mathcal{M}^C$ to the study of linear representations of a quiver with relations $(Q,\Omega)$. Therefore the following question is raised in [Sim01, Section 8]:

**Question.** *Is any basic coalgebra, over an algebraically closed field, isomorphic to the path coalgebra of a quiver with relations?.*

This is an analogue for coalgebras of Gabriel's theorem. In Chapter 2 we consider this problem. For that aim, we separate the admissible subcoalgebras of a path coalgebra into two classes depending on whether the coalgebra is generated by paths or not. Obviously, the first class is easy to study and we focus our efforts on the second one. For this purpose, we establish a general framework using the weak* topology on the dual algebra to treat the problem in an elementary context. In particular, we describe the path coalgebra of a quiver with relations $(Q,\Omega)$ as the orthogonal space $\Omega^{\perp}$ of the ideal $\Omega$. Then the former problem may be rewritten as follows: *for any admissible subcoalgebra $C \leq KQ$, is there a relation ideal $\Omega$ of the algebra $KQ$ such that $\Omega^{\perp} = C$?* The result is proven in [Sim05] for the family of coalgebras $C$ such that the Gabriel quiver $Q_C$ of $C$ is intervally finite. Unfortunately, that proof does not hold for arbitrary coalgebras, as a class of counterexamples given in Section 2.5 shows. Moreover, there it is proven a criterion allowing us to decide whether or not a coalgebra is of that kind:

**Criterion** (2.5.11)**.** *Let $C$ be an admissible subcoalgebra of a path coalgebra $KQ$. Then $C$ is not the path coalgebra of a quiver with relations if and only if there exist infinite different paths $\{\gamma_i\}_{i\in\mathbb{N}}$ in $Q$ such that:*

$(a)$ *All of them have common source and common sink.*

$(b)$ *None of them is in* $C$.

$(c)$ *There exist elements* $a_j^n \in K$ *for all* $j, n \in \mathbb{N}$ *such that the set* $\{\gamma_n + \sum_{j>n} a_j^n \gamma_j\}_{n\in\mathbb{N}}$ *is contained in* $C$.

Nevertheless, it is worth noting that for all counterexamples found in Section 2.5, the category of finitely generated comodules has very bad properties. This bad behavior is similar to the notion of **wildness** given for the finitely generated module categories of some finite dimensional algebras, meaning that this category is so big that it contains (via an exact representation embedding) the category of all finite dimensional representations of the noncommutative polynomial algebra $K\langle x, y\rangle$. As it is well-known, the category of finite dimensional modules over $K\langle x, y\rangle$ contains (again via an exact representation embedding) the category of all finitely generated representations for any other finite dimensional algebra, and thus it is not realistic aiming to give an explicit description of this category (or, by extension, of any wild algebra). The counterpart to the notion of wild algebra is the one of **tameness**, a tame algebra being one whose indecomposable modules of finite dimension are parametrised by a finite number of one-parameter families for each dimension vector. A classical result in representation theory of algebras (the **Tame-Wild Dichotomy**, see [Dro79]) states that any finite dimensional algebra over an algebraically closed field is either of tame module type or of wild module type. We refer the reader to [Sim92] for basic definitions and properties about module type of algebras.

Analogous concepts were defined by Simson in [Sim01] for coalgebras. In [Sim05], it was proven a weak version of the Tame-Wild Dichotomy that goes as follows: over an algebraically closed field, if $C$ is coalgebra of tame comodule type, then $C$ is not of wild comodule type. The full version remains open:

**Conjecture.** *Any coalgebra, over an algebraically closed field, is either of tame comodule type, or of wild comodule type, and these types are mutually exclusive.*

As, according to Criterion (2.5.11) above, the coalgebras which are not path coalgebras of quivers with relations are close to be wild, we may reformulate the problem stated above as follows:

**Question.** *Is any basic coalgebra of tame comodule type, over an algebraically closed field, isomorphic to the path coalgebra of a quiver with relations?*

In order to handle this problem, in Chapter 3 we develop the localization in coalgebras. The category $\mathcal{M}^C$ of right comodules over a coalgebra $C$ is a locally finite Grothendieck category in which the theory of localization as described by Gabriel in [Gab62] can be applied. Namely, for any **localizing subcategory** $\mathcal{T} \subseteq \mathcal{M}^C$, we can construct a new category $\mathcal{M}^C/\mathcal{T}$ (the **quotient category**) and a pair of adjoint functors $T : \mathcal{M}^C \to \mathcal{M}^C/\mathcal{T}$ and $S : \mathcal{M}^C/\mathcal{T} \to \mathcal{M}^C$, the **quotient and the section functor**. The keypoint of the theory lies in the fact that the **quotient category becomes a comodule category**, and then it is better understood than in the case of modules over an arbitrary algebra. Another good comes from that there are several descriptions of the localizing subcategories by means of different concepts. From the general theory of localization in Grothendieck categories, it is well-known that there exists a one-to-one correspondence between localizing subcategories of $\mathcal{M}^C$ and sets of **indecomposable injective** right $C$-comodules, and, as a consequence, sets of **simple** right $C$-comodules, see [Gre76], [Lin75], [NT94] and [NT96]. More concretely, a localizing subcategory is determined by an injective right $C$-comodule $E = \oplus_{j\in J} E_j$, where $J \subseteq I_C$ (therefore the associated set of indecomposable injective comodules is $\{E_j\}_{j\in J}$). Then $\mathcal{M}^C/\mathcal{T} \simeq \mathcal{M}^D$, where $D$ is the coalgebra of coendomorphism $\mathrm{Cohom}_C(E,E)$, and the quotient and section functors are $\mathrm{Cohom}_C(E,-)$ and $-\Box_D E$, respectively. We also recall that, by [Sim07], the quotient and the section functors define an equivalence of categories between $\mathcal{M}^D$ and the category $\mathcal{M}^C_E$ of $E$-copresented right $C$-comodules, that is, the right $C$-comodules $M$ which admits an exact sequence

$$0 \longrightarrow M \longrightarrow E_0 \longrightarrow E_1 ,$$

where $E_0$ and $E_1$ are direct sums of direct summands of the comodule $E$. Following [CGT02] and [JMNR06], we may get a bijection between localizing subcategories of $\mathcal{M}^C$ and equivalence classes of **idempotent elements** of the dual algebra. That fact allows us to describe the quotient category as the category of comodules over the coalgebra $eCe$, whose structure is given by

$$\Delta_{eCe}(exe) = \sum_{(x)} ex_{(1)}e \otimes ex_{(2)}e \quad \text{and} \quad \epsilon_{eCe}(exe) = \epsilon_C(x)$$

for any $x \in C$, where $\Delta_C(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$, using the sigma-notation of [Swe69]. This is frequently used when dealing with pointed coalgebras since, in that case, the localization is somehow **combinatoric** since it is done by keeping and removing suitable vertices and arrows of the Gabriel quiver according to the associated idempotent. Moreover, the "localized" coalgebras may be described by certain manipulation of the quiver. In this direction, we introduce cells and tails of a quiver and prove the following:

**Theorem** (3.6.3 and 3.6.8). *Let $C$ be an admissible subcoalgebra of a path coalgebra $KQ$ of a quiver $Q$. Let $e_X$ be the idempotent of $C^*$ associated to a subset of vertices $X$. The following statements hold:*

(*a*) *the localized coalgebra $e_X C e_X$ is an admissible subcoalgebra of the path coalgebra $KQ^{ex}$, where $Q^{ex}$ is the quiver whose set of vertices is $(Q^{ex})_0 = X$ and the number of arrows from $x$ to $y$ is $\dim_K K\mathcal{C}ell_X^Q(x,y) \cap C$ for all $x, y \in X$.*

(*b*) *The localizing subcategory $\mathcal{T}_X$ of $\mathcal{M}^C$ is colocalizing if and only if the $K$-dimension of the $K$-vector space $K\mathcal{T}ail_X^Q(x) \cap C$ is finite for all $x \in X$.*

From a more theoretical point of view, since we wish to relate the representation theory of a coalgebra and its "localized" coalgebras, Sections 3.4 and 3.5 are devoted to the study of the behavior of the localizing functors. Surprisingly, this behavior strongly depends on the comportment of simple and indecomposable injective comodules.

**Theorem** (3.5.2). *Let $C$ be a coalgebra and $\mathcal{T}_e \subseteq \mathcal{M}^C$ be a localizing subcategory associated to an idempotent $e \in C^*$. The following conditions are equivalent:*

(*a*) *$\mathcal{T}_e$ is a stable subcategory.*

(*b*) *$T(E_x) = 0$ for any $x \notin I_e$.*

(*c*) $T(E_x) = \begin{cases} \overline{E}_x & \text{if } x \in I_e, \\ 0 & \text{if } x \notin I_e. \end{cases}$

(*d*) *$\mathrm{Hom}_C(E_y, E_x) = 0$ for all $x \in I_e$ and $y \notin I_e$.*

(*e*) *$\mathcal{K} = \{S \in (\Gamma_C)_0 \mid eS = S\}$ is a right link-closed subset of $(\Gamma_C)_0$, i.e., there is no arrow $S_x \to S_y$ in $\Gamma_C$, where $T(S_x) = S_x$ and $T(S_y) = 0$.*

(*f*) *There is no path in $\Gamma_C$ from a vertex $S_x$ to a vertex $S_y$ such that $T(S_x) = S_x$ and $T(S_y) = 0$.*

(*g*) *$e$ is a left semicentral idempotent in $C^*$.*

*If $T_e$ is a colocalizing subcategory this is also equivalent to*

(*h*) *$H(S_x) = S_x$ for any $x \in I_e$.*

**Proposition** (3.5.6)**.** *Let $C$ be a coalgebra and $\mathcal{T}_e \subseteq \mathcal{M}^C$ be a localizing subcategory associated to an idempotent $e \in C^*$. The following conditions are equivalent:*

($a$) *$\mathcal{T}_{1-e}$ is a stable subcategory.*

($b$) *$T(E_x) = E_x$ for any $x \in I_e$.*

($c$) *There is no path in $\Gamma_C$ from a vertex $S_y$ to a vertex $S_x$ such that $T(S_x) = S_x$ and $T(S_y) = 0$.*

($d$) *$\mathcal{K} = \{S \in (\Gamma_C)_0 \mid eS = S\}$ is a left link-closed subset of $(\Gamma_C)_0$, i.e., there is no arrow $S_y \to S_x$ in $\Gamma_C$, where $T(S_x) = S_x$ and $T(S_y) = 0$.*

($e$) *$e$ is a right semicentral idempotent in $C^*$.*

($f$) *The torsion subcomodule of a right $C$-comodule $M$ is $(1-e)M$*

($g$) *$S(S_x) = S_x$ for all $x \in I_e$.*

In Chapter 4 we conjugate the results obtained previously in order to obtain some applications to Representation Theory of Coalgebras. The main one is an acyclic version for coalgebras of Gabriel's theorem:

**Corollary** (4.4.3)**.** *Let $Q$ be an acyclic quiver and let $K$ be an algebraically closed field.*

($a$) *Any tame admissible subcoalgebra $C'$ of the path coalgebra $KQ$ is isomorphic to the path coalgebra $C(Q,\Omega)$ of a quiver with relation $(Q,\Omega)$.*

($b$) *The map $\Omega \mapsto C(Q,\Omega)$ defines a one-to-one correspondence between the set of relation ideals $\Omega$ of the path $K$-algebra $KQ$ and the set of admissible subcoalgebras $H$ of the path coalgebra $KQ$. The inverse map is given by $H \mapsto H^{\perp}$.*

In order to do that, first we need an analysis of the tameness and wildness of a coalgebra in the framework of localization. The drawback of treating the tameness from a general point of view lies in the fact that the section functor does not preserve finite dimensional comodules, or equivalently, it does not preserve the finite dimension of the simple comodules. Therefore there are no functors between the categories of finite dimensional comodules defined in a natural way. Once we assume this property as a hypothesis, we are able to prove that the localization process preserves tameness.

**Corollary** (4.2.9)**.** *Let $C$ be a coalgebra and $e \in C^*$ an idempotent element such that $S$ preserves finite dimensional comodules. If $C$ is tame then $eCe$ is tame.*

Wildness is much more complicated to study. For that reason, a particular case is treated: when the coalgebra $eCe$ is a subcoalgebra of $C$. We prove that this situation corresponds to the localization by a split idempotent (see [Lam06]). Therefore we deal with the description of that kind of idempotents. Actually, we prove the following characterization for pointed coalgebras.

**Lemma** (4.3.8)**.** *Let $Q$ be a quiver and $C$ be an admissible subcoalgebra of $KQ$. Let $e_X \in C^*$ be the idempotent associated to a subset of vertices $X$. Then $e_X$ is split in $C^*$ if and only if $I_p \subseteq X$ for any path $p$ in* $\mathrm{PSupp}(e_X C e_X)$.

Finally, Chapter 5 is devoted to look into certain kind of coalgebras by applying the topics developed all along this work. Mainly, we care for hereditary and serial coalgebras. About hereditariness, this is a well-known kind of coalgebras which have been studied with satisfactory results in many papers, see [Chi02], [JLMS05], [JMNR06] or [NTZ96]. Since, in view of the previous chapters, pointed coalgebras are of importance, then path coalgebras are treated extensively. For instance, we prove that colocalizing categories are also perfect colocalizing:

**Theorem** (5.1.9)**.** *Let $Q$ be a quiver and $e_X$ be the idempotent in $(KQ)^*$ associated to a subset $X \subseteq Q_0$. The following conditions are equivalent:*

$(a)$ *The localizing subcategory $\mathcal{T}_X$ of $\mathcal{M}^{KQ}$ is colocalizing.*

$(b)$ *The localizing subcategory $\mathcal{T}_X$ of $\mathcal{M}^{KQ}$ is perfect colocalizing.*

$(c)$ *$\mathcal{T}ail_X^Q(x)$ is a finite set for all $x \in X$. That is, there are at most a finite number of paths starting at the same point whose only vertex in $X$ is the first one.*

With respect to serial coalgebras, a systematic study of them is initiated in [CGT04], where, in particular, it is shown that any serial indecomposable coalgebra over an algebraically closed field is Morita-Takeuchi equivalent to a subcoalgebra of a path coalgebra of a quiver which is either a cycle or a chain (finite or infinite). We take advantage of the valued Gabriel quivers associated to a coalgebra to characterize indecomposable serial coalgebras over any field.

**Theorem** (5.3.4)**.** *Let $C$ be a indecomposable basic coalgebra over an arbitrary field $K$. Then $C$ is serial if and only if the right (and then also the left) valued Gabriel quiver of $C$ is one of the following valued quivers:*

$(a)$ ${}_{\infty}\mathbb{A}_{\infty}:$ $\cdots\cdots \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \cdots\cdots$

$(b)$ $\mathbb{A}_{\infty}:$ $\circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \cdots\cdots$

$(c)$ ${}_{\infty}\mathbb{A}:$ $\cdots\cdots \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ$

$(d)$ $\mathbb{A}_n:$ $\circ \longrightarrow \circ \longrightarrow \circ \cdots\cdots \circ \longrightarrow \circ \longrightarrow \circ$ *n vertices,* $n \geq 1$

$(e)$ $\widetilde{\mathbb{A}}_n:$

$$\begin{tikzcd}
 & \circ \arrow[dl] & \circ \arrow[l] & \circ \arrow[l] & \\
\circ \arrow[dr] & & & & \circ \arrow[ul] \\
 & \circ \arrow[r] & \circ \arrow[r, dotted, no head] & \circ \arrow[ur] &
\end{tikzcd}$$

*n vertices,* $n \geq 1$

It is observed in [CGT04] that a consequence of [EG71, Corollary 3.2] is that the finite dual coalgebra of a hereditary noetherian prime algebra over a field is serial. In Section 5.3.4, in conjunction with localization techniques, we reconsider this result of Eisenbud and Griffith from the "coalgebraic" point of view:

**Corollary** (5.3.18)**.** *If $C$ is a subcoalgebra of a prime, hereditary and strictly quasi-finite (left and right) coalgebra over an arbitrary field, then $C$ is serial.*

# Chapter 1

# Preliminaries

This chapter contains some of the background material that will be used throughout this work. Namely, after a few categorical remarks, we introduce the notation and terminology on coalgebras and we recall some basic facts about their representation theory. We assume that the reader is familiar with elementary category theory and ring theory, and some homological concepts such as injective and projective objects; anyhow we refer to [AF91], [Mac71], [Pop73] and [Wis91] for questions on these subjects. All rings considered have identity and modules are unitary. By a field we mean a commutative division ring.

## 1.1 Some categorical remarks

This section is devoted to establish some categorical definitions and properties which we assume for a category of comodules in what follows. For further information see, for example, [Mac71], [Pop73] or [Wis91].

A category $\mathcal{C}$ is said to be **abelian** if the following conditions are satisfied:

($a$) There exists the direct sum of any finite set of objects of $\mathcal{C}$.

($b$) For each pair of objects $X$ and $Y$ of $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is equipped with an abelian group structure such that the composition of morphisms in $\mathcal{C}$ is bilinear.

($c$) $\mathcal{C}$ has a zero object.

($d$) Each morphism $f : X \to Y$ in $\mathcal{C}$ admits a kernel $(\mathrm{Ker}\, f, u)$ and a cokernel $(\mathrm{Coker}\, f, p)$, and the unique morphism $\overline{f}$ making commutative the diagram

$$\begin{tikzcd} \operatorname{Ker} f \arrow[r, "u"] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "p"] & \operatorname{Coker} f \\ & \operatorname{Coker} u \arrow[r, "\overline{f}"] & \operatorname{Ker} p \arrow[u] & \end{tikzcd}$$

is an isomorphism.

Throughout this work we fix a field $K$. We say that $\mathcal{C}$ is a $K$**-category** if, for each pair of objects $X$ and $Y$ of $\mathcal{C}$, the set $\operatorname{Hom}_{\mathcal{C}}(X,Y)$ is equipped with a $K$-vector space structure such that the composition of morphisms in $\mathcal{C}$ is a $K$-bilinear map. An abelian category $\mathcal{C}$ is said to be a **Grothendieck category** if it has arbitrary direct sums, a set of generators and direct limits are exact. Moreover, if each object of the set of generators has finite length then $\mathcal{C}$ is known as a **locally finite category**.

**Proposition 1.1.1.** *[Gab62] Let $\mathcal{C}$ be a locally finite $K$-category. Then it verifies the following assertions:*

(*a*) *The category $\mathcal{C}$ has injective envelopes.*

(*b*) *The direct sum of injective objects is injective.*

(*c*) *Each object of $\mathcal{C}$ is an essential extension of its socle (the sum of all its simple subobjects).*

(*d*) *An injective object $E$ of $\mathcal{C}$ is indecomposable if and only if its socle is a simple object.*

(*e*) *If $\{S_i\}_{i\in I}$ is a complete set of isomorphism classes of simple objects of $\mathcal{C}$ and $E_i$ is the injective envelope of $S_i$ for each $i \in I$, then $\{E_i\}_{i\in I}$ is a complete set of isomorphism classes of indecomposable injective objects of $\mathcal{C}$.*

(*f*) *With the above notation, each injective object $E$ of $\mathcal{C}$ is isomorphic to a direct sum $\bigoplus_{i\in I} E_i^{\alpha_i}$, where each $\alpha_i$ is a non-negative integer. Furthermore, this sum is uniquely determined by the set $\{\alpha_i\}_{i\in I}$.*

(*g*) *With the above notation, $E = \bigoplus_{i\in I} E_i^{\alpha_i}$ is an injective cogenerator of $\mathcal{C}$ if and only if $\alpha_i > 0$ for all $i \in I$.*

Let $\mathcal{C}$ be a locally finite $K$-category. We say that $\mathcal{C}$ is of **finite type** if, for each pair of objects $X$ and $Y$ of $\mathcal{C}$ of finite length, the vector space $\operatorname{Hom}_{\mathcal{C}}(X,Y)$ has finite dimension over $K$.

**Proposition 1.1.2.** *[Tak77] Let $\mathcal{C}$ be a locally finite $K$-category. The following conditions are equivalent:*

(*a*) *$\mathcal{C}$ is of finite type.*

(*b*) *For each simple object $S$ of $\mathcal{C}$, the vector space $\mathrm{Hom}_{\mathcal{C}}(S,S)$ is finite dimensional over $K$.*

An object $F$ of a $K$-category of finite type $\mathcal{C}$ is said to be **quasi-finite** if, for each object $X$ of $\mathcal{C}$ of finite length, the vector space $\mathrm{Hom}_{\mathcal{C}}(X,F)$ has finite dimension over $K$.

**Proposition 1.1.3.** *[Tak77] Let $\mathcal{C}$ be a $K$-category of finite type and $F$ be an object of $\mathcal{C}$. The following sentences are equivalent:*

(*a*) *$F$ is quasi-finite.*

(*b*) *For each simple object $S$ of $\mathcal{C}$, the vector space $\mathrm{Hom}_{\mathcal{C}}(S,F)$ is finite dimensional over $K$.*

(*c*) *With the notation of Proposition 1.1.1, the socle of $F$ is isomorphic to $\bigoplus_{i\in I} S_i^{\alpha_i}$, where the non-negative integers $\alpha_i$ are finite for all $i \in I$.*

**Corollary 1.1.4.** *[Tak77] Let $\mathcal{C}$ be a $K$-category of finite type. Then, with the notation of Proposition 1.1.1, $\bigoplus_{i\in I} E_i$ is a quasi-finite injective cogenerator of $\mathcal{C}$.*

## 1.2 Coalgebras and comodules

Let us now define the main objects of our study, that is, coalgebras and their category of comodules. Recall that the category of comodules over a coalgebra is a particular case of a category of finite type so all definitions and results of the last section remain valid here. Following [Abe77] and [Swe69], by a $K$-**coalgebra** we mean a triple $(C, \Delta, \epsilon)$, where $C$ is a $K$-vector space and $\Delta : C \to C \otimes C$ and $\epsilon : C \to K$ are $K$-linear maps, called **comultiplication** and **counit**, respectively; such that the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C\otimes C \arrow[d, "\Delta\otimes I"] \\ C\otimes C \arrow[r, "I\otimes\Delta"] & C\otimes C\otimes C \end{tikzcd}
\qquad
\begin{tikzcd} & C\otimes C \arrow[dl, "I\otimes\epsilon"'] \arrow[dr, "\epsilon\otimes I"] & \\ C\otimes K & & K\otimes C \\ & C \arrow[ul, "\cong"] \arrow[ur, "\cong"'] \arrow[uu, "\Delta"] & \end{tikzcd}$$

(Coassociativity) (Counit)

In what follows we denote the coalgebra $(C, \Delta, \epsilon)$ simply by $C$.

A $K$-vector subspace $V$ of $C$ is said to be a **subcoalgebra** of $C$ if $\Delta(V) \subseteq V \otimes V$. If $\Delta(V) \subseteq V \otimes C$ (resp. $\Delta(V) \subseteq C \otimes V$) we say that $V$ is a **right** (resp. **left**) **coideal**. Finally, it is called a **coideal** if $\Delta(V) \subseteq V \otimes C + C \otimes V$ and $\epsilon(V) = 0$. Note that a right and left coideal is not a coideal but a subcoalgebra. Let $S$ be a subset of a coalgebra $C$, the vector space obtained from the intersection of all subcoalgebras of $C$ containing $S$ is called the **subcoalgebra generated** by $S$. The following theorem asserts that this set is also a subcoalgebra.

**Theorem 1.2.1.** *[Swe69]*

$(a)$ *The intersection of subcoalgebras is again a subcoalgebra.*

$(b)$ *Any subcoalgebra generated by a finite set is finite dimensional.*

$(c)$ *Any simple subcoalgebra of a coalgebra is finite dimensional.*

The following result is often called the **Fundamental Coalgebra Structure Theorem** and shows the locally finite nature of a coalgebra, see [Mon93] and [Swe69].

**Theorem 1.2.2.** *Any $K$-coalgebra is a directed union of its finite dimensional subcoalgebras.*

Given two $K$-coalgebras $C$ and $D$, a **morphism** of $K$-coalgebras $f : C \to D$ is a $K$-linear map such that the following diagrams are commutative:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\ C \otimes C \arrow[r, "f\otimes f"] & D \otimes D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\epsilon_C"'] & & D \arrow[dl, "\epsilon_D"] \\ & K & \end{tikzcd}$$

If $f : C \to D$ is a morphism of coalgebras, it is easy to prove that $\operatorname{Ker} f$ is a coideal of $C$ and $\operatorname{Im} f$ is a subcoalgebra of $D$.

Let $C$ be a $K$-coalgebra. A right $C$-**comodule** is a pair $(M, \omega)$, where $M$ is a $K$-vector space and $\omega : M \to C \otimes M$ is a $K$-linear map making commutative the following diagrams:

$$\begin{tikzcd} M \arrow[r, "\omega"] \arrow[d, "\omega"'] & M \otimes C \arrow[d, "\omega\otimes I"] \\ M \otimes C \arrow[r, "I\otimes\Delta"] & M \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\omega"] \arrow[dr, "\cong"'] & M \otimes C \arrow[d, "I\otimes\epsilon"] \\ & M \otimes K \end{tikzcd}$$

In what follows we denote the right $C$-comodule $(M, \omega)$ simply by $M$, or by $M_C$.

Given two right $C$-comodules $M$ and $N$, a morphism of right $C$-comodules $f : M \to N$ is a $K$-linear map such that the following diagram is commutative:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\omega_M"'] & N \arrow[d, "\omega_N"] \\ M \otimes C \arrow[r, "f\otimes I"] & N \otimes C \end{tikzcd}$$

From now on we identify every comodule with the identity map defined on it, so we use the notation $f \otimes M$, $f \otimes 1_M$ or simply $f \otimes I$, it doesn't matter which. We denote by $\mathcal{M}^C$ the category of right $C$-comodules and morphisms of right $C$-comodules and by $\mathcal{M}^C_{qf}$ and $\mathcal{M}^C_f$, the full subcategories of $\mathcal{M}^C$ whose objects are quasi-finite and finite dimensional right $C$-comodules, respectively. Symmetrically we define the corresponding categories of left $C$-comodules. We denote by ${}^C\mathcal{M}_{qf}$, ${}^C\mathcal{M}_f$ and ${}^C\mathcal{M}$ the category of quasi-finite, finite dimensional and all left $C$-comodules, respectively.

**Example 1.2.3.** *Let $C$ be a $K$-coalgebra, $V$ a $K$-vector space and $M$ a right $C$-comodule. Then $V \otimes M$ has a structure of right $C$-comodule with comultiplication $I \otimes \omega_M$. It is easy to prove that we have an isomorphism* $\mathrm{Hom}_C(V \otimes M, N) \cong \mathrm{Hom}_K(V, \mathrm{Hom}_C(M, N))$ *for any right $C$-comodule $N$.*

Let $C$ and $D$ be $K$-coalgebras. A $(C, D)$-**bicomodule** is a $K$-vector space $M$ with a structure of left $C$-comodule $(M, \omega)$ and a structure of right $D$-comodule $(M, \rho)$ such that the following diagram is commutative:

$$\begin{tikzcd} M \arrow[r, "\omega"] \arrow[d, "\rho"'] & C \otimes M \arrow[d, "I\otimes\rho"] \\ M \otimes D \arrow[r, "\omega\otimes I"] & C \otimes M \otimes D \end{tikzcd}$$

The reader should note that this means that $\rho$ is a morphism of left $C$-comodules or, equivalently, $\omega$ is a morphism of right $D$-comodules. A morphism of $(C, D)$-bicomodules is a $K$-linear map which is a morphism of left $C$-comodules and of right $D$-comodules. We denote by ${}^C\mathcal{M}^D$ the category of $(C, D)$-bicomodules.

Here we list some important properties of a category of comodules, see [Mon93] and [Swe69] for details.

**Proposition 1.2.4.** *Let $C$ be a $K$-coalgebra. Then the following assertions hold:*

(*a*) *$\mathcal{M}^C$ is an abelian $K$-category of finite type.*

(*b*) *$\mathcal{M}^C_f$ is a skeletally small abelian Krull-Schmidt $K$-category.*

(*c*) *$\mathcal{M}^C$ has enough injective objects.*

(*d*) *The coalgebra $C$, viewed as a right $C$-comodule, is a quasi-finite injective cogenerator in $\mathcal{M}^C$.*

(*e*) *A direct sum of indecomposable right $C$-comodules is injective if and only if each direct summand is injective.*

(*f*) *Every right $C$-comodule is the directed union of its finite dimensional subcomodules.*

(*g*) *Each simple right $C$-comodule has finite dimension.*

**Remark 1.2.5.** *In general, the category $\mathcal{M}^C$ has no enough projectives and sometimes it has no non-zero projective objects.*

Throughout we denote by $\{S_i\}_{i\in I_C}$ a complete set of pairwise non-isomorphic simple right $C$-comodules and by $\{E_i\}_{i\in I_C}$ a complete set of pairwise non-isomorphic indecomposable injective right $C$-comodules.

Let $(M,\rho)$ be a right $C$-comodule. There exists a unique minimal subcoalgebra $\mathrm{cf}(M)$ of $C$ such that $\rho(M)\subseteq M\otimes \mathrm{cf}(M)$, that is, such that $M$ is a right $\mathrm{cf}(M)$-comodule. This coalgebra $\mathrm{cf}(M)$ is called the **coefficient space** of $M$.

**Proposition 1.2.6.** *[Gre76] Let $C$ be a $K$-coalgebra and $m_i = \dim_{D_i} S_i$ for any $i\in I_C$, where $D_i$ is the division algebra $\mathrm{End}_C(S_i)$. Then:*

(*a*) *Each simple subcoalgebra of $C$ is isomorphic to $\mathrm{cf}(S_i)$ for some $i\in I_C$.*

(*b*) *$\mathrm{cf}(S_i) = S_i\oplus\cdots\oplus S_i = S_i^{m_i}$ as right $C$-comodules for each $i\in I_C$.*

(*c*) *$\mathrm{Corad}(C) = C_0 = \bigoplus_{i\in I_C}\mathrm{cf}(S_i) = \bigoplus_{i\in I_C} S_i^{m_i}$.*

We finish this section giving an important characterization of the categories of comodules.

**Theorem 1.2.7.** *[Tak77] Let $\mathcal{C}$ be an abelian $K$-category. $\mathcal{C}$ is of finite type if and only if $\mathcal{C}$ is $K$-linearly equivalent to $\mathcal{M}^C$ for some $K$-coalgebra $C$.*

## 1.3 The cotensor product

Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian $K$-categories. A functor $T : \mathcal{A} \to \mathcal{B}$ is said to be $K$-**linear** if the map $T_{X,Y} : \mathrm{Hom}_{\mathcal{A}}(X,Y) \to \mathrm{Hom}_{\mathcal{B}}(T(X),T(Y))$ defined by $T_{X,Y}(f) = T(f)$ is $K$-linear for any objects $X$ and $Y$ of $\mathcal{A}$. Let now $S : \mathcal{A} \to \mathcal{B}$ and $T : \mathcal{B} \to \mathcal{A}$ be two functors. We say that $S$ is **left adjoint** to $T$, or $T$ is **right adjoint** to $S$, if there exists a natural isomorphism $\mathrm{Hom}_{\mathcal{B}}(S(-),-) \backsimeq \mathrm{Hom}_{\mathcal{A}}(-,T(-))$. In this case, $S$ is right exact and preserves colimits and $T$ is left exact and preserves limits.

In the particular case of categories of modules over $K$-algebras, we have an important example of adjoint functors: the tensor functor and the Hom functor. Suppose that $R$ is a $K$-algebra, $M$ is a right $R$-module and $N$ is a left $R$-module. Then we may introduce the tensor product $M_R \otimes_R {}_RN$ as the cokernel (coequalizer) of the maps

$$M \otimes_K R \otimes_K N \overset{\mu_M \otimes I}{\underset{I \otimes \mu_N}{\rightrightarrows}} M \otimes_K N \dashrightarrow M \otimes_R N,$$

where $\mu_M$ and $\mu_N$ are the structure maps of $M$ and $N$ as $R$-modules. Furthermore, if $S$ is another $K$-algebra and $N$ is an ($R$,$S$)-bimodule then $M \otimes_R N$ has a structure of right $S$-module. Thus we may define a functor $- \otimes_R N : \mathrm{Mod}_R \to \mathrm{Mod}_S$ which is left adjoint to $\mathrm{Hom}_S(N,-)$, that is, there is a natural isomorphism $\mathrm{Hom}_R(M, \mathrm{Hom}_S(N,T)) \cong \mathrm{Hom}_S(M \otimes_R N, T)$ for any right $R$-module $M$ and any right $S$-module $T$.

Let us come back to coalgebras. In what follows we assume that all coalgebras are over the field $K$. We would like to obtain a similar situation to the one above, i.e., a pair of functors between categories of comodules (probably over different coalgebras) with adjoint properties. Let $C$ be a coalgebra, $M$ be a right $C$-comodule and $N$ be a left $C$-comodule. Following [EM66], we recall that the **cotensor product** of $M$ and $N$, $M_C \square_{CC}N$, is the kernel (equalizer) of the maps

$$M \square_C N \dashrightarrow M \otimes_K N \overset{\omega_M \otimes I}{\underset{I \otimes \omega_N}{\rightrightarrows}} M \otimes_K C \otimes_K N,$$

where $\omega_M$ and $\omega_N$ are the structure maps of $M$ and $N$ as right $C$-comodule and left $C$-comodule, respectively. We collect here some properties of the cotensor product.

**Proposition 1.3.1.** *[Tak77] Let $C$ be a coalgebra, $M$ a right $C$-comodule and $N$ a left $C$-comodule. Then the following assertions hold:*

*(a) If $C = K$ then $M \square_C N = M \otimes_K N$.*

(*b*) *The cotensor product is associative.*

(*c*) *The functors* $M\Box_C-$ *and* $-\Box_C N$ *are left exact and preserve direct sums.*

(*d*) *We have isomorphisms* $M\Box_C(N\otimes_K W)\cong(M\Box_C N)\otimes_K W$ *and* $(W\otimes_K M)\Box_C N\cong W\otimes_K(M\Box_C N)$ *for any* $K$*-vector space* $W$*.*

(*e*) *The functor* $M\Box_C-$ *(resp.* $-\Box_C N$*) is exact if and only if* $M$ *(resp.* $N$*) is an injective right (resp. left)* $C$*-comodule .*

(*f*) *There are isomorphisms* $M\Box_C C\cong M$ *and* $C\Box_C N\cong N$*.*

Let now $C$, $D$ and $E$ be three coalgebras, $M$ an $(E,C)$-bicomodule and $N$ a $(C,D)$-bicomodule. Then $M\Box_C N$ acquires a structure of $(E,D)$-bicomodule with structure maps

$$\rho_M\Box I : M\Box_C N\to(E\otimes_K M)\Box_C N\cong E\otimes_K(M\Box_C N)$$

and

$$I\Box\rho_N : M\Box_C N\to M\Box_C(N\otimes_K D)\cong(M\Box_C N)\otimes_K D.$$

Therefore we may consider a functor $-\Box_C N:\mathcal{M}^C\to\mathcal{M}^D$. Unfortunately, in general, $-\Box_C N$ does not have a left adjoint functor.

**Theorem 1.3.2.** *[Tak77] Let* $C$ *and* $D$ *be two coalgebras and* $M$ *be a* $(D,C)$*-bicomodule. Then the functor* $-\Box_D M$ *has a left adjoint functor if and only if* $M$ *is a quasi-finite right* $C$*-comodule.*

If $M$ is a quasi-finite right $C$-comodule, we denote the left adjoint functor of $-\Box_D M$ by $\mathrm{Cohom}_C(M,-)$. The functor $\mathrm{Cohom}_C(M,-)$ has a similar behavior to the usual Hom functor of algebras.

**Proposition 1.3.3.** *[Tak77] Let* $C, D$ *and* $E$ *be three coalgebras. Let* $M$ *and* $N$ *be a* $(D,C)$*-bicomodule and an* $(E,C)$*-bicomodule, respectively, such that* $M$ *is quasi-finite as right* $C$*-comodule. Then:*

(*a*) *We have* $\mathrm{Cohom}_C(M,N)=\varinjlim\mathrm{Hom}_C(N_\lambda,M)^*=\varinjlim(M\Box_C N_\lambda^*)^*$*, where* $\{N_\lambda\}_\lambda$ *is the family of finite dimensional subcomodules of* $N_C$*.*

(*b*) *The vector space* $\mathrm{Cohom}_C(M,N)$ *is an* $(E,D)$*-bicomodule.*

(*c*) *The functor* $\mathrm{Cohom}_C(M,-)$ *is right exact and preserves direct sums.*

(*d*) *The functor* $\mathrm{Cohom}_C(M,-)$ *is exact if and only if* $M$ *is injective as right* $C$*-comodule.*

**Remark 1.3.4.** *The set* $\mathrm{Coend}_C(M) = \mathrm{Cohom}_C(M, M)$ *has a structure of coalgebra and then* $M$ *becomes a* $(\mathrm{Coend}_C(M), C)$*-bicomodule, see [Tak77] for details.*

Symmetrically, ${}^D M^C$ is quasi-finite as left $D$-comodule if and only if the functor $M\square_C - : {}^C\mathcal{M} \to {}^D\mathcal{M}$ has a left adjoint functor. In this case we denote that functor by $\mathrm{Cohom}_D(-, M)$.

As a consequence, we prove the Krull-Remak-Schmidt-Azumaya theorem for quasi-finite comodules. We need the following lemmata:

**Lemma 1.3.5.** *Let* $E$ *be an indecomposable injective right* $C$*-comodule. Then* $\mathrm{Hom}_C(E, E) = \mathrm{End}_C(E)$ *is a local ring.*

*Proof.* Let $f \in \mathrm{End}_C(E)$. It holds that $\mathrm{Ker}\ f \cap \mathrm{Ker}\ (id_E - f) = 0$. Since $E$ is indecomposable, $\mathrm{Ker}\ f = 0$ or $\mathrm{Ker}\ (id_E - f) = 0$. If $f$ is injective then there exists a map $g$ such that

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "id_E"'] & E \arrow[dl, dashed, "g"] \\ E & \end{tikzcd}$$

is commutative and then the sequence $E \xrightarrow{f} E \longrightarrow \mathrm{Coker}\ f$ splits. Therefore $E = E \oplus \mathrm{Coker}\ f$. Thus $\mathrm{Coker}\ f = 0$ and $f$ is bijective.

Otherwise, proceeding as above, $id_E - f$ is bijective so $f$ is quasi-regular. Then $f$ is in the radical. This proves that $\mathrm{End}_C(E)$ is local. $\square$

Let $M$ be a quasi-finite right $C$-comodule, we denote by $\mathrm{add}\, M$ the category of direct summands of arbitrary direct sums of copies of $M$. Let us consider the coalgebra $D = \mathrm{Cohom}_C(M, M)$. Then the Cohom and the cotensor functor can be restricted to $\mathrm{Cohom}_C(M, -) : \mathrm{add}\, M \to \mathrm{add}\, D$ and $-\square_D M : \mathrm{add}\, D \to \mathrm{add}\, M$, respectively.

**Lemma 1.3.6.** *[CKQ02] Let* $M$ *be a quasi-finite right* $C$*-comodule and let* $D$ *be the coalgebra* $\mathrm{Cohom}_C(M, M)$*. Then the functors*

$$\mathrm{add}\, M \underset{-\square_D M}{\overset{\mathrm{Cohom}_C(M,-)}{\rightleftarrows}} \mathrm{add}\, D$$

*are inverse equivalences of categories.*

**Corollary 1.3.7.** *Let* $M$ *be an indecomposable quasi-finite right* $C$*-comodule then* $\mathrm{End}_C(M)$ *is a local ring.*

*Proof.* By Lemma 1.3.6, $\mathrm{Cohom}_C(M, -) : \mathrm{add}\, M \to \mathrm{add}\, D$ is an equivalence. Since $D$ is quasi-finite then $\mathrm{add}\, D = \mathcal{I}^D$, the category of quasi-finite injective right $D$-comodules. Therefore it is enough to prove it for indecomposable injective comodules. But this is proved in Lemma 1.3.5. $\square$

**Theorem 1.3.8** (Krull-Remak-Schmidt-Azumaya Theorem)**.** *Let $C$ be a coalgebra and $M$ be a quasi-finite right $C$-comodule. Then two decompositions of $M$ as direct sum of indecomposable right $C$-comodules are essentially the same, that is, if $M = \bigoplus_{i\in I} M_i = \bigoplus_{j\in J} N_j$, where all $M_i's$ and $N_j's$ are indecomposable right $C$-comodules, then $I = J$ and there exists a bijective correspondence $\sigma : I \to J$ such that $M_i \cong N_{\sigma(i)}$ for all $i \in I$.*

*Proof.* It is a consequence of the previous results and [Gab62, p. 338, Theorem 1]. $\square$

## 1.4 Equivalences between comodule categories

Let $C$ and $D$ be two coalgebras and $M$ and $N$ be a $(C, D)$-bicomodule and a $(D, C)$-bicomodule, respectively. Let $f : C \to M\square_D N$ and $g : D \to N\square_C M$ be two bicomodule maps. We say that $(C, D, M, N, f, g)$ is a **Morita-Takeuchi context** if the following diagrams commute:

$$\begin{tikzcd} M \arrow[r, "\cong"] \arrow[d, "\cong"'] & M\square_D D \arrow[d, "I\square g"] \\ C\square_C M \arrow[r, "f\square I"] & M\square_D N\square_C M \end{tikzcd} \qquad \begin{tikzcd} N \arrow[r, "\cong"] \arrow[d, "\cong"'] & N\square_C C \arrow[d, "I\square f"] \\ D\square_D N \arrow[r, "g\square I"] & N\square_C M\square_D N \end{tikzcd}$$

A Morita-Takeuchi context $(C, D, M, N, f, g)$ is said to be **injective** if $f$ is injective. If $f$ and $g$ are isomorphisms then we say that $C$ and $D$ are **Morita-Takeuchi equivalent**.

**Proposition 1.4.1.** *[Tak77] Let $(C, D, M, N, f, g)$ be an injective Morita-Takeuchi context. Then the following statements hold:*

*(a) The map $f$ is an isomorphism.*

*(b) The comodules $M_D$ and ${}_DN$ are quasi-finite and injective.*

*(c) The comodules ${}_CM$ and $N_C$ are cogenerators.*

*(d)* $\mathrm{Cohom}_D(M, D) \cong N$ *as $(D, C)$-bicomodules and* $\mathrm{Cohom}_C(N, C) \cong M$ *as $(C, D)$-bicomodules.*

(e) $\mathrm{Coend}_D(M) \cong C$ *and* $\mathrm{Coend}_C(N) \cong D$ *as coalgebras.*

**Example 1.4.2.** *Suppose that $D$ is a coalgebra and $M$ is a quasi-finite right $D$-comodule. Denote by $C$ the coalgebra $\mathrm{Coend}_D(M)$ and by $N$ the $(D, C)$-bicomodule $\mathrm{Cohom}_D(M, D)$. Then $\mathrm{Hom}_D(D, N\Box_C M) \cong \mathrm{Hom}_C(N, N)$. Now, consider $g : D \longrightarrow N\Box_C M$ the associated morphism to $id_Y$ via the equivalence, and let $f$ be the morphism*

$$f : C \cong \mathrm{Cohom}_D(M, M\Box_D D) \to M\Box_D \mathrm{Cohom}_D(M, D) = M\Box_D N.$$

*Then $(C, D, M, N, f, g)$ is a Morita-Takeuchi context known as the Morita-Takeuchi context associated to $M_D$. Clearly, $f$ is injective if and only if $M_D$ is injective and $g$ is injective if and only if $M_D$ is a cogenerator. As a consequence,*

**Proposition 1.4.3.** *[Tak77] Let $M_D$ be a quasi-finite $D$-comodule and let $(C, D, X, Y, f, g)$ be the Morita-Takeuchi context associated to $M$. Then $C$ and $D$ are Morita-Takeuchi equivalent if and only if $M$ is an injective cogenerator of the category $\mathcal{M}^D$.*

We may use Morita-Takeuchi contexts in order to know whenever two categories of comodules are equivalent.

**Theorem 1.4.4.** *[Tak77] Let $M$ be a $(C, D)$-bicomodule which is quasi-finite as right $D$-comodule. The following conditions are equivalent:*

(a) *The functor $-\Box_C M : \mathcal{M}^C \to \mathcal{M}^D$ is an equivalence of categories.*

(b) *The functor $M\Box_D - : {}^D\mathcal{M} \to {}^C\mathcal{M}$ is an equivalence of categories.*

(c) *$M_D$ is a quasi-finite injective cogenerator and* $\mathrm{Coend}_D(M) \cong C$ *as coalgebras.*

(d) *${}_CM$ is a quasi-finite injective cogenerator and* $\mathrm{Coend}_C(M) \cong D$ *as coalgebras.*

(e) *There exists a Morita-Takeuchi context $(C, D, M, N, f, g)$, where $f$ and $g$ are injective.*

(f) *There exists a Morita-Takeuchi context $(D, C, N', M, f', g')$, where $f'$ and $g'$ are injective.*

*If these conditions hold, there is an isomorphism between the $(C, D)$-bicomodules* $\mathrm{Cohom}_D(M, D)$ *and* $\mathrm{Cohom}_C(M, C)$. *If we denote it by $N$ then $-\Box_D N$ and $N\Box_C -$ are the quasi-inverse functors of $-\Box_C M$ and $M\Box_D -$, respectively.*

**Corollary 1.4.5.** *Two coalgebras are Morita-Takeuchi equivalent if and only if their categories of right comodules are equivalent.*

The reader could ask about what happen when two categories of comodules are equivalent but the functor is not of the form $-\Box_C M$ where $M$ is a bicomodule. The answer is simple: that situation cannot appear.

**Theorem 1.4.6.** *[Tak77] Let $T : \mathcal{M}^C \to \mathcal{M}^D$ be a $K$-linear functor. If $T$ is left exact and preserves direct sums then there exists a $(C, D)$-bicomodule $M$ such that $T \cong -\Box_C M$.*

We have seen that quasi-finite injective cogenerators play an important rôle in the equivalences between categories of comodules. We recall from Section 1.1 that this kind of comodules has an easy description.

**Proposition 1.4.7.** *Let $C$ be a coalgebra and $\{E_i\}_{i\in I_C}$ be a complete set of pairwise non-isomorphic indecomposable injective right $C$-comodules. A right $C$-comodule $E$ is a quasi-finite injective cogenerator of $\mathcal{M}^C$ if and only if $E = \bigoplus_{i\in I_C} E_i^{\alpha_i}$, where $\alpha_i$ is a finite cardinal number greater than zero for all $i \in I_C$.*

## 1.5 Basic and pointed coalgebras

Any coalgebra $C$ is a quasi-finite injective cogenerator of its category $\mathcal{M}^C$ of right $C$-comodules. Then, by the last section, $C$ has a decomposition, as right $C$-comodule,

$$C_C = \bigoplus_{i\in I_C} E_i^{\alpha_i} \ ,$$

where each $\alpha_i$ is a finite positive integer; that is, its socle has a decomposition

$$\operatorname{soc} C = \bigoplus_{i\in I_C} S_i^{\alpha_i} \ .$$

The coalgebra $C$ is called **basic** if $\alpha_i = 1$ for all $i \in I_C$, i.e., if $\operatorname{soc} C = \bigoplus_{i\in I_C} S_i$, where $S_i \ncong S_j$ for $i \neq j$. Following this definition, we may obtain an immediate consequence:

**Proposition 1.5.1.** *The following conditions are equivalent:*

*(a)* *$C$ is basic.*

*(b)* $C = \bigoplus_{i\in I_C} E_i$.

(*c*) *$C$ is a minimal injective cogenerator of the category $\mathcal{M}^C$.*

The main reason to study basic coalgebras comes from the fact that, in order to classify coalgebras by means of its category of comodules, it is enough to consider only this kind of coalgebras, see for example [CM97] and [Sim01].

**Theorem 1.5.2.** *Let $C$ be an arbitrary coalgebra. Then there exits a unique (up to isomorphism) basic coalgebra $D$ such that $\mathcal{M}^C \simeq \mathcal{M}^D$.*

*Proof.* Suppose that $C = \bigoplus_{i\in I_C} E_i^{\alpha_i}$. We consider the comodule $E = \bigoplus_{i\in I_C} E_i$. By Proposition 1.4.7, $E$ is a quasi-finite injective cogenerator and, by Theorem 1.4.4, the functor $-\square_D E$ defines an equivalence between the categories $\mathcal{M}^D$ and $\mathcal{M}^C$, where $D = \mathrm{Cohom}_C(E,E)$. Thus we only need to prove that $D$ is a basic coalgebra. Let $\{E_i'\}_{i\in I_D}$ be a complete set of indecomposable injective right $D$-comodules. Since $-\square_D E$ is an equivalence, we may number them in order to do that $E_i'\square_D E = E_i$ for all $i \in I_C = I_D$. Now, suppose that $D = \bigoplus_{i\in I_D} E_i'^{t_i}$. Then $E \cong D\square_D E \cong \bigoplus_{i\in I_D} E_i'^{t_i}\square_D E \cong \bigoplus_{i\in I_D}(E_i'\square_D E)^{t_i} = \bigoplus_{i\in I_D} E_i^{t_i}$ and therefore, by Krull-Remak-Schmidt-Azumaya Theorem, $t_i = 1$ for all $i \in I_C$.

Let now $H$ be another basic coalgebra such that $\mathcal{M}^C \simeq \mathcal{M}^H$. Then there exists an equivalence $-\square_{D\,D}Q_H : \mathcal{M}^D \to \mathcal{M}^H$, where $Q$ is a quasi-finite injective cogenerator of $\mathcal{M}^H$. Since the equivalences preserve the minimal quasi-finite injective cogenerator then $Q \cong D\square_D Q = H$ because $D$ and $H$ are basic. Then, by Theorem 1.4.4, $\mathrm{Cohom}_H(H,H) \cong D$ as coalgebras. Consider the inverse equivalence and then $\mathrm{Cohom}_D(D,D) \cong H$. Finally, if $D = \varinjlim D_\gamma$, where $\{D_\gamma\}_\gamma$ is the set of its finite dimensional subcoalgebras, then $H \cong \mathrm{Cohom}_D(D,D) = \varinjlim \mathrm{Hom}_D(D_\gamma, D)^* \cong \varinjlim \mathrm{Hom}_H(D_\gamma\square_D H, H)^* = \mathrm{Cohom}_H(H,H) \cong D$. □

**Corollary 1.5.3.** *Any coalgebra is Morita-Takeuchi equivalent to a basic coalgebra.*

When the field $K$ is algebraically closed, the structure of the coalgebra is much simpler. The following definition is of importance all along the work. A coalgebra is said to be **pointed** if every simple comodule is one dimensional.

**Proposition 1.5.4.** *Every pointed coalgebra is basic.*

*Proof.* Let $C$ be a coalgebra such that $\mathrm{soc}\, C = \bigoplus_{i\in I_C} S_i^{t_i}$. Since $S_i^* = \mathrm{Hom}_K(S_i,K) \cong \mathrm{Hom}_C(S_i,C) \cong \mathrm{Hom}_C(S_i,\mathrm{soc}\, C) \cong \mathrm{Hom}_C(S_i,S_i)^{t_i}$ then $\dim_K S_i = \dim_K S_i^* = t_i \dim_K \mathrm{End}_C(S_i)$ because $\dim_K S_i$ is finite. Therefore $t_i = \frac{\dim_K S_i}{\dim_K \mathrm{End}_C(S_i)}$. Now, if $C$ is pointed then $\dim_K S_i = 1$ and thus $\dim_K \mathrm{End}_C(S_i) = t_i = 1$. □

There are easy examples of non-pointed basic coalgebras:

**Example 1.5.5.** *Let $\mathbb{C}$ be the $\mathbb{R}$-vector space of the complex numbers endowed with a structure of coalgebra given by the following formulae:*

- $\Delta(1) = 1 \otimes 1 - i \otimes i$ *and* $\Delta(i) = i \otimes 1 + 1 \otimes i$
- $\epsilon(1) = 1$ *and* $\epsilon(i) = 0$.

*Then $\mathbb{C}$ is a non-pointed and basic (actually, it is simple) coalgebra.*

**Corollary 1.5.6.** *Let $K$ be an algebraically closed field and $C$ be a $K$-coalgebra. Then $C$ is basic if and only if $C$ is pointed.*

*Proof.* If $C$ is basic then $t_i = 1$. Now, every $K$-division algebra is one dimensional so $\dim_K \mathrm{End}_C(S_i) = 1$. Thus $\dim_K S_i = 1$. □

**Corollary 1.5.7.** *Every coalgebra over an algebraically closed field is Morita-Takeuchi equivalent to a pointed coalgebra.*

## 1.6 Path coalgebras

In representation theory of coalgebras an important rôle is played by path coalgebras. This is the analogous situation to the path algebra associated to a quiver (see [ASS05], [ARS95] and [GR92]). In this section we give a brief approach to them. They will be studied deeper in the next chapters. Following [Gab72] (see also [ASS05] or [ARS95]), by a **quiver**, $Q$, we mean a quadruple $(Q_0, Q_1, s, t)$, where $Q_0$ is the set of **vertices** (or **points**), $Q_1$ is the set of **arrows** and, for each arrow $\alpha \in Q_1$, the vertices $s(\alpha)$ and $t(\alpha)$ are the **source** (or **start point** or **origin**) and the **sink** (or **end point** or **tail**) of $\alpha$, respectively. We denote an arrow $\alpha$ such that $s(\alpha) = i$ and $t(\alpha) = j$ as $\alpha : i \to j$ or $i \xrightarrow{\alpha} j$. If $i = j$ we say that $\alpha$ is a **loop**.

If $i$ and $j$ are two vertices in a quiver $Q$, an (oriented) **path** in $Q$ of length $m$ from $i$ to $j$ is a formal composition of arrows

$$p = \alpha_m \cdots \alpha_2 \alpha_1,$$

where $s(\alpha_1) = i$, $t(\alpha_m) = j$ and $t(\alpha_{k-1}) = s(\alpha_k)$ for $k = 2, \ldots, m$. To any vertex $i \in Q_0$, we attach a **trivial path** of length 0, say $e_i$ or simply $i$, starting and ending at $i$ such that $\alpha e_i = \alpha$ (resp. $e_j \beta = \beta$) for any arrow $\alpha$ (resp. $\beta$) with $s(\alpha) = i$ (resp. $t(\beta) = i$). We identify the set of vertices and the set of trivial paths. A **cycle** is a path which starts and ends at the same vertex.

Let $KQ$ be the $K$-vector space generated by the set of all paths in $Q$. Then $KQ$ can be endowed with a structure of (non necessarily unitary) $K$-algebra with multiplication induced by the concatenation of paths, that is,

$$(\alpha_m \cdots \alpha_2\alpha_1)(\beta_n \cdots \beta_2\beta_1) = \begin{cases} \alpha_m \cdots \alpha_2\alpha_1\beta_n \cdots \beta_2\beta_1 & \text{if } t(\beta_n) = s(\alpha_1), \\ 0 & \text{otherwise;} \end{cases}$$

$KQ$ is the **path algebra** of the quiver $Q$. The algebra $KQ$ can be graded by

$$KQ = KQ_0 \oplus KQ_1 \oplus \cdots \oplus KQ_m \oplus \cdots ,$$

where $Q_m$ is the set of all paths of length $m$; $Q_0$ is a complete set of primitive orthogonal idempotents of $KQ$. If $Q_0$ is finite then $KQ$ is unitary and it is clear that $KQ$ has finite dimension if and only if $Q$ is finite and has no cycles.

An ideal $\Omega \subseteq KQ$ is called an **ideal of relations** or a **relation ideal** if $\Omega \subseteq KQ_2 \oplus KQ_3 \oplus \cdots = KQ_{\geq 2}$. An ideal $\Omega \subseteq KQ$ is **admissible** if it is a relation ideal and there exists a positive integer, $m$, such that $KQ_m \oplus KQ_{m+1} \oplus \cdots = KQ_{\geq m} \subseteq \Omega$. By a **quiver with relations** we mean a pair $(Q, \Omega)$, where $Q$ is a quiver and $\Omega$ a relation ideal of $KQ$. If $\Omega$ is admissible then $(Q, \Omega)$ is said to be a **bound quiver** (for more details see [ASS05] and [ARS95]).

The path algebra $KQ$ can be viewed as a graded $K$-coalgebra with comultiplication induced by the decomposition of paths, that is, if $p = \alpha_m \cdots \alpha_1$ is a path from the vertex $i$ to the vertex $j$, then

$$\Delta(p) = e_j \otimes p + p \otimes e_i + \sum_{i=1}^{m-1} \alpha_m \cdots \alpha_{i+1} \otimes \alpha_i \cdots \alpha_1 = \sum_{\eta\tau = p} \eta \otimes \tau$$

and for a trivial path, $e_i$, we have $\Delta(e_i) = e_i \otimes e_i$. The counit of $KQ$ is defined by the formula

$$\epsilon(\alpha) = \begin{cases} 1 & \text{if } \alpha \in Q_0, \\ 0 & \text{if } \alpha \text{ is a path of length} \geq 1. \end{cases}$$

The coalgebra $(KQ, \Delta, \epsilon)$ is called the **path coalgebra** of the quiver $Q$.

**Proposition 1.6.1.** *Let $Q$ be a quiver and $KQ$ the path coalgebra of $Q$. Then the following assertions hold:*

(a) $KQ = KQ_0 \oplus KQ_1 \oplus \cdots \oplus KQ_n \oplus \cdots$ *is a graded $K$-coalgebra.*

(b) *The subcoalgebras $KQ_0 \subseteq KQ_0 \oplus KQ_1 \subseteq KQ_0 \oplus KQ_1 \oplus KQ_2 \subseteq \cdots$ give the coradical filtration of $KQ$.*

(c) *Every simple right $KQ$-comodule is isomorphic to $Ke_i$ for some trivial path $e_i$.*

(d) *$KQ$ is pointed.*

(e) *$Soc(KQ) = \bigoplus_{i \in Q_0} Ke_i$.*

(f) *For each $i \in Q_0$, the injective envelope of the simple right $KQ$-comodule $S_i = Ke_i$ is generated by the set of all paths in $Q$ ending at $i$.*

Let us now introduce path coalgebras in another way. This point of view allow us to associate a path coalgebra to any pointed coalgebra. Following [Nic78], let $C$ be a coalgebra and $M$ be a $(C, C)$-bicomodule. Then we may construct the **cotensor coalgebra**

$$CT_C(M) = C \oplus M \oplus M \square_C M \oplus M \square_C M \square_C M \oplus \cdots$$

Since the cotensor product of $M$, $n$-times, is usually denoted by $M^{\square n}$, we write $CT_C(M) = \bigoplus_n M^{\square n}$. The comultiplication in $CT_C(M)$ is given by the formula

$$\Delta(m_1 \otimes m_2 \otimes \cdots \otimes m_n) = \omega^l(m_1) \otimes m_2 \otimes \cdots \otimes m_n +$$
$$+ \sum_{i=1}^{n-1} (m_1 \otimes \cdots \otimes m_i) \otimes (m_{i+1} \otimes \cdots \otimes m_n) + m_1 \otimes \cdots \otimes m_{n-1} \otimes \omega^r(m_n),$$

where $\omega^l$ and $\omega^r$ are the structure maps of $M$ as left and right $C$-comodule, respectively. The counit is given by $\epsilon = \epsilon_C \circ \pi$, where $\pi$ is the projection of $CT_C(M)$ onto $C$.

**Example 1.6.2.** *Let $Q$ be a quiver. Then it is easy to see from the definition that $KQ \cong CT_{KQ_0}(KQ_1)$. Furthermore, each piece $(KQ_1)^{\square n}$ is isomorphic to $KQ_n$.*

An element $x \in C$ is said to be a **group-like element** if $\Delta_C(x) = x \otimes x$. It is not hard to prove that the set of group-like elements, $\mathcal{G}(C)$, is bijective with the set of one dimensional subcoalgebras (which are simple) by the map $x \mapsto Kx$, see [Swe69]. If $C$ is pointed then all simple subcoalgebras are 1-dimensional, so the set of group-like elements generates the coradical of $C$. Let now $x$ and $y$ be two group-like elements. We say that $c \in C$ is a $(x, y)$-**primitive element** if $\Delta_C(c) = y \otimes c + c \otimes x$. We denote the vector space of $(x, y)$-primitive elements of $C$ by $P^C_{x,y}$. Note that the vector space $T^C_{x,y} = K(x - y) \subseteq P^C_{x,y}$. These elements are called the trivial $(x, y)$-primitive elements. We denote the vector space formed by the non-trivial $(x, y)$-primitive elements $P^C_{x,y}/T^C_{x,y}$ by $P'^C_{x,y}$.

**Lemma 1.6.3.** *[Mon93] Let $C$ be a pointed coalgebra and*

$$C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots \subseteq C_n \subseteq \cdots$$

*its coradical filtration. Then $C_1 = \bigoplus_{x,y\in\mathcal{G}(C)} P^C_{x,y}$. Consequently, $C_1/C_0 = \bigoplus_{x,y\in\mathcal{G}(C)} P'^C_{x,y}$.*

Observe that $C_0$ is a coalgebra and $C_1/C_0$ is a $(C_0, C_0)$-bicomodule with structure maps $\omega^l(c) = y \otimes c$ and $\omega^r(\alpha) = c \otimes x$ for each $c \in P'^C_{x,y}$. Therefore, for each pointed coalgebra $C$, we may associate the cotensor coalgebra $CT_{C_0}(C_1/C_0)$.

**Proposition 1.6.4.** *Every pointed coalgebra $C$ is a subcoalgebra of the cotensor coalgebra $CT_{C_0}(C_1/C_0)$.*

*Proof.* By [Nic78], the cotensor coalgebra $CT_{C_0}(C_1/C_0)$ verifies that if $C'$ is a coalgebra, $h : C' \to C_0$ and $q : C \to C'$ are coalgebras maps, and $f : C \to C_1/C_0$ is a $(C, C)$-bicomodule map with $f(\operatorname{soc} C) = 0$; then there exists a unique coalgebra map $F : C \to CT_{C_0}(C_0/C_1)$ such that the diagrams

$$\begin{tikzcd} CT_C(M) \arrow[d, "\pi"'] & D' \arrow[l, "F"'] \arrow[d, "q"] \\ C & C' \arrow[l, "h"] \end{tikzcd} \qquad \begin{tikzcd} CT_C(M) \arrow[d, "p"'] & \\ M & D' \arrow[l, "f"] \arrow[ul, "F"'] \end{tikzcd}$$

are commutative, where $\pi$ and $p$ are the standard projections. Furthermore, the map $F$ is exactly $h \circ q + \sum_{n\geq 0} T_n(f)\Delta_{n-1}$.

For our purpose, we choose $C' = C_0$, $h = id$, $q$ the projection from $C = C_0 \oplus I$ onto $C_0$ and $f : C = C_0 \oplus I \to C_1/C_0$ the linear projection from $I$ to $C_1/C_0$ extended to $C_0$ by taking $f(C_0) = 0$. Then $F|_{C_1} = id$ and therefore $F$ is injective (see [Nic78], [Rad78] and [Mon93] for details). $\square$

Given a pointed coalgebra $C$, we may construct a quiver $Q$ as follows: $Q_0$ is the set of group-like elements and, for each $x, y \in Q_0$, the number of arrows from $x$ to $y$ equals $\dim_K P'^C_{x,y}$. This quiver is called the **Gabriel quiver** of $C$. Also it is known as the **Ext-quiver** of $C$ because of the vector space $P'^C_{x,y} \cong Ext^1_C(Kx, Ky)$.

**Lemma 1.6.5.** *Let $C$ be a pointed coalgebra and $Q$ be the Gabriel quiver of $C$. Then $CT_{C_0}(C_1/C_0) \cong KQ$.*

*Proof.* We have $KQ_0 \cong C_0$ as coalgebras and $KQ_1 \cong C_1/C_0$ as $(C_0, C_0)$-bicomodules. Thus $CT_{C_0}(C_1/C_0) \cong CT_{KQ_0}(KQ_1) \cong KQ$. $\square$

As a consequence of Proposition 1.6.4 and Lemma 1.6.5, we obtain the main result of this section. We recall from [Woo97] that a subcoalgebra of a path coalgebra $KQ$ is said to be **admissible** if it contains the subcoalgebra generated, as vector space, by the set of all vertices and all arrows of $Q$, that is, if it contains $KQ_0 \oplus KQ_1$ (see ). On the other hand, a subcoalgebra $C$ of a path coalgebra $KQ$ is called a **relation subcoalgebra** (cf. [Sim05]) if $C$ satisfies the following two conditions:

$(a)$ $C$ is admissible.

$(b)$ $C = \bigoplus_{x,y\in Q_0} C_{xy}$, where $C_{xy} = C \cap KQ(x,y)$ and $Q(x,y)$ is the set of all paths in $Q$ from $x$ to $y$.

**Theorem 1.6.6.** *[Woo97] Let $C$ be a pointed coalgebra. Then $C$ is isomorphic to an admissible subcoalgebra of the path coalgebra of its Gabriel quiver.*

# Chapter 2

# Path coalgebras of quivers with relations

Path algebras of bound quivers are one of the major tools in the representation theory of finite dimensional algebras. Indeed, a very well-known result of Gabriel (see for instance [ASS05], [ARS95], [GR92] and references therein) asserts that any basic algebra, over an algebraically closed field, is isomorphic to a quotient of the path algebra of its Gabriel quiver. This gives rise to a combinatorial approach to the calculation and description of the modules, which is employed, up to some extent, in many of the present developments of the theory. The main aim of this chapter is to study the possibility of an analogous result for coalgebras, through the notion of path coalgebra of a quiver with relations defined by Simson in [Sim01]. For this purpose, we establish a general framework using the weak* topology on the dual algebra to treat the problem in an elementary context. In conjunction with the calculation of a more manageable basis of an admissible coalgebra, this allow us to give a criterion for deciding whether or not an admissible subcoalgebra is the path coalgebra of a quiver with relations.

## 2.1 Duality

After seing the definition of the coalgebra structure, it is rather natural to have in mind that there should be certain kind of duality between algebras and coalgebras. In this section we remind this duality. We also point out some known facts about it in order to apply them all along this work.

Let $(C, \Delta, \epsilon)$ be a coalgebra, then we equip the dual vector space $C^* = \mathrm{Hom}_K(C, K)$ with an algebra structure as follows:

- The product $m$ is the composition of the maps

$$\begin{tikzcd} C^* \otimes C^* \arrow[r, "\rho"] \arrow[rr, bend right, "m"'] & (C \otimes C)^* \arrow[r, "\Delta^*"] & C^*, \end{tikzcd}$$

  where $\rho$ is defined by $\rho(f \otimes g)(v \otimes w) = f(v)g(w)$ for any $f, g \in C^*$ and $u, v \in C$. That is, for each $f, g \in C^*$, $m(f \otimes g) = (f \otimes g) \circ \Delta$. This product is known as the **convolution product**. We shall denote $m(f \otimes g)$ by $f * g$, or simply by $fg$.

- The unit is $u = \epsilon^* : K \to C^*$.

**Proposition 2.1.1.** *[Tak77] The triple $(C^*, m, u)$ is an algebra. We call it the **dual algebra** of $C$.*

We may relate the vector subspaces of $C$ and its dual algebra. Let $c \in C$. The **orthogonal space** to $c$ is the vector space $c^\perp = \{f \in C^* \mid f(c) = 0\}$. More generally, for any subset $S \subseteq C$, we may define the orthogonal space to $S$ to be the vector space

$$S^\perp = \{\, f \in C^* \mid f(S) = 0\}.$$

On the other hand, for any subset $T \subseteq C^*$, the **orthogonal space** to $T$ in $C$ is defined by the formula

$$T^\perp = \{\, c \in C \mid f(c) = 0 \text{ for all } f \in T\}.$$

We say that $T \subseteq C^*$ is **closed** if $T^{\perp\perp} = T$.

**Proposition 2.1.2.** *[Swe69]*

*(a) If $D \subseteq C$ is a subcoalgebra then $D^\perp$ is an ideal of $C^*$.*

*(b) If $I \subseteq C^*$ is an ideal then $I^\perp$ is a subcoalgebra of $C$.*

*(c) $D \subseteq C$ is a subcoalgebra if and only if $D^\perp$ is an ideal of $C^*$. In this case, $C^*/D^\perp \cong D^*$ as algebras.*

**Proposition 2.1.3.** *[Swe69]*

*(a) If $J \subseteq C$ is a right (left) coideal then $J^\perp$ is a right (left) ideal of $C^*$.*

*(b) If $I \subseteq C^*$ is a right (left) ideal then $I^\perp$ is a right (left) coideal of $C$.*

*(c) $J \subseteq C$ is a right (left) coideal if and only if $J^\perp$ is a right (left) ideal of $C^*$.*

**Proposition 2.1.4.** *[Swe69]*

(*a*) *If $J \subseteq C$ is a coideal then $D^{\perp}$ is a subalgebra of $C^*$.*

(*b*) *If $I \subseteq C^*$ is a subalgebra then $I^{\perp}$ is a coideal of $C$.*

(*c*) *$J \subseteq C$ is a coideal if and only if $D^{\perp}$ is a subalgebra of $C^*$.*

**Remark 2.1.5.** *In general, if $(A, m, u)$ is an algebra, its dual vector space $A^*$ does not have to be a coalgebra. This fact comes true if $A$ is finite dimensional, since, in such a case, the aforementioned map $\rho$ is bijective. Namely, we may set $\Delta = \rho^{-1} \circ m^*$ and $\epsilon = u^*$, and then $(A^*, \Delta, \epsilon)$ becomes a coalgebra. Hence we have an equivalence between the category of finite dimensional coalgebras and finite dimensional algebras over a field*

$$\mathcal{F}in\mathcal{D}im\mathcal{A}lg_K \xleftrightarrow{(-)^*} \mathcal{F}in\mathcal{D}im\mathcal{C}oalg_K$$

*On the other hand, by the Fundamental Coalgebra Structure Theorem, every coalgebra is a directed union of its family of finite dimensional subcoalgebras. So we may see a coalgebra as a direct limit of finite dimensional algebras. For that reason, coalgebras might be considered as an intermediate structure between finite dimensional and infinite dimensional algebras.*

A coalgebra $C$ may be endowed with a right and left $C^*$-module structure using the actions $\leftharpoonup$ and $\rightharpoonup$ defined by

$$c \leftharpoonup f = \sum_{(c)} f(c_{(1)})c_{(2)} \quad \text{and} \quad f \rightharpoonup c = \sum_{(c)} f(c_{(2)})c_{(1)},$$

where $f \in C^*$ and $c \in C$ verifies that $\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$ using the sigma-notation of Sweedler (see [Swe69]). For simplicity we write $cf$ and $fc$ instead of $c \leftharpoonup f$ and $f \rightharpoonup c$, respectively.

A right $C$-comodule $(M, \omega)$ acquires a structure of left $C^*$-module $(M, \rho)$ (which is called the **rational** $C^*$ structure), where $\rho$ is the composition

$$C^* \otimes M \xrightarrow{I \otimes \omega} C^* \otimes M \otimes C \xrightarrow{T \otimes I} M \otimes C^* \otimes C \xrightarrow{I \otimes e} M \otimes K \cong M,$$

where $T : C^* \otimes M \to M \otimes C^*$ is the **flip map** defined by $T(f \otimes m) = m \otimes f$ for any $f \in C^*$ and $m \in M$, and $e$ is the evaluation map. That is, using the sigma-notation,

$$fm = \rho(f \otimes m) = \sum_{(m)} f(m_{(1)})m_{(0)},$$

where $f \in C^*$ and $m \in M$ verifies that $\omega(m) = \sum_{(m)} m_{(0)} \otimes m_{(1)}$. Observe that if $M = C$, we obtain the aforementioned structure of $C^*$-module. Symmetrically, for any left $C$-comodule, we may define on it a right $C^*$-module structure.

The reader should consider the question of which modules arise in the above fashion from comodules. The solution comes from the so-called **rational modules** (or **discrete modules** in the terminology of [Sim01], or **pseudo-compact** in the sense of [Gab62]). Let $(M, \rho)$ be a left $C^*$-module and $\omega : M \to \mathrm{Hom}_{C^*}(C^*, M)$ be the linear map defined by $\omega(m)(f) = \rho(f \otimes m)$ for any $f \in C^*$ and $m \in M$. There exist the following injective maps:

$$\begin{array}{ccccccc}
M \otimes C & \longrightarrow & M \otimes C^{**} & \xrightarrow{f} & \mathrm{Hom}_{C^*}(C^*, M) & & \\
m \otimes c & \longmapsto & m \otimes c^{**} & \longmapsto & f_{m\otimes c^{**}} : C^{**} & \longrightarrow & M \\
 & & & & c^* & \longmapsto & f_{m\otimes c^{**}}(c^*) = c^{**}(c^*)m
\end{array}$$

Then $M$ is called rational if $\omega(M) \subseteq M \otimes C$.

**Proposition 2.1.6.** *Let $(M, \rho)$ be a rational left $C^*$-module. Then $(M, \omega)$ is a right $C$-comodule.*

This produces an equivalence of categories, $\mathcal{M}^C \simeq \mathrm{Rat}(C^*)$, between the category of right $C$-comodules and the category of rational left $C^*$-modules.

## 2.2 Pairings and weak* topology

This is a technical section devoted to developing some basic facts on topologies induced by pairing of vector spaces which will be useful in what follows. For further information see [Abe77], [HR73], [Rad74a] and [Rad74b].

Let $V$ and $W$ be vector spaces over a field $K$. A **pairing** $(V, W)$ of $V$ and $W$ is a bilinear map $\langle -, - \rangle : V \times W \to K$. A pairing $\langle -, - \rangle$ is **non degenerate** if the following properties hold

$$\begin{cases} \text{if } \langle v, w \rangle = 0 \text{ for all } v \in V, \text{ then } w = 0, \\ \text{if } \langle v, w \rangle = 0 \text{ for all } w \in W, \text{ then } v = 0. \end{cases}$$

This means that the linear maps $\sigma : V \longrightarrow W^*$ and $\tau : W \longrightarrow V^*$ defined by $\sigma(v)(w) = \langle v, w \rangle$ and $\tau(w)(v) = \langle v, w \rangle$ for all $v \in V$ and $w \in W$ are injective. Throughout this section all pairings will be non-degenerate.

A well-known example of a non degenerate pairing is the dual pairing, $(V, V^*)$, given by the evaluation map $\langle v, f \rangle = f(v)$ for all $v \in V$, $f \in V^*$.

Given a pairing, $(V, W)$, we may relate subspaces of $V$ and $W$ through the dual pairing, compare with the former section. Let $v \in V$. The **orthogonal complement** to $v$ is the set $v^\perp = \{f \in V^* \mid f(v) = 0\}$. More generally, for any subset $S \subseteq V$, we define the orthogonal complement to $S$ to be the space

$$S^\perp = \{\ f \in V^* \mid f(S) = 0\}.$$

Since $W$ can be embebed in $V^*$ by the pairing, we may consider the orthogonal subspace to $S$ in $W$

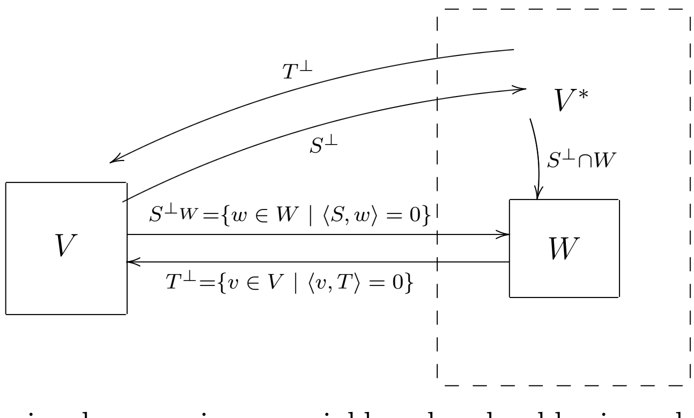
$$S^{\perp_W} = S^\perp \cap W = \{w \in W \mid \langle S, w\rangle = 0\}.$$

On the other hand, for any subset $T \subseteq V^*$, the **orthogonal complement** to $T$ in $V$ is defined by the formula

$$T^{\perp_V} = \{\ v \in V \mid f(v) = 0 \text{ for all } f \in T\},$$

and if $T \subseteq W$, then we write $T^{\perp_V} = \{\ v \in V \mid \langle v, w\rangle = 0 \text{ for all } w \in T\}$. For simplicity we write $\perp$ instead of $\perp_V$.

The following diagram summarizes the above discussion:




The following lemma gives a neighbourhood subbasis and a neighbourhood basis of a topology on $V^*$. We call it the **weak* topology** on $V^*$, see [Abe77], [Rad74a] and [Rad74b].

**Lemma 2.2.1.** *Let $f$ be a linear map in $V^*$.*

(*a*) *The set $\mathcal{U}_f = \{\ f + v^\perp \mid v \in V\}$ is a neighbourhood subbasis of $f$ for a topology on $V^*$.*

(*b*) *The sets $\mathcal{B}^f_{x_1,\dots,x_n} = \{g \in V^* \mid g(x_i) = f(x_i)\ \forall i = 1, \dots, n\} \subseteq V^*$, for any $x_1, \dots, x_n \in V$ and $n \in \mathbb{N}^*$, form a neighbourhood basis at $f$ for the topology on $V^*$ defined in (a).*

*Proof.* $(a)$ This is straightforward.

$(b)$ The finite intersections of elements of a neighbourhood subbasis form a neighbourhood basis and it is easy to check that

$$f + x^{\perp} = \{g \in V^* \mid g(x) = f(x)\},$$

for any $x \in V$.

□

If we view $W$ as a subspace of the vector space $V^*$, the induced topology on $W$ is called the **$V$-topology**. In the next proposition we collect some useful properties of the weak* topology.

**Proposition 2.2.2.** *Let $(V, W)$ be a pairing of $K$-vector spaces.*

$(a)$ *The weak* topology is the weakest topology on $V^*$ which makes continuous the elements of $V$, that is, it is the initial topology for the elements of $V$.*

$(b)$ *The closed subspaces on the weak* topology are $S^{\perp}$, where $S$ is a subspace of $V$.*

$(c)$ *The closure of a subspace $T$ of $V^*$ (in the weak* topology) is $T^{\perp\perp}$.*

$(d)$ *The closed subspaces on the $V$-topology are $S^{\perp_W}$, where $S$ is a subspace of $V$.*

$(e)$ *The closure of a subspace $T$ of $W$ (in the $V$-topology) is $T^{\perp\perp_W}$.*

$(f)$ *Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of subspaces of $V$. Then*

$$\left(\sum_{\lambda\in\Lambda} A_\lambda\right)^{\perp} = \bigcap_{\lambda\in\Lambda} A_\lambda^{\perp} \quad \text{and} \quad \left(\sum_{\lambda\in\Lambda} A_\lambda\right)^{\perp_W} = \bigcap_{\lambda\in\Lambda} A_\lambda^{\perp_W}.$$

$(g)$ *Any finite dimensional subspace of $W$ is closed.*

*Proof.* $(a)$ Let $\mathcal{T}$ be the initial topology for the elements of $V$, and $\mathcal{W}$ the weak* topology on $V^*$. Let $k \in K$ and $\mathrm{ev}_y \in V$ be the evaluation on $y$. Then

$$(\mathrm{ev}_y)^{-1}(k) = \{f \in V^* \mid f(y) = k\}.$$

But, given $g \in (\mathrm{ev}_y)^{-1}(k)$, we obtain that $g \in g + y^{\perp} \subseteq (\mathrm{ev}_y)^{-1}(k)$ so $(\mathrm{ev}_y)^{-1}(k)$ is an open set in weak* topology and thus $\mathcal{T} \subseteq \mathcal{W}$. Conversely, given $f \in V^*$ and $x \in V$, a neighbourhood of $f$ in weak* topology is $f + x^{\perp} = \mathrm{ev}_x^{-1}(f(x))$, which is open in $\mathcal{T}$ and thus $\mathcal{W} \subseteq \mathcal{T}$.

($b$) Let $S \subseteq V$, if $f \notin S^\perp$ then there exists $x \in S$ such that $f(x) \neq 0$. Thus $(f + x^\perp) \cap S^\perp = \emptyset$ and $f \notin \overline{S^\perp}$. Conversely, let $T$ be a closed subspace; it suffices to prove that $T^{\perp\perp} \subseteq T$. Fix $f \in T^{\perp\perp}$ and $x \in V$; if $x \in T^\perp$ then $f(x) = 0$, hence $0 \in (f + x^\perp) \cap T$. If, on the contrary, $x \notin T^\perp$ then there exists $g \in T$ such that $g(x) \neq 0$, therefore $\frac{f(x)}{g(x)} g \in (f + x^\perp) \cap T$. This shows that $f \in \overline{T} = T$.

($c$) $T^{\perp\perp}$ is a closed set satisfying $T \subseteq T^{\perp\perp}$, therefore $\overline{T} \subseteq T^{\perp\perp}$. We can now proceed analogously to the proof of ($b$) in order to show $T^{\perp\perp} \subseteq \overline{T}$.

($d$) The $V$-topology on $W$ is induced by the weak* topology on $V^*$ so $S^{\perp_W} = S^\perp \cap W$ is closed. If $T$ is closed, then $T = \overline{T}^W = \overline{T} \cap W = T^{\perp\perp} \cap W = T^{\perp\perp_W}$.

($e$) The proof is straightforward from ($d$).

($f$) We have

$$\begin{aligned} f \in \bigcap_{\lambda \in \Lambda} A_\lambda^\perp &\Leftrightarrow f(A_\lambda) = 0 \quad \forall \lambda \in \Lambda, \\ &\Leftrightarrow f(\textstyle\sum_{\lambda\in\Lambda} A_\lambda) = 0, \\ &\Leftrightarrow f \in \left(\textstyle\sum_{\lambda\in\Lambda} A_\lambda\right)^\perp. \end{aligned}$$

($g$) See [Abe77, Chapter 2].

□

**Lemma 2.2.3.** *Let $(V, W)$ be a pairing of $K$-vector spaces.*

($a$) *Let $A$ be a subspace of $V$. Then $A^{\perp\perp} = A$.*

($b$) *Let $A$ be a finite dimensional subspace of $V$. Then $A^{\perp_W\perp} = A$.*

($c$) *Let $\{T_i\}_{i\in I}$ be a family of subspaces of $V^*$. Then*

$$\left(\sum_{i\in I} T_i\right)^\perp = \bigcap_{i\in I} T_i^\perp.$$

*Proof.* ($a$) $f(A) = 0$ for each $f \in A^\perp$ and so $A \subseteq A^{\perp\perp}$. Converselly, let $v \notin A \subsetneq V$. There exists $f \in V^*$ such that $f(A) = 0$ and $f(v) \neq 0$. By Proposition 2.2.2, $A^\perp$ is closed so $A^{\perp\perp\perp} = A^\perp$ and therefore, $\forall g \in V^*$, $g(A) = 0 \Leftrightarrow g(A^{\perp\perp}) = 0$, which implies that $v \notin A^{\perp\perp}$.

($b$) See, for instance, [Abe77, Theorem 2.2.1].

$(c)$ We have

$$\begin{aligned} v \in \bigcap_{i\in I} T_i^{\perp} &\Leftrightarrow f(v) = 0 \quad \forall f \in T_i \ \forall i \in I, \\ &\Leftrightarrow f(v) = 0 \ \forall f \in \sum_{i\in I} T_i, \\ &\Leftrightarrow v \in \left(\sum_{i\in I} T_i\right)^{\perp}. \end{aligned}$$

□

## 2.3 Basis of a relation subcoalgebra

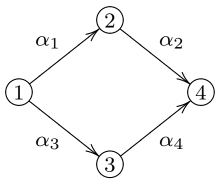
The aim of this section is to obtain a more manageable basis for a relation subcoalgebra of a path coalgebra. For more information and technical properties of subcoalgebras see [JMR].

Let $Q = (Q_0, Q_1)$ be a quiver and $C$ a subcoalgebra of $KQ$. Fix a path $p = \alpha_n\alpha_{n-1}\cdots\alpha_1$ in $Q$; a **subpath** of $p$ is a path, $q$, such that either $q$ is a vertex of $p$ or $q$ is a non-trivial path $\alpha_i\alpha_{i+1}\cdots\alpha_j$, where $1 \leq j \leq i \leq n$.

**Lemma 2.3.1.** *[JMR] Let $C \subseteq KQ$ be a subcoalgebra, and $p$ be a path in $C$. Then all subpaths of $p$ are in $C$.*

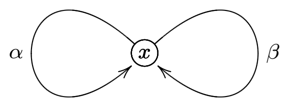
This result could lead the reader to ask if any subcoalgebra is generated by a set of paths. Unfortunately this is not true as the following examples show.

**Example 2.3.2.** *Let $Q$ be the quiver*




*The subspace of $KQ$ generated by $\{e_1, e_2, e_3, e_4, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_2\alpha_1 + \alpha_4\alpha_3\}$ is a subcoalgebra of $KQ$ which cannot be generated by paths.*

**Example 2.3.3.** *Let $Q$ be the quiver*




*The subcoalgebra $C = K\{e_x, \alpha + \beta\}$ is not generated by paths.*

One may observe that, in the preceding examples, the basic elements which are not paths have the common property of being a linear combination of paths with the same source and the same sink. The next proposition asserts that, in general, every subcoalgebra of a path coalgebra has this property.

**Proposition 2.3.4.** *Let $Q$ be a quiver and $C \subseteq KQ$ a subcoalgebra. Then there exists a $K$-linear basis of $C$ such that each basic element is a linear combination of paths with common source and common sink.*

*Proof.* See [JMR, Proposition 2.8]. □

**Corollary 2.3.5.** *Any admissible subcoalgebra of a path coalgebra is a relation subcoalgebra.*

Proposition 2.3.4 is the key-tool which allows us to give a more precise description of the basis of a relation subcoalgebra. Throughout this section, we assume that $C$ is an admissible subcoalgebra of certain path coalgebra $KQ$ and that $\mathcal{B}$ is a $K$-linear basis of $C$ as in Proposition 2.3.4. By definition, $C$ contains the set of all vertices, $V = \{e_i\}_{i \in Q_0}$, and the set of all arrows, $F = \{\alpha\}_{\alpha \in Q_1}$, therefore we rearrange the elements of the basis $\mathcal{B}$ as follows:

$$\mathcal{B} = V \cup F \cup \{G_{ij}^{\tau} \mid \tau \in \mathrm{T}_{ij} \text{ and } i, j \in Q_0\},$$

where, for all $\tau \in \mathrm{T}_{ij}$, the element $G_{ij}^{\tau}$ is a $K$-linear combination of paths with length greater than one which start at $i$ and end at $j$.

We now assume that $D = \{p_\lambda\}_{\lambda \in \Lambda}$ is the set of all paths of length greater than one in $C$. Proceeding as before we can rewrite

$$\mathcal{B} = V \cup F \cup D \cup \{R_{ij}^{\upsilon} \mid \upsilon \in \mathrm{U}_{ij} \text{ and } i, j \in Q_0\},$$

where, for all $\upsilon \in \mathrm{U}_{ij}$, the element $R_{ij}^{\upsilon}$ is a $K$-linear combination of at least two paths of length greater than one which start at $i$ and end at $j$. Obviously, the paths involved in the linear combinations $R_{ij}^{\upsilon}$ are not in $C$, for any $\upsilon \in \mathrm{U}_{ij}$ and $i, j \in Q_0$.

For the convenience we introduce some notation. In that follows we denote by $\mathcal{Q} = Q_0 \cup Q_1 \cup \cdots \cup Q_n \cup \cdots$ the set of all paths in $Q$. Let $a$ be an element of $KQ$. Then we may write $a = \sum_{p \in \mathcal{Q}} a_p p$, for some $a_p \in K$. We define **the path support** of $a$ to be $\mathrm{PSupp}(a) = \{p \in \mathcal{Q} \mid a_p \neq 0\}$. In this way, for any set $S \subseteq KQ$, we define $\mathrm{PSupp}(S) = \bigcup_{a \in S} \mathrm{PSupp}(a)$.

**Definition 2.3.6.** *Let $S$ be a set in $KQ$. $S$ is called **connected** if $\mathrm{PSupp}(S_1) \cap \mathrm{PSupp}(S_2) \neq \emptyset$ for any subsets $S_1, S_2 \subseteq S$ such that $S_1 \cup S_2 = S$ and $S_1 \cap S_2 = \emptyset$. A subset $S' \subset S$ is a **connected component** of $S$ when $S'$ is connected and $\mathrm{PSupp}(S') \cap \mathrm{PSupp}(S \backslash S') = \emptyset$.*

Therefore we can break down each set $S_{ij} = \{R^v_{ij}\}_{v\in \mathrm{U}_{ij}}$ into its connected components and then write the basis $\mathcal{B}$ of $C$ as

$$\mathcal{B} = V \cup F \cup D \cup \bigcup_{\phi\in\Phi} \Upsilon_\phi,$$

where, for any $\phi \in \Phi$, the set $\Upsilon_\phi$ is a connected set of $K$-linear combinations of at least two paths such that $\mathrm{PSupp}(\Upsilon_\phi) \subset KQ_{\geq 2}$ and $\mathrm{PSupp}(\Upsilon_{\phi_1}) \cap \mathrm{PSupp}(\Upsilon_{\phi_2}) = \emptyset \Leftrightarrow \phi_1 \neq \phi_2$.

As a final reduction, it will be useful to distinguish those sets $\Upsilon_\phi$ which are finite. Thus the basis $\mathcal{B}$ of $C$ can be written as

$$\mathcal{B} = V \cup F \cup D \cup \bigcup_{\gamma\in\Gamma} \Pi_\gamma \cup \bigcup_{\beta\in B} \Sigma_\beta,$$

where $\Pi_\gamma$ is a finite set for all $\gamma \in \Gamma$ and $\Sigma_\beta$ is infinite for all $\beta \in B$.

## 2.4 Path coalgebras of quivers with relations

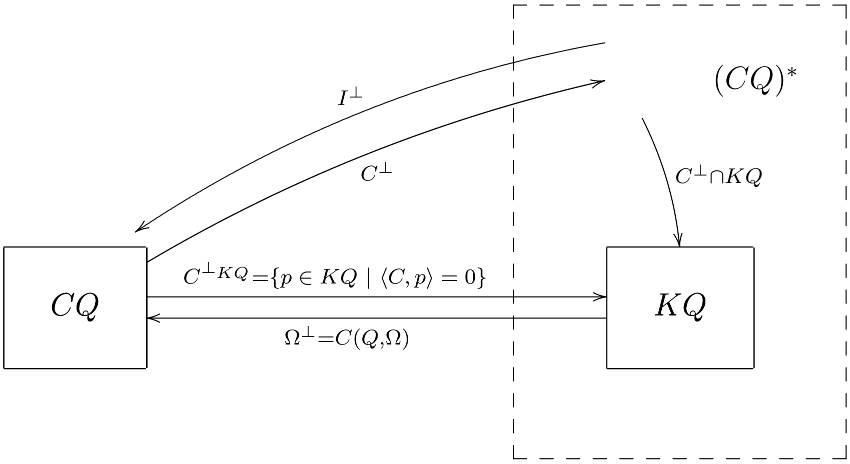
In this section we study the notion of the path coalgebra of a quiver with relations introduced by Simson in [Sim01] and [Sim05]. For the convenience of the reader we shall denote by $CQ$ and by $KQ$ the path coalgebra and the path algebra associated to a quiver $Q$, respectively (despite that the underlying vector space is the same).

**Definition 2.4.1.** *Let $(Q, \Omega)$ be a quiver with relations. The* ***path coalgebra*** *of $(Q, \Omega)$ is defined by the subspace of $CQ$,*

$$C(Q, \Omega) = \{a \in CQ \mid \langle a, \Omega\rangle = 0\}$$

*where $\langle -, -\rangle : CQ \times KQ \longrightarrow K$ is the bilinear map defined by $\langle v, w\rangle = \delta_{v,w}$ (the Kronecker delta) for any two paths $v, w \in \mathcal{Q}$.*

This notion may be reformulated in the notation of the Section 2.2. It is clear that $\langle -, -\rangle$ is a non-degenerate pairing between $CQ$ and $KQ$, therefore we have the following picture:




First we prove the following result.

**Lemma 2.4.2.** *If $Q$ is any quiver, then the injective morphism $KQ \hookrightarrow (CQ)^*$ defined by the pairing $\langle -, - \rangle$ of* 2.4.1 *is a morphism of algebras.*

*Proof.* Recall that in the dual algebra $(CQ)^* := \mathrm{Hom}_K(CQ, K)$ the (convolution) product is defined by

$$(f * g)(p) = \sum_{p = p_2 p_1} f(p_2) g(p_1), \text{ for any } f, g \in (CQ)^* \text{ and any } p \in \mathcal{Q}.$$

Fix $p \in \mathcal{Q}$ and let $p^* : CQ \to K$ be the linear map defined by $p^*(q) = \delta_{p,q}$ for any $q \in \mathcal{Q}$. It is enough to prove that $(pq)^* = p^* * q^*$ for any two paths $p, q \in \mathcal{Q}$. To prove this, let $r$ be a path in $Q$. Then:

$$(p^* * q^*)(r) = \sum_{r = r_2 r_1} \delta_{p, r_2} \delta_{q, r_1} = \begin{cases} 0 & \text{if } r \neq pq \\ 1 & \text{if } r = pq \end{cases} = (pq)^*(r),$$

and so $(pq)^* = p^* * q^*$. $\square$

It may be helpful to point out that, in general, the algebra $KQ$ does not have identity (it has identity if and only if $Q_0$ is finite) and the map defined above is an injective morphism of non necessarily unitary algebras. Therefore the situation is the following:

$$KQ \hookrightarrow (KQ)_1 \hookrightarrow (CQ)^*,$$

where $(KQ)_1 = KQ \oplus K \cdot 1$ is the unification of $KQ$.

**Lemma 2.4.3.** *$KQ$ is dense in $(CQ)^*$ in the weak* topology on $(CQ)^*$. Consequently, $KQ$ is dense in $(KQ)_1$ and $(KQ)_1$ is dense in $(CQ)^*$ in the weak* topology on $(CQ)^*$.*

*Proof.* This is a particular case of Lemma 2.5.2 below. It is enough to consider $C = 0$, obviously, $(KQ)^\perp = 0$ and then $0^\perp \cap KQ = KQ$ is dense in $0^\perp = (CQ)^*$. $\square$

From now on we will make no distinction between elements of $KQ$ and linear maps $f : CQ \to K$ with finite path support, that is, $f(p) = 0$, for almost all $p$ in $\mathcal{Q}$. On the other hand, it is convenient to note that any element $g \in (CQ)^*$ can be written as a formal sum $g = \sum_{p \in \mathcal{Q}} a_p p$, where $a_p = g(p) \in K$.

**Corollary 2.4.4.** *Let $Q$ be a quiver and $C$ an admissible subcoalgebra of $CQ$. Then $C^{\perp_{KQ}}$ is a relation ideal of $KQ$.*

*Proof.* Since $C^\perp$ is an ideal of $(CQ)^*$, $C^\perp \cap KQ = C^{\perp_{KQ}}$ is an ideal of $KQ$ by Lemma 2.4.2. If $c \in KQ_0 \oplus KQ_1$, then $c \in C$ since $C$ is an admissible subcoalgebra. Therefore $\langle c, C\rangle \neq 0$, so $c \notin C^{\perp_{KQ}}$, which completes the proof. $\square$

The following result, proved in [Sim05], justifies the preceding definition of the path coalgebra of a quiver with relations.

**Proposition 2.4.5.** *Let $Q$ be a quiver and $\Omega$ a relation ideal of $KQ$, then $C(Q,\Omega) = \Omega^\perp$ is an admissible subcoalgebra of $CQ$.*

Let us finish the section highlighting that the importance of the definition presented by Simson lies in the fact that it is consistent with the classical case. Just as modules over path algebras of bound quivers, comodules over path coalgebras of quivers with relations can be described by means of representations of the corresponding quiver. We recall that a $K$-**linear representation** (cf. [ASS05], [ARS95] and [Sim92]) of a quiver $Q$ is a system

$$X = (X_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1},$$

where $X_i$ is a $K$-vector space and $\varphi_\alpha : X_i \to X_j$ is a $K$-linear map for any $\alpha : i \to j$. Given two $K$-linear representations of $Q$, $(X_i, \varphi_\alpha)$ and $(Y_i, \psi_\alpha)$, a morphism

$$f : (X_i, \varphi_\alpha) \longrightarrow (Y_i, \psi_\alpha)$$

of representations of $Q$ is a system $f = (f_i)_{i \in Q_0}$ of $K$-linear maps $f_i : X_i \longrightarrow Y_i$ for any $i \in Q_0$ such that, for any $\alpha : i \to j$ in $Q_1$, the following diagram is commutative

$$\begin{tikzcd} X_i \arrow[r, "\varphi_\alpha"] \arrow[d, "f_i"'] & X_j \arrow[d, "f_j"] \\ Y_i \arrow[r, "\psi_\alpha"] & Y_j \end{tikzcd}$$

We denote by $\mathrm{Rep}_K(Q)$ the Grothendieck $K$-category of $K$-linear representations of $Q$, and by $\mathrm{Rep}_K^{\mathrm{lf}}(Q)$ the full Grothendieck $K$-subcategory of $\mathrm{Rep}_K(Q)$ formed by locally finite-dimensional representations, that is, directed unions of representations of finite length. A linear representation $X$ of $Q$ is said to be of **finite length** if $X_i$ is a finite dimensional vector space for all $i \in Q_0$ and $X_i = 0$ for almost all indices $i$. Finally, we denote by $\mathrm{rep}_K(Q) \supseteq \mathrm{rep}_K^{\mathrm{lf}}(Q)$ the full subcategories of $\mathrm{Rep}_K(Q)$ formed by finitely generated representations and representations of finite length, respectively.

A linear representation $X$ of $Q$ is said to be **nilpotent** if there exists an integer $m \geq 1$ such that the composed linear map

$$X_{i_0} \xrightarrow{\varphi_{\alpha_1}} X_{i_1} \xrightarrow{\varphi_{\alpha_2}} X_{i_2} \longrightarrow \cdots \longrightarrow X_{i_{m-1}} \xrightarrow{\varphi_{\alpha_m}} X_{i_m}$$

is zero for any path $\alpha_m \alpha_{m-1} \cdots \alpha_1$ in $Q$ of length $m$. We denote by $\mathrm{nilrep}_K^{\mathrm{lf}}(Q)$ the full subcategory of $\mathrm{rep}_K(Q)$ formed by all nilpotent representations of finite length, and by $\mathrm{Rep}_K^{\mathrm{lnlf}}(Q)$ the full subcategory of $\mathrm{Rep}_K(Q)$ of all locally nilpotent representations that are locally finite, that is, directed unions of representations from $\mathrm{nilrep}_K^{\mathrm{lf}}(Q)$.

Given a quiver with relations $(Q, \Omega)$, a linear representation of $(Q, \Omega)$ is a linear representation $X = (X_i, \varphi_\alpha)$ of $Q$ which verifies that if

$$p = \sum_{i=1}^{n} \lambda_i \alpha_{m_i}^i \cdots \alpha_1^i$$

is in $\Omega$, then $\sum_{i=1}^{n} \lambda_i \varphi_{\alpha_{m_i}^i} \cdots \varphi_{\alpha_1^i} = 0$. As above, we may define the categories $\mathrm{Rep}_K(Q, \Omega)$, $\mathrm{Rep}_K^{\mathrm{lf}}(Q, \Omega)$, $\mathrm{rep}_K(Q, \Omega)$, $\mathrm{rep}_K^{\mathrm{lf}}(Q, \Omega)$, $\mathrm{Rep}_K^{\mathrm{lnlf}}(Q, \Omega)$ and $\mathrm{nilrep}_K^{\mathrm{lf}}(Q, \Omega)$, see [Sim01], [Sim05] or [Woo97] for details.

**Theorem 2.4.6** ([Sim05], Theorem 3.5)**.** *Let $(Q, \Omega)$ be a quiver with relations. There are category isomorphisms*

$$\mathcal{M}^{C(Q,\Omega)} \cong \mathrm{Rep}_K^{\mathrm{lnlf}}(Q, \Omega) \ \textit{ and } \ \mathcal{M}_f^{C(Q,\Omega)} \cong \mathrm{nilrep}_K^{\mathrm{lf}}(Q, \Omega)$$

*between the category of $C(Q, \Omega)$-comodules (resp. finite dimensional $C(Q, \Omega)$-comodules) and locally finite locally nilpotent representations (resp. nilpotent representations of finite length) of $(Q, \Omega)$.*

## 2.5 When is a coalgebra a path coalgebra of a quiver with relations?

It is well-known that, over a algebraically closed field, a finite dimensional algebra $A$ is isomorphic to $KQ_A/\Omega$, where $Q_A$ is the Gabriel quiver of $A$ and $\Omega$ is an admissible ideal of $KQ$. In [Sim01], it is suggested, as an open problem, to relate the admissible subcoalgebras of a path coalgebra $CQ$ and the relation ideals of the path algebra $KQ$, through the above-mentioned notion of path coalgebra of a quiver with relations. That is, for any admissible subcoalgebra $C \leq CQ$, is there a relation ideal $\Omega \leq KQ$ such that $C = C(Q,\Omega)$? In other words, in the notation of Section 2.2, for any admissible subcoalgebra $C \leq CQ$, is there a relation ideal $\Omega$ of $KQ$ such that $\Omega^{\perp} = C$? Note that if $C$ has finite dimension, then, by Lemma 2.2.3, $(C^{\perp_{KQ}})^{\perp} = C$ and the result follows. This yields a reduction of the problem:

**Problem 2.5.1.** *Verify the relation $\Omega^{\perp} = C$ for the ideal $\Omega = C^{\perp_{KQ}}$.*

**Lemma 2.5.2.** *Let $Q$ be a quiver and $C$ a vector subspace of $CQ$. Then the following conditions are equivalent.*

*(a) There exists a subspace $\Omega$ of $KQ$ such that $\Omega^{\perp} = C$.*

*(b) $C^{\perp_{KQ}}$ is dense in $C^{\perp}$ in the weak* topology on $(CQ)^*$.*

*(c) $(C^{\perp_{KQ}})^{\perp} = C$.*

*Proof.* $(a) \Rightarrow (b)$. Since $C = \Omega^{\perp}$, it follows that $C^{\perp} = \Omega^{\perp\perp}$ is the closure of $\Omega$ in weak* topology by Proposition 2.2.2. Thus $\Omega \subset C^{\perp} \cap KQ = C^{\perp_{KQ}} \subset C^{\perp}$ and, by Proposition 2.2.3, $C = C^{\perp\perp} \subset (C^{\perp_{KQ}})^{\perp} \subset \Omega^{\perp} = C$. Therefore $C = (C^{\perp_{KQ}})^{\perp}$ and thus $C^{\perp} = (C^{\perp_{KQ}})^{\perp\perp} = \overline{C^{\perp_{KQ}}}$.

$(b) \Rightarrow (c)$. Since $C^{\perp} = (C^{\perp_{KQ}})^{\perp\perp}$, we have $C^{\perp\perp} = (C^{\perp_{KQ}})^{\perp\perp\perp}$ and, by Proposition 2.2.3, $C = (C^{\perp_{KQ}})^{\perp}$.

$(c) \Rightarrow (a)$. It is trivial. □

We now assume that $C$ is an admissible subcoalgebra of $CQ$. If we consider the basis of $C$,

$$\mathcal{B} = V \cup F \cup D \cup \bigcup_{\gamma\in\Gamma} \Pi_{\gamma} \cup \bigcup_{\beta\in B} \Sigma_{\beta},$$

built in Section 2.3, then we have

$$C = KV \oplus KF \oplus KD \oplus \left(\bigoplus_{\gamma\in\Gamma} K\Pi_{\gamma}\right) \oplus \left(\bigoplus_{\beta\in B} K\Sigma_{\beta}\right) \tag{2.1}$$

as $K$-vector space. Since the subsets into which we have partitioned $\mathcal{B}$ have disjoint path supports, it is easily seen that $\Omega^{\perp} = C$ if and only if each direct summand $C_i$ of (2.1) is the orthogonal complement $\Omega_i^{\perp}$ of a subspace $\Omega_i$ and, in this case, $\Omega = \bigcap \Omega_i$. There are two trivial cases:

CASE 1. It is immediate that $KV = K(\mathcal{Q}\backslash V)^{\perp}$, $KF = K(\mathcal{Q}\backslash F)^{\perp}$ and $KD = K(\mathcal{Q}\backslash D)^{\perp}$.

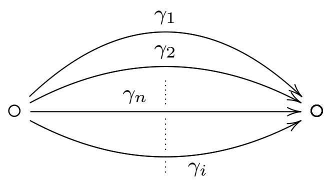
CASE 2. For each $\gamma \in \Gamma$, $K\Pi_{\gamma}$ is a finite dimensional subspace and so, by Lemma 2.2.3, $K\Pi_{\gamma} = ((K\Pi_{\gamma})^{\perp_{KQ}})^{\perp}$. As a consequence we get:

**Corollary 2.5.3.** *With the above notation, $C = \Omega^{\perp}$ if and only if $\Sigma_{\beta} = (\Sigma_{\beta})^{\perp_{KQ}\perp}$ for each $\beta \in B$.*

In particular, this implies the following proposition proved in [Sim05].

**Proposition 2.5.4.** *Let $Q$ be a quiver without cycles such that the set of paths in $Q$ from $i$ to $j$ is finite, for all $i, j \in Q_0$. Then the map $C \longmapsto C^{\perp_{KQ}}$ define a bijection between the set of all relation subcoalgebras of $CQ$ and the set of all admissible ideals of $KQ$. The inverse map is defined by $\Omega \longmapsto \Omega^{\perp}$, for any relation ideal $\Omega$ of $KQ$.*

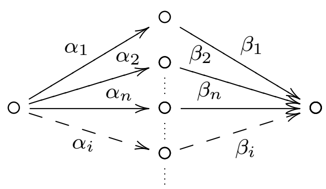
Therefore, we can reduce Problem 2.5.1 to the situation of a quiver $Q$ with the following structure

$Q \equiv$  length($\gamma_i$)$> 1$, $i \in I$, $I$ infinite (2.2)

and $C$ an admissible subcoalgebra generated, as vector space, by $V \cup F \cup D \cup \Sigma$, where $\Sigma$ is an infinite connected set with $\mathrm{PSupp}(\Sigma) = \{\gamma_i\}_{i \in I}$. We may assume that $\gamma_i \notin C$ for all $i \in I$. Then the question is: when the equality $\Sigma = \Sigma^{\perp_{KQ}\perp}$ holds?

Let us first show that, at least, there is an example of an admissible subcoalgebra $C \subseteq CQ$ such that $C$ is not of the form $C = \Omega^{\perp}$, where $\Omega$ is a relation ideal of $KQ$.

**Example 2.5.5.** *Let $Q$ be the quiver*

 *$\gamma_i = \beta_i \alpha_i$ for all $i \in \mathbb{N}$* (2.3)

*and let $H$ be the admissible subcoalgebra of $CQ$ as in* (2.2) *with*

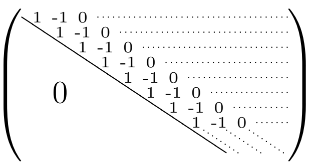
$$\Sigma = \{\gamma_i - \gamma_{i+1}\}_{i \in \mathbb{N}}.$$

*Assume that $x = \sum_{i \geq 1} a_i \gamma_i$ belongs to $H^{\perp}$ and $a_i = 0$ for $i \geq n$ we have some $n \in \mathbb{N}$. Then $\langle \gamma_i - \gamma_{i+1}, x \rangle = a_i - a_{i+1} = 0$ for all $i \in \mathbb{N}$, so $a_i = a_{i+1}$ for all $i \in \mathbb{N}$. But $a_n = 0$ and it follows that $x = 0$. Hence $H^{\perp_{KQ}} = 0$.*
*By a similar argument $H^{\perp} = \langle f \rangle$, where $f(\gamma_i) = 1$ for all $i \in \mathbb{N}$. That is, $f \equiv \sum_{i \geq 1} \gamma_i$. Obviously, $H^{\perp_{KQ}}$ is not dense in $H^{\perp}$.*

Here we present a positive example:

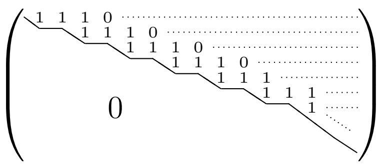
**Example 2.5.6.** *Let $Q$ be the quiver of* (2.3)*, and $C$ the admissible subcoalgebra generated by $\Sigma = \{\gamma_{2n-1} + \gamma_{2n} + \gamma_{2n+1}\}_{n \geq 1}$. A straightforward calculation shows that $\Omega^{\perp} = C$, where $\Omega = \langle \{\gamma_1 - \gamma_2, \{\gamma_{2n} - \gamma_{2n+1} + \gamma_{2n+2}\}_{n \geq 1}\} \rangle$.*

We now analyze them deeply to provide a criterium which allows us to know, when an admissible relation subcoalgebra of $CQ$ is the path coalgebra $C(Q, \Omega)$ of a quiver with relations. First, it is convenient to see Examples 2.5.5 and 2.5.6 from a more graphic point of view. We write the elements of $\Sigma$ in a matrix form. Thus we have the associated infinite matrices



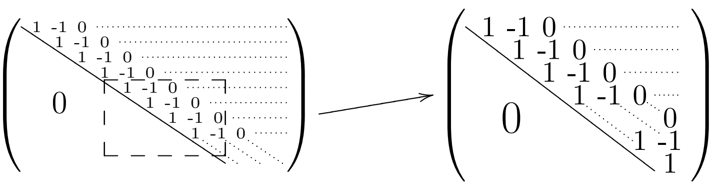
Example 2.5.5

and



Example 2.5.6

We may observe that Example 2.5.5 has an infinite diagonal of non zero elements. Let $h \in H^{\perp_{KQ}}$. Then $h$ must have finite path support, and so, if we want to know $h$, we only have to solve a finite linear system of equations with associated matrix



but zero is the unique solution.

In this way we obtain a class of admissible subcoalgebras which are not path coalgebras of quivers with relations:

**Definition 2.5.7.** *Let $Q$ be a quiver as in* (2.2)*, $C$ be an admissible subcoalgebra generated by a connected set $\Sigma$ with* $\mathrm{PSupp}(\Sigma) = \{\gamma_i\}_{i\in I}$ *and $\gamma_i \notin C$ for all $i \in I$. We say that $C$ has the* ***infinite diagonal property*** *(IDP for short) if there exists a subset $\Sigma' \subseteq \Sigma$ with* $\mathrm{PSupp}(\Sigma') = \{\gamma_n\}_{n\in\mathbb{N}}$ *such that by means of elementary transformations, $\Sigma'$ can be reduced to $\{\gamma_n + \sum_{j>n} a_j^n \gamma_j\}_{n\in\mathbb{N}}$, where $a_j^n \in K$ for all $j, n \in \mathbb{N}$.*

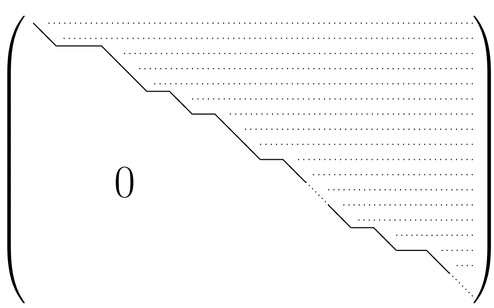
**Proposition 2.5.8.** *Let $Q$ be a quiver as in* (2.2) *and $C$ be an admissible subcoalgebra generated by a connected set $\Sigma$ with* $\mathrm{PSupp}(\Sigma) = \{\gamma_i\}_{i\in I}$*. Suppose that $\gamma_i \notin C$, for each $i \in I$. If $C$ has IDP, then there is no relation ideal $\Omega \subseteq KQ$ such that $C = C(Q,\Omega)$.*

*Proof.* Let $\Sigma' = \{\gamma_n + \sum_{j>n} a_j^n \gamma_j\}_{n\in\mathbb{N}} \subseteq \Sigma$. Assume that the assertion is not true, i.e., there is a relation ideal $\Omega \subseteq KQ$ such that $C = C(Q,\Omega)$. By Lemma 2.5.2, $C^{\perp_{KQ}}$ is dense in $C^{\perp}$. Since $\gamma_1 \notin C$, there exists a linear map $g \in C^{\perp}$ such that $g(\gamma_1) \neq 0$. By the density of $C^{\perp_{KQ}}$ in $C^{\perp}$, there exists a linear map $h$ with finite path support such that $h(\gamma_1) = g(\gamma_1)$. Defining $x_i := h(\gamma_i)$, for any $i \in \mathbb{N}$, we obtain that $h(\Sigma') = 0$ is the infinite system of linear equations $\{x_n + \sum_{j>n} a_j x_j = 0\}_{n\in\mathbb{N}}$. Since $h$ has finite path support, there exists an integer $m$ such that $x_k = 0$, for $k \geq m$. Hence $x_1, \ldots x_m$ satisfy the finite system of linear equations

$$\begin{aligned} x_1 + a_2^1 x_2 + \cdots + a_m^1 x_m &= 0 \\ x_2 + \cdots + a_m^2 x_m &= 0 \\ &\vdots \\ x_m &= 0 \end{aligned}$$

which has the unique solution $x_m = x_{m-1} = \cdots = x_1 = h(\gamma_1) = 0$, and we get a contradiction. $\square$

We claim that Example 2.5.6 does not have IDP. This means that for any infinite countable subset $\Sigma' \subseteq \Sigma$, the associated matrix can be reduced to a matrix of a "staircase" form




(2.4)

that is, for any positive integer $n$, the first $n$ rows have at least $n$ variables and there is an integer $m > n$ such that the first $m$ rows have more than $m$ variables. We can prove that for any linear map $f \in C^{\perp}$ and any finite set $\{\gamma_1, \ldots, \gamma_n\}$ of paths in $Q$, we obtain a linear map $g \in C^{\perp}$ such that $f(\gamma_i) = g(\gamma_i)$ for all $i = 1, \ldots, n$. That is, $C^{\perp_{KQ}}$ is dense on $C^{\perp}$.

**Proposition 2.5.9.** *Under the assumptions of Proposition* 2.5.8*, if $C$ fails IDP, then there exists a relation ideal $\Omega$ such that $C = C(Q, \Omega)$.*

*Proof.* It suffices to show that $\Sigma^{\perp_{KQ}}$ is dense in $\Sigma^{\perp}$, that is, given $f \in \Sigma^{\perp}$ and $\gamma_1, \ldots, \gamma_n \in \mathrm{PSupp}(\Sigma)$ there exists $h \in \Sigma^{\perp}$, with finite path support, such that $h(\gamma_i) = f(\gamma_i)$ for all $i = 1, \ldots, n$. We give the proof only for $n = 1$; the general case is analogous and left to the reader.

We know that $h(\Sigma) = 0$ produces an infinite system of linear equations with variables $\{h(\gamma_i) = x_i\}_{i \in I}$. We rewrite the system in the following way:

Step 1. Fix an equation, say $E_1$, such that the coefficient of $x_1$ is not zero. We may assume that it is the only one with this property. Suppose that

$$E_1 \equiv x_1 + a_2^1 x_2 + \cdots + a_{r_1}^1 x_{r_1} + \cdots + a_m^1 x_m = 0,$$

where $a_2^1, \ldots, a_m^1$ are non zero and $x_1, \ldots, x_{r_1 - 1}$ do not appear in any other equation of the system.

Step 2. We take now $x_{r_1}$. There is at least one equation, say $E_2$, different from $E_1$, in which the coefficient of $x_{r_1}$ is not zero. We eliminate it from the remaining equations different from $E_1$. Choose variables $x_{r_1+1}, \ldots, x_{r_2-1}$ which only appear in $E_1$ or $E_2$, and the system starts as

$$\begin{aligned} x_1 + a_2^1 x_2 + \cdots + a_{r_1}^1 x_{r_1} + \cdots + a_m^1 x_m &= 0 \\ x_{r_1} + \cdots + a_m^2 x_m + \cdots + a_l^2 x_l &= 0. \end{aligned}$$

Step 3. We do the same with $x_{r_2}$ to obtain

$$\begin{aligned} x_1 + \cdots + a_{r_1}^1 x_{r_1} + \cdots + a_{r_2}^1 x_{r_2} + \cdots + a_m^1 x_m &= 0 \\ x_{r_1} + \cdots + a_{r_2}^2 x_{r_2} + \cdots\cdots + a_l^2 x_l &= 0 \\ x_{r_2} + a_{r_2+1}^3 x_{r_2+1} + \cdots + a_h^3 x_h &= 0. \end{aligned}$$

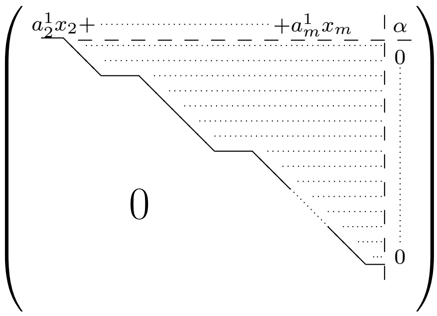
Step 4. We continue in this fashion. When we finish with the variables of $E_1$, we proceed with the variables of $E_2$ and so on. The reader should observe that the variables $x_1, \ldots, x_{r_1}, x_{r_1+1}, \ldots, x_{r_i}$ only appear in the equations $E_1, E_2, \ldots, E_{i+1}$, for all $i \in \mathbb{N}$.

There are two cases to consider:

Case 1. This process stops after a finite number of steps. Then we consider $x_\alpha = 0$, for all variables outside the finite subsystem which we have obtained. Since any equation has at least two variables, the subsystem has more variables than equations and maximal range. This follows that there is a solution for $x_1 = -f(\gamma_1)$.

CASE 2. This process is infinite. Then we stop after finding a variable $x_{r_k}$ where $r_k$ is the minimal integer such that $r_k > n$ and $r_{k+1} - r_k > 1$ (it is possible because $C$ fails IDP). Roughly speaking, this means that we stop this process on the first 'step' (horizontal segments in (2.4)) after processing the variables of $E_1$.

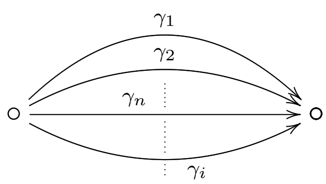
We consider $x_i = 0$, for all $i \neq 1, \ldots, r_k + 1$, and therefore it suffices to prove that the finite system of $k+1$ equations and $r_k$ variables




has a solution, where $\alpha = -f(\gamma_1)$. But this is clearly true, because $r_k \geq k+1$ and the matrix of coefficients has maximal range. $\square$

Let $Q$ be a quiver as in (2.2) and $C$ be an admissible subcoalgebra as in the assumptions of Proposition 2.5.8. Let us suppose that there exists a subset $\Sigma' \subseteq \Sigma$ such that $\Sigma' = \{\gamma_n + \sum_{j>n} a_j^n \gamma_j\}_{n\in\mathbb{N}}$, where $a_j^n \in K$ for all $j, n \in \mathbb{N}$, and $\gamma_i = \alpha_{n_i}^i \alpha_{n_i-1}^i \cdots \alpha_2\alpha_1$, for all $i \in \mathbb{N}$. We may consider the subquiver $Q' = (Q'_0, Q'_1)$, where $Q'_0 = \{t(\alpha_j^i), s(\alpha_j^i)\}_{j=1,\ldots,n_i}^{i\in\mathbb{N}}$ and $Q'_1 = \{\alpha_j^i\}_{j=1,\ldots,n_i}^{i\in\mathbb{N}}$. Then $C$ contains the admissible subcoalgebra of $CQ'$ generated by $\Sigma'$.

Therefore we turn to the case of a quiver $Q$ with the following structure:

$Q \equiv$ 

length$(\gamma_i) > 1$, $i \in \mathbb{N}$ (2.5)

and $C$ an admissible subcoalgebra of $CQ$ generated by an infinite countable connected set $\Sigma = \{\gamma_n + \sum_{j>n} a_j^n \gamma_j\}_{n\in\mathbb{N}}$, where $a_j^n \in K$ for all $j, n \in \mathbb{N}$. We may suppose that $\gamma_i \notin C$ for all $i \in \mathbb{N}$.

Under these conditions, we denote by $\mathbb{H}_Q^n$ the class of admissible subcoalgebras of $CQ$ such that $\dim_K(\langle \mathrm{PSupp}(\Sigma)\rangle / \langle \Sigma \rangle) = n$ and by $\mathbb{H}_Q$ the class of admissible subcoalgebras of $CQ$ such that the dimension of the vector space $\dim_K(\langle \mathrm{PSupp}(\Sigma)\rangle/\langle\Sigma\rangle) = \infty$. Finally, we set

$$\mathbb{H}_Q^\infty = \mathbb{H}_Q \cup \bigcup_{n\in\mathbb{N}} \mathbb{H}_Q^n$$

**Theorem 2.5.10.** *[JMN05] Let $Q$ be any quiver and $C$ be an admissible subcoalgebra of $CQ$. There exists a relation ideal $\Omega$ of $KQ$ such that $C = C(Q,\Omega)$ if and only if there is no subquiver $\Gamma$ of $Q$ such that $C$ contains a subcoalgebra in $\mathbb{H}_\Gamma^\infty$.*

*Proof.* This follows from Proposition 2.5.8 and 2.5.9, and the arguments mentioned above. □

As a consequence of Theorem 2.5.10 we may stated the desired criterion characterizing path coalgebras of quivers with relations.

**Corollary 2.5.11** (Criterion)**.** *Let $C$ be an admissible subcoalgebra of a path coalgebra $CQ$. Then $C$ is not the path coalgebra of a quiver with relations if and only if there exist an infinite number of different paths $\{\gamma_i\}_{i\in\mathbb{N}}$ in $Q$ such that:*

(*a*) *All of them have common source and common sink.*

(*b*) *None of them is in $C$.*

(*c*) *There exist elements $a_j^n \in K$ for all $j,n\in\mathbb{N}$ such that the set $\{\gamma_n + \sum_{j>n} a_j^n\gamma_j\}_{n\in\mathbb{N}}$ is contained in $C$.*

**Remark 2.5.12.** *The reader could ask if an admissible subcoalgebra $C$ of $CQ_C$, which contains a subcoalgebra in $\mathbb{H}_\Gamma^\infty$ can be written as $C(Q',\Omega')$, where $Q'$ is a quiver which is not the Gabriel quiver of $C$. The answer is negative. We know that there exists an injective map $f : C \longrightarrow CQ$ such that $f|_{C_1} = id$. If there is a quiver $Q'$ and an inclusion $C \xrightarrow{i} CQ'$, the following diagram conmutes:*

$$\begin{tikzcd} C_1 \arrow[r, leftrightarrow, "id"] \arrow[dr, hook, "i"'] & CQ_1 \arrow[d, hook, dashed, "i"] \\ & CQ'_1 \end{tikzcd}$$

*We need the following lemma to finish our remark.*

**Lemma 2.5.13.** *Let $f : C \to D$ be a morphism of coalgebras.*

(a) *If $e$ is a group-like element of $C$ then $f(e)$ is a group-like element of $D$.*

(b) *If $f$ is injective and $x$ is a non-trivial $(e, d)$-primitive element of $C$ then $f(x)$ is a non-trivial $(f(e), f(d))$-primitive element of $D$.*

*Thus, since $CQ_1$ and $CQ'_1$ are generated by the set of all vertices and arrows of $Q$ and $Q'$, respectively, using Lemma 2.5.13, we conclude that $Q$ is a subquiver of $Q'$; so it contains some coalgebra in $\mathbb{H}_\Gamma^\infty$.*

As a consequence, we get a negative answer to the following open problem considered by Simson in [Sim01] and [Sim05]: *Is any basic coalgebra, over an algebraically closed field, isomorphic to the path coalgebra of a quiver with relations?*

# Chapter 3

# Localization in coalgebras

Localization is an old theory which has been developed by many authors and from different points of view. Among them, the most renowned procedure is the localization in rings as a systematic method of adding multiplicative inverses to a ring. In the noncommutative case, a very satisfactory results have been achieved by Chatters, Goldie, Goodearl, Small, Warfield and others, see for instance the books [GW89] and [MR87]. In a higher level of abstraction, in [Gab62], Gabriel describe the localization in abelian and Grothendieck categories. This is presented as a functor onto a new category, the quotient category, which has a right adjoint, the section functor. In this chapter we develop the ideas of Gabriel in comodule categories (Grothendieck categories of finite type). The key-point of the theory lies in the fact that the quotient category becomes a comodule category, and then it is better understood than in the case of modules over an arbitrary algebra. Therefore, to any coalgebra $C$, we may attach a set of "localized" coalgebras which can provide information about $C$ or, as we show in the next chapter, its category of comodules $\mathcal{M}^C$. Pointed coalgebras are a very special case. This is due to fact that the "localized" coalgebras can be described by certain manipulation of the Gabriel quiver. For that reason, we devote the last section to their study in depth.

## 3.1 Some categorical remarks about localization

Let $\mathcal{C}$ be an abelian category. A full subcategory $\mathcal{A}$ of $\mathcal{C}$ is said to be **dense** if, for each exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$, it satisfies that $X$ belongs to $\mathcal{A}$ if and only if $X'$ and $X''$ belong to $\mathcal{A}$. For any dense subcategory $\mathcal{A}$ of $\mathcal{C}$, there exists an abelian category $\mathcal{C}/\mathcal{A}$ and an exact functor $T : \mathcal{C} \to \mathcal{C}/\mathcal{A}$ such that $T(X) = 0$, for each $X \in \mathcal{A}$, satisfying the following universal property: for any exact functor $H : \mathcal{C} \to \mathcal{C}'$ such that $H(X) = 0$ for each $X \in \mathcal{A}$, there exists a unique functor $\overline{H} : \mathcal{C}/\mathcal{A} \to \mathcal{C}'$ such that $H = \overline{H}T$. The category $\mathcal{C}/\mathcal{A}$ is called the **quotient category** of $\mathcal{C}$ with respect to $\mathcal{A}$, and the functor $T$ is known as the **quotient functor**, see [Gab62] for an explicit description of $\mathcal{C}/\mathcal{A}$ and $T$.

Following [Gab62], a dense subcategory $\mathcal{A}$ of $\mathcal{C}$ is called **localizing** if the quotient functor $T : \mathcal{C} \to \mathcal{C}/\mathcal{A}$ has a right adjoint functor $S : \mathcal{C}/\mathcal{A} \to \mathcal{C}$. The functor $S$ is called the **section** functor of $T$.

**Lemma 3.1.1.** *[Gab62] Let $\mathcal{A}$ be a localizing subcategory of an abelian category $\mathcal{C}$. With the above notation, the following statements hold:*

(*a*) *$T$ is an exact functor.*

(*b*) *$S$ is a left exact functor.*

(*c*) *$T$ is full and $S$ is a fully faithful functor.*

(*d*) *The equivalence $TS \simeq 1_{\mathcal{C}/\mathcal{A}}$ holds.*

Conversely, if $T : \mathcal{C} \to \mathcal{C}'$ is an exact functor between abelian categories and $S : \mathcal{C}' \to \mathcal{C}$ is a fully faithful functor right adjoint to $T$, then the dense subcategory $\mathrm{Ker}\,(T)$ of $\mathcal{C}$, whose class of objects is $\{X \in \mathcal{C} \mid T(X) = 0\}$, is a localizing subcategory of $\mathcal{C}$ and $\mathcal{C}'$ is equivalent to $\mathcal{C}/\mathrm{Ker}\,(T)$, see [Pop73, 4.4.9].

If $\mathcal{C}$ is a Grothendieck category, localizing subcategories have an easier description.

**Proposition 3.1.2.** *[Gab62] A dense subcategory $\mathcal{A}$ of a Grothendieck category $\mathcal{C}$ is localizing if and only if it is closed under direct sums, or equivalently, each object $X \in \mathcal{C}$ contains a subobject $\mathcal{A}(X)$ which is maximal among the subobjects of $X$ belonging to $\mathcal{A}$.*

We say that a localizing subcategory is **perfect localizing** if the composition functor $Q = ST : \mathcal{C} \to \mathcal{C}$ is exact, or equivalently, by [Gab62, Chapter III, Corollary 3], if the section functor $S$ is exact.

There exists a dual notion of localizing subcategory. Namely, if $\mathcal{C}$ is an abelian category, a dense subcategory $\mathcal{A}$ of $\mathcal{C}$ is said to be **colocalizing** if the functor $T : \mathcal{C} \to \mathcal{C}/\mathcal{A}$ has a left adjoint functor $H : \mathcal{C}/\mathcal{A} \to \mathcal{C}$, see [NT96]. Throughout we call $H$ the **colocalizing functor**.

**Lemma 3.1.3.** *[NT96] Let $\mathcal{A}$ be a colocalizing subcategory of a Grothendieck category $\mathcal{C}$. Then $\mathcal{A}$ is a localizing subcategory of $\mathcal{C}$.*

**Lemma 3.1.4.** *[NT96] Let $\mathcal{A}$ be a colocalizing subcategory of $\mathcal{C}$. Then*

$(a)$ *$X \in \mathcal{A}$ if and only if $\operatorname{Hom}_{\mathcal{C}}(H(Y), X) = 0$ for any $Y \in \mathcal{C}/\mathcal{A}$.*

$(b)$ *The colocalizing functor $H$ is a fully faithful and right exact functor.*

$(c)$ *The equivalence $TH \simeq 1_{\mathcal{C}/\mathcal{A}}$ holds.*

A colocalizing subcategory $\mathcal{A}$ of $\mathcal{C}$ is said to be **perfect colocalizing** if the colocalization functor $H : \mathcal{C}/\mathcal{A} \to \mathcal{C}$ is exact.

## 3.2 Localizing subcategories of a category of comodules

The category $\mathcal{M}^C$ of right comodules over a coalgebra $C$ is a locally finite Grothendieck category in which the theory of localization, as described in the previous section, can be applied. The localizing subcategories of $\mathcal{M}^C$ have been studied in several papers with satisfactory results, see [Gre76], [JMN07], [JMNR06], [Lin75], [NT94] and [NT96]. The reason of such advances comes from that they can be parameterized by different ways, and, consequently, there exist several approaches to the theory. Let us recall briefly some of these relationships that there exist between localizing subcategories and other concepts. In particular, we highlight the one with idempotents elements of the dual algebra, which will be of importance in the next chapter.

$(a)$ *Coidempotent coalgebras.* A subcoalgebra $A$ of $C$ is said to be **coidempotent** if $A \wedge A = A$. In [NT94], a bijective correspondence between localizing subcategories of $\mathcal{M}^C$ and coidempotent subcoalgebras of $C$ is established. Namely, the authors associate to any localizing subcategory $\mathcal{T}$ the subcoalgebra $\mathcal{T}(C) = \sum_{M \in \mathcal{T}} \operatorname{cf}(M)$; and to any coidempotent subcoalgebra $A$ of $C$ the closed subcategory $\mathcal{T}_A$ whose class of objects is $\{M \in \mathcal{M}^C \mid \operatorname{cf}(M) \subseteq A\}$.

$(b)$ *Injective comodules.* From the general theory of localization in Grothendieck categories (cf. [Gab62]), there exists a bijective correspondence between localizing subcategories of a Grothendieck category $\mathcal{C}$ and equivalence classes of injective objects of $\mathcal{C}$. We recall that two injective objets $E_1$ and $E_2$ are said to be **equivalent** if $E_i$ can be embedded in a direct product of copies of $E_j$ for $i,\ j \in \{1, 2\}$. The above correspondence associates to any injective object $E$ the localizing subcategory $\mathcal{T}_E = \{M \in \mathcal{C} \mid \operatorname{Hom}_{\mathcal{C}}(M, E) = 0\}$. If we deal with a comodule category $\mathcal{M}^C$, the inverse is given as follows. For any localizing subcategory $\mathcal{T}$ of

$\mathcal{M}^C$, it maps $\mathcal{T}$ to the injective right $C$-comodule $E = S(D)$, where $D$ is an injective cogenerator of $\mathcal{M}^C/\mathcal{T}$.

$(c)$ *Indecomposable injective comodules.* This is a refinement of the previous case. Since two injective right $C$-comodules are equivalent if and only if in their decompositions, as a direct sum of indecomposable injectives, appear the same indecomposable injective comodules (probably with different multiplicity) then every equivalence class of injective right $C$-comodules is uniquely determined by a set of isomorphism classes of indecomposable injective right $C$-comodules.

$(d)$ *Simple comodules.* To any indecomposable injective right $C$-comodule, we can attach a simple right $C$-comodule defined by its socle. Conversely, given a simple comodule, its injective envelope is an indecomposable injective comodule. Therefore we have a bijection between sets of indecomposable injective comodules and sets of simple comodules.

$(e)$ *Idempotents of the dual algebra.* Given two idempotent elements $f, g \in C^*$, we say that $f$ is **equivalent** to $g$ if the injective right $C$-comodules $Cf$ and $Cg$ are equivalent in the sense defined above. On the other hand, it is easy to see that every injective right $C$-comodule $E$ is of the form $E = Ce$ for some idempotent $e \in C^*$. Therefore there exists a bijective correspondence between equivalence classes of injective comodules and equivalence classes of idempotent elements of the dual algebra. Observe that the set $\{E_i\}_{i \in I_C}$ of indecomposable injective comodules is given by the set $\{e_i\}_{i \in I_C}$ of primitive orthogonal idempotents of $C^*$. That is, $E_i = Ce_i$ for any $i \in I_C$. Then a localizing subcategory of $\mathcal{M}^C$ is determinated by a suitable choice of some primitive orthogonal idempotents in $C^*$.

Given an idempotent element $e \in C^*$, we denote the localizing subcategory associated to $e$ by $\mathcal{T}_e$ and by $I_e$ the subset of $I_C$ such that $\{S_i\}_{i \in I_e}$ is the subset of simple (or, $\{E_i\}_{i \in I_e}$ is the subset if indecomposable injective) comodules associated to this idempotent element. We refer the reader to [CGT02], [JMN07], [JMNR06] and [Woo97] for details.

Following [Gab62], if $\mathcal{T}$ is a localizing subcategory of a Grothendieck category, then $\mathcal{C}/\mathcal{T}$ is also Grothendieck. Furthermore, if $\mathcal{C}$ is of finite type, then so is $\mathcal{C}/\mathcal{T}$. In other words, a quotient category $\mathcal{M}^C/\mathcal{T}$ is a category of comodules $\mathcal{M}^D$ for certain coalgebra $D$. Our aim now is to provide a description of the coalgebra $D$ and of the localizing functors associated to the localizing subcategory. Let us first pay attention to this problem from the point of view of the Morita-Takeuchi contexts.

**Theorem 3.2.1.** *[JMNR06] Let $\mathcal{T}$ be a localizing subcategory of $\mathcal{M}^C$ and $X$ be a injective quasifinite right $C$-comodule such that $\mathcal{T} = \mathcal{T}_X$. Consider the injective Morita-Takeuchi context $(D, C, X, Y, f, g)$ defined by $X$. Then the functors*

$$T = -\square_C Y : \mathcal{M}^C \to \mathcal{M}^D \text{ and } S = -\square_D X : \mathcal{M}^D \to \mathcal{M}^C$$

*define a localization of $\mathcal{M}^C$ with respect to the localizing subcategory $\mathcal{T}$. In particular, $\mathcal{M}^C/\mathcal{T}$ is equivalent to $\mathcal{M}^D$.*

*Proof.* Since $X_C$ is injective and quasifinite, the funtor $S = -\square_D X$ has an exact left adjoint functor $\mathrm{Cohom}_C(X, -)$. This functor preserves direct sums so, for every $N \in \mathcal{M}^C$, there is an isomorphism $\mathrm{Cohom}_C(X, N) \cong N\square_C \mathrm{Cohom}_C(X, C) = N\square_C Y$. Therefore, we obtain a natural isomorphism $\mathrm{Cohom}_C(X, -) \cong -\square_C Y = T$ and thus $S$ is right adjoint of $T$.

Now, we have to show that $\mathrm{Ker}\,(T) = \mathcal{T}_X$. Let us point out that $X = S(D)$. Then, by the adjunction, for every $M \in \mathcal{M}^C$, There is a bijection $\mathrm{Hom}_C(M, X) \longleftrightarrow \mathrm{Hom}_D(T(M), D)$. Thus $M \in \mathcal{T}_X$ if and only if $\mathrm{Hom}_C(M, X) = 0$ if and only if $\mathrm{Hom}_D(T(M), D) = 0$ if and only if $T(M) = 0$. $\square$

**Remark 3.2.2.** *By virtue of the previous theorem and Proposition 1.4.1, we have an isomorphism $D \cong \mathrm{Cohom}_C(X)$, where $X$ is an injective right $C$-comodule. Therefore, we may understand the theory of localization as an evolution of the Morita-Takeuchi equivalences described in Chapter 1. Namely, the coalgebra $C$ is Morita-Takeuchi equivalent to $D = \mathrm{Coend}_C(E)$ for any injective cogenerator $E$, i.e., for any $E = \oplus_{i\in I_C} E_i^{\alpha_i}$, where $\alpha_i > 0$. Therefore, from the point of view of its representation theory, we may look into $C$ by means of $D$ and by taking advances of its possible good properties (for instance, $D$ could be basic). Now, if we drop the condition of $E$ being a cogenerator, i.e., not any indecomposable injective comodule appears as a direct summand of $E$. Then $D$ and $C$ are no longer equivalent. However, there exists a weaker relation: $D$ is a "localized" coalgebra of $C$. Therefore, it is rather natural to inquire oneself which properties of $C$ we may infer from the ones of its "localized" coalgebras. Next chapters are devoted to this aim.*

Let us now show the description of localizing categories by means of idempotents. Let $e$ be an idempotent in $C^*$ and $\mathcal{T}_e$ be the localizing subcategory associated to $e$. From the above considerations, $\mathcal{M}^C/\mathcal{T}_e$ is an abelian category of finite type and therefore $\mathcal{M}^C/\mathcal{T}_e \cong \mathcal{M}^D$ for some coalgebra $D$. We may give an explicit description of $D$. Let us consider the subspace $eCe \subseteq C$. Then it can be endowed with a coalgebra structure given by the formulae:

$$\Delta_{eCe}(exe) = \sum_{(x)} ex_{(1)}e \otimes ex_{(2)}e \quad \text{and} \quad \epsilon_{eCe}(exe) = \epsilon_C(x)$$

for any $x \in C$, where $\Delta_C(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ using the sigma-notation of [Swe69].

**Lemma 3.2.3.** *[Woo97] There exists a equivalence between the categories $\mathcal{M}^C/\mathcal{T}_e$ and $\mathcal{M}^{eCe}$.*

If $M$ is a right $C$-comodule, the vector space $eM$ has a natural structure of right $eCe$-comodule given by the formula:

$$\omega_{eM}(em) = \sum_{(m)} em_{(0)} \otimes em_{(1)}e,$$

for any $m \in M$, where $\omega_M(m) = \sum_{(m)} m_{(0)} \otimes m_{(1)}$ in the sigma-notation. Now, there is a natural right $eCe$-comodule isomorphism $eM \cong M\square_C eC$, defined by $ex \mapsto x_{(0)} \otimes ex_{(1)}$. This means that the functor $-\square_C eC$ is naturally isomorphic to the functor from $\mathcal{M}^C$ to $\mathcal{M}^{eCe}$ defined by $M \mapsto eM$. Observe that if $M = Ce$, we obtain a natural isomorphism $eCe \cong Ce\square_C eC$, see [CGT02] for details.

Since $Ce$ is a quasifinite injective right $C$-comodule, we may consider the injective Morita-Takeuchi context associated to $Ce$, which, by [CGT02], is $(eCe, C, Ce, eC, f, g)$, where $f : eCe \cong Ce\square_C eC$ is the aforementioned isomorphism and $g : C \to eC\square_{eCe} Ce$ is defined by $g(x) = \sum ex_{(1)} \otimes x_{(2)}e$ for any $x \in C$. Hence there exist isomorphisms $eC \cong \mathrm{Cohom}_C(Ce, C)$ and $eCe \cong \mathrm{Coend}_C(Ce)$. Thus we can rewrite Theorem 3.2.1 as follows:

**Theorem 3.2.4.** *The functors*

$$T = -\square_C eC = e(-) : \mathcal{M}^C \to \mathcal{M}^{eCe} \text{ and } S = -\square_{eCe} Ce : \mathcal{M}^{eCe} \to \mathcal{M}^C$$

*define a localization of $\mathcal{M}^C$ with respect to the localizing subcategory $\mathcal{T}_e$.*

**Corollary 3.2.5.** *[CGT02] The functor $T$ is naturally equivalent to the* Cohom *functor* $\mathrm{Cohom}_C(Ce, -)$.

**Corollary 3.2.6.** *A localizing subcategory $\mathcal{T}_e$ is perfect localizing if and only if $Ce$ is an injective left $eCe$-comodule.*

**Remark 3.2.7.** *Note that, as a consequence of Theorem 3.2.4, we obtain an easy description of the localizing subcategory:*

$$\mathcal{T}_e = \mathrm{Ker}\,(T) = \{M \in \mathcal{M}^C \mid M\square_C eC = 0\} = \{M \in \mathcal{M}^C \mid eM = 0\}.$$

**Remark 3.2.8.** *If $e \in C^*$ is an idempotent, for a simple right $C$-comodule $S$, we have exactly two possibilities:*

(*a*) $eS = 0$, *in this case* $e \cdot \mathrm{cf}(S) = 0$ *and* $e_{|\mathrm{cf}(S)} = 0$, *or*

(*b*) $eS = S$, *in this case* $e \cdot \mathrm{cf}(S) = \mathrm{cf}(S)$ *and* $e_{|\mathrm{cf}(S)} = \epsilon_{|\mathrm{cf}(S)}$.

*Thus the class* $\mathcal{T}_e$ *is the localizing subcategory of* $\mathcal{M}^C$ *determined by the subset* $I_e = \{i \in I_C \mid eS_i = S_i\}$. *Furthermore, a complete set of pairwise non-isomorphic simple comodules of the quotient category is* $\{S_i\}_{i \in I_e}$. *It is not difficult to see that the coidempotent subcoalgebra determined by* $\mathcal{T}_e$ *is the biggest subcoalgebra of* $C$ *annihilated by* $e$. *Note that, for any subcoalgebra* $A$ *of* $C$, $eA = 0$ *if and only if* $Ae = 0$.

We now turn to colocalizing subcategories. From Theorem 3.2.4 we may deduce easily the following result.

**Proposition 3.2.9.** *Let* $e \in C^*$ *be an idempotent and* $\mathcal{T}_e$ *be its associated localizing subcategory in* $\mathcal{M}^C$. *Then* $\mathcal{T}_e$ *is a colocalizing subcategory if and only if* $eC$ *is a quasi-finite right* $eCe$*-comodule.*

*Proof.* By Theorem 1.3.2, the functor $T = -\square_C eC : \mathcal{M}^C \to \mathcal{M}^{eCe}$ has a left adjoint functor if and only if $eC$ is quasi-finite as right $eCe$-comodule. □

In the same direction we may characterize, also in terms of idempotents, perfect colocalizing subcategories.

**Proposition 3.2.10.** *Let* $e \in C^*$ *be an idempotent element and let* $\mathcal{T}_e$ *be the associated localizing subcategory in* $\mathcal{M}^C$. *Then* $\mathcal{T}_e$ *is a perfect colocalizing subcategory if and only if* $eC$ *is a quasifinite injective right* $eCe$*-comodule.*

*Proof.* Observe that the left adjoint of $T = -\square_C eC : \mathcal{M}^C \to \mathcal{M}^{eCe}$ is $H = \mathrm{Cohom}_{eCe}(eC, -)$. By Proposition 1.3.3, $H$ is exact if and only if $eC$ is an injective right $eCe$-comodule. □

**Remark 3.2.11.** *The reader can compare the last two propositions with [NT96, Proposition 3.1] and [NT96, Proposition 4.1], where colocalizing and perfect colocalizing subcategories of* $\mathcal{M}^C$ *are characterized in terms of the biggest subcoalgebra of* $C$ *annihilated by* $e$.

**Proposition 3.2.12.** *Let* $C$ *be a coalgebra and* $\mathcal{T}$ *be a perfect colocalizing subcategory of* $\mathcal{M}^C$. *Then* $\mathcal{T}$ *is a perfect localizing subcategory of* $\mathcal{M}^C$.

*Proof.* If $\mathcal{T} = \mathcal{T}_e$ is a perfect colocalizing subcategory of $\mathcal{M}^C$, $eC$ is an quasifinite injective right $eCe$-comodule. Thus, the functor $-\square_C eC$ has an exact left adjoint, namely $\mathrm{Cohom}_{eCe}(eC, -)$. On the other hand, since $Ce$ is a quasifinite injective right $C$-comodule, $-\square_{eCe} Ce$ admits an exact left adjoint, namely $\mathrm{Cohom}_C(Ce, -)$. Then the composed functor $-\square_C (eC \square_{eCe} Ce) =$

$(-\square_{eCe}Ce) \circ (-\square_C eC)$ has an exact left adjoint functor $\mathrm{Cohom}_{eCe}(eC,-) \circ \mathrm{Cohom}_C(Ce,-)$. Therefore $eC\square_{eCe}Ce$ is an quasifinite injective right $C$-comodule and $\mathcal{T}_e$ is a perfect localizing subcategory of $\mathcal{M}^C$. $\square$

A symmetric version of all this section may be done for left comodules. In particular, the localization by means of idempotents is described as follows: for each localizing subcategory $\mathcal{T}'$ of ${}^C\mathcal{M}$, there exists a unique (up to equivalence) idempotent $e$ in $C^*$ such that the localizing functors are equivalent to

$$ {}^C\mathcal{M} \underset{S=-\square_{eCe}eC}{\overset{T=(-)e=-\square_C Ce}{\rightleftarrows}} {}^C\mathcal{M}/\mathcal{T}' \,, $$

where ${}^C\mathcal{M}/\mathcal{T}'$ is equivalent to ${}^{eCe}\mathcal{M}$.

## 3.3 Quivers associated to a coalgebra

Associating a graphical structure to a certain mathematical object is a very common strategy in the literature. Sometimes, it provides us a nice method for replacing the object with a simpler one and improving our intuition about its properties. When dealing with representation theory of coalgebras, the quivers associated to a coalgebra play a prominent rôle in order to study their structure in depth. For instance, we remind from Chapter 1 and Chapter 2 that the Gabriel quiver of a pointed coalgebra becomes a powerful tool. Unfortunately, for an arbitrary coalgebra, these results are no longer valid. This section deals with several quivers that we may attach to any coalgebra. In particular, we study some algebraic properties which we may deduce from their shapes, aiming to generalize the case of pointed coalgebras.

### 3.3.1 Definitions

Let us consider an arbitrary coalgebra $C$. We may associate to $C$ a quiver $\Gamma_C$ known as the **right Ext-quiver** of $C$, see [Mon95]. We recall that the set of vertices of $\Gamma_C$ is the set of pairwise non-isomorphic simple right $C$-comodules $\{S_i\}_{i\in I_C}$ and, for two vertices $S_i$ and $S_j$, there exists a unique arrow $S_j \to S_i$ in $\Gamma_C$ if and only if $\mathrm{Ext}^1_C(S_j,S_i) \neq 0$.

This quiver admits a generalization by means of the notion of right **valued Gabriel quiver** $(Q^C, d^C)$ of $C$, namely, following [KS05], the valued quiver whose set of vertices is $\{S_i\}_{i\in I_C}$ and such that there exists a unique valued arrow

$$ S_i \xrightarrow{(d'_{ij},d''_{ij})} S_j $$

if and only if $\mathrm{Ext}^1_C(S_i, S_j) \neq 0$ and,

$$d'_{ij} = \dim_{G_i}\mathrm{Ext}^1_C(S_i, S_j) \text{ and } d''_{ij} = \dim_{G_j}\mathrm{Ext}^1_C(S_i, S_j),$$

as right $G_i$-module and as left $G_j$-module, respectively, where $G_i$ denotes the division algebra of endomorphisms $\mathrm{End}_C(S_i)$ for any $i \in I_C$. The (non-valued) Gabriel quiver of $C$ is obtained taking the same set of vertices and the number of arrows from a vertex $S_j$ to a vertex $S_i$ is $\dim_{\mathrm{End}_C(S_j)}\mathrm{Ext}^1_C(S_j, S_i)$ as right $\mathrm{End}_C(S_j)$-module. Obviously, if $C$ is pointed (or $K$ is algebraically closed) then it is isomorphic to the one used by Woodcock and $C$ is a subcoalgebra of the path coalgebra of its (non-valued) Gabriel quiver.

In [Sim06], the valued Gabriel quiver of $C$ is described through the notion of **irreducible** morphisms between indecomposable injective right $C$-comodules. Let us denote by $\mathrm{inj}^C$ (respect. ${}^C\mathrm{inj}$) the full subcategory of $\mathcal{M}^C$ (respect. ${}^C\mathcal{M}$) formed by socle-finite (i.e., comodules whose socle is finite-dimensional) injective right (respect. left) $C$-comodules. Let $E$ and $E'$ be two comodules in $\mathrm{inj}^C$. A morphism $f : E \to E'$ is said to be irreducible if $f$ is not an isomorphism and given a factorization

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "g"'] & & E' \\ & Z \arrow[ur, "h"'] & \end{tikzcd}$$

of $f$, where $Z$ is in $\mathrm{inj}^C$, $g$ is a section, or $h$ is a retraction. Set again $G_i = \mathrm{End}_C(S_i)$ for any $i \in I_C$. Analogously to the case of finite-dimensional algebras, there it is proven that the set of irreducible morphism $\mathrm{Irr}_C(E_i, E_j)$ between two indecomposable injective right $C$-comodules $E_i$ and $E_j$ is isomorphic, as $G_j$-$G_i$-bimodule, to the quotient $\mathrm{rad}_C(E_i, E_j)/\mathrm{rad}^2_C(E_i, E_j)$. We recall that, for each two indecomposable injective right $C$-comodules $E_i$ and $E_j$, the **radical** of $\mathrm{Hom}_C(E_i, E_j)$ is the $K$-subspace $\mathrm{rad}_C(E_i, E_j)$ of $\mathrm{Hom}_C(E_i, E_j)$ generated by all non-isomorphisms. Observe that if $i \neq j$, then $\mathrm{rad}_C(E_i, E_j) = \mathrm{Hom}_C(E_i, E_j)$. The square of $\mathrm{rad}_C(E_i, E_j)$ is defined to be the $K$-subspace

$$\mathrm{rad}^2_C(E_i, E_j) \subseteq \mathrm{rad}_C(E_i, E_j) \subseteq \mathrm{Hom}_C(E_i, E_j)$$

generated by all composite homomorphisms of the form

$$E_i \xrightarrow{f} E_k \xrightarrow{g} E_j,$$

where $f \in \mathrm{rad}_C(E_i, E_k)$ and $g \in \mathrm{rad}_C(E_k, E_j)$. The $m$th power $\mathrm{rad}^m_C(E_i, E_j)$ of $\mathrm{rad}_C(E_i, E_j)$ is defined analogously, for each $m > 2$.

**Lemma 3.3.1.** *[Sim06, Theorem 2.3(a)] Let $C$ be a basic coalgebra and set $G_i = \mathrm{End}_C(S_i)$ for each $i \in I_C$. There is an arrow*

$$S_i \xrightarrow{(d'_{ij}, d''_{ij})} S_j$$

*in the right valued Gabriel quiver $(Q_C, d_C)$ of $C$ if and only if $\mathrm{Irr}_C(E_j, E_i) \neq 0$ and*

$$d'_{ij} = \dim_{G_j} \mathrm{Irr}_C(E_j, E_i) \quad \textit{and} \quad d''_{ij} = \dim_{G_i} \mathrm{Irr}_C(E_j, E_i),$$

*as right $G_j$-module and as left $G_i$-module, respectively.*

We may proceed analogously with left $C$-comodules and obtain the left-hand version of this quivers. Nevertheless, as we show in the following proposition, they are opposite to each other.

**Proposition 3.3.2.** *Let $C$ be a basic coalgebra. The right valued Gabriel quiver $(Q_C, d_C)$ of $C$ is the opposite valued quiver of the left valued Gabriel quiver $({}_CQ, {}_Cd)$ of $C$. Consequently, the left (non-valued) Gabriel quiver of $C$ is the opposite of the right (non-valued) Gabriel quiver of $C$.*

*Proof.* Denote by $\{E_i\}_{i\in I_C}$ and $\{F_i\}_{i\in I_C}$ the set of pairwise non-isomorphic indecomposable injective right and left $C$-comodules, respectively. We recall from [CKQ02] that there exists a duality $D : \mathrm{inj}^C \to {}^C\mathrm{inj}$ given by $D = \mathrm{Cohom}_C(-, C)$ whose inverse (which we also denote by $D$) is given by $D = \mathrm{Cohom}_{C^{op}}(-, C^{op})$. Let us denote $D(E_i) = F_i$ (and then $D(F_i) = E_i$) for each indecomposable injective right $C$-comodule $E_i$. Following [CGT02], if $S_i$ is the socle of $E_i$ and $E_i = Ce_i$ for some idempotent $e_i \in C^*$,

$$F_i = D(E_i) = \mathrm{Cohom}_C(Ce_i, C) = e_iC,$$

then $\mathrm{soc}\, F_i = S_i$. Summarizing, $E_i$ and $F_i$ are the right and the left injective envelopes of $S_i$, respectively.

Now, since $\mathrm{Hom}_C(E_i, E_j) \cong \mathrm{Hom}_C(F_j, F_i)$, for each two indecomposable injective right comodules $E_i$ and $E_j$, it is easy to see that also $\mathrm{Irr}_C(E_i, E_j) \cong \mathrm{Irr}_C(F_j, F_i)$ and then $\dim_K \mathrm{Irr}_C(E_i, E_j) = \dim_K \mathrm{Irr}_C(F_j, F_i)$. Thus

$$\begin{aligned} \dim_{G_i} \mathrm{Irr}_C(E_i, E_j) &= \frac{\dim_K \mathrm{Irr}_C(E_i, E_j)}{\dim_K S_i} \\ &= \frac{\dim_K \mathrm{Irr}_C(F_j, F_i)}{\dim_K S_i} \\ &= \dim_{G_i} \mathrm{Irr}_C(F_j, F_i) \end{aligned}$$

Analogously, $\dim_{G_j} \mathrm{Irr}_C(E_i, E_j) = \dim_{G_j} \mathrm{Irr}_C(F_j, F_i)$. Therefore, there exists an arrow

$$S_i \xrightarrow{(d'_{ij}, d''_{ij})} S_j$$

in $(Q_C, d_C)$ if and only if there exists an arrow

$$S_j \xrightarrow{(d''_{ij}, d'_{ij})} S_i$$

in $({}_CQ, {}_Cd)$ and the result follows. $\square$

### 3.3.2 The geometry of the Ext-quiver

Throughout we assume that the aforementioned quivers are connected, that is, by [Sim06], $C$ is indecomposable as coalgebra. Let us take into consideration some geometric properties of them. Since the discussion below is independent of the chosen quiver, throughout this section and the next one we deal with the Ext-quiver $\Gamma_C$. Given a vertex $S_x$, we say that the vertex $S_y$ is an **immediate predecessor** (respectively, a **predecessor**) of $S_x$ if there exists an arrow $S_y \to S_x$ in $\Gamma_C$ (respectively, a path from $S_y$ to $S_x$ in $\Gamma_C$).

**Lemma 3.3.3.** *$S_y$ is an immediate predecessor of $S_x$ if and only if $S_y \subseteq \operatorname{soc}(E_x/S_x)$.*

*Proof.* Let us consider the short exact sequence $S_x \hookrightarrow E_x \to E_x/S_x$. Then we obtain the exact sequence

$$0 \longrightarrow \operatorname{Hom}_C(S_y, S_x) \longrightarrow \operatorname{Hom}_C(S_y, E_x) \longrightarrow$$
$$\longrightarrow \operatorname{Hom}_C(S_y, E_x/S_x) \longrightarrow \operatorname{Ext}_1^C(S_y, S_x) \longrightarrow 0$$

Since $\operatorname{Hom}_C(S_y, S_x) \cong \operatorname{Hom}_C(S_y, E_x)$ then $\operatorname{Hom}_C(S_y, E_x/S_x) \cong \operatorname{Ext}^1_C(S_y, S_x)$ and the result follows. $\square$

We may generalize the former definition by means of the socle filtration. Following [Gre76], any right $C$-comodule $M$ has a filtration

$$0 \subset \operatorname{soc} M \subset \operatorname{soc}^2 M \subset \cdots \subset M$$

called the **Loewy series** , where, for $n > 1$, $\operatorname{soc}^n M$ is the unique subcomodule of $M$ satisfying that $\operatorname{soc}^{n-1} M \subset \operatorname{soc}^n M$ and

$$\frac{\operatorname{soc}^n M}{\operatorname{soc}^{n-1} M} = \operatorname{soc}\left(\frac{M}{\operatorname{soc}^{n-1} M}\right).$$

**Lemma 3.3.4.** *[Gre76] Let $M$ and $N$ be right $C$-comodules.*

(a) *Let $n$ be a positive integer. If $0 \subseteq R_1 \subseteq R_2 \subseteq \cdots \subseteq R_n$ is a chain of subcomodules of $M$ such that $R_i/R_{i-1}$ is semisimple for all $i = 1, \ldots, n$ then $R_n \subseteq \operatorname{soc}^n M$.*

(*b*) *If $N$ is a subcomodule of $M$ then* $\mathrm{soc}^n N = \mathrm{soc}^n M \cap N$ *for all* $n \geq 1$.

(*c*) *If* $f : N \to M$ *is a morphism of right $C$-comodules then* $f(\mathrm{soc}^n N) \subseteq \mathrm{soc}^n M$ *for all* $n \geq 1$.

(*d*) *If* $M = \bigoplus_\lambda M_\lambda$ *then* $\mathrm{soc}^n M = \bigoplus_\lambda \mathrm{soc}^n M_\lambda$ *for all* $n \geq 1$.

We shall need the following result.

**Lemma 3.3.5.** *Let $M$ be a right $C$-comodule. For any $n \geq 1$, the Loewy series of $M/\mathrm{soc}^n M$ is the chain*

$$0 \subset \frac{\mathrm{soc}^{n+1} M}{\mathrm{soc}^n M} \subset \frac{\mathrm{soc}^{n+2} M}{\mathrm{soc}^n M} \subset \cdots \subset \frac{\mathrm{soc}^{n+t} M}{\mathrm{soc}^n M} \subset \cdots$$

*Proof.* For each $t > 0$, denote by $N_t$ the $C$-comodule $\mathrm{soc}^{n+t} M/\mathrm{soc}^n M$ and by $N$ the $C$-comodule $M/\mathrm{soc}^n M$. The case $t = 1$ follows from the definition of the Loewy series. Assume now that the statement holds for $t - 1$. Then $N_t$ is a subcomodule of $N$ such that $\mathrm{soc}^{t-1} N \subset N_t$ and

$$\frac{N_t}{\mathrm{soc}^{t-1} N} \cong \frac{\mathrm{soc}^{n+t} M}{\mathrm{soc}^{n+t-1} M} \cong \mathrm{soc}\left(\frac{M}{\mathrm{soc}^{n+t-1} M}\right) \cong \mathrm{soc}\left(\frac{N}{\mathrm{soc}^{t-1} N}\right).$$

Thus $\mathrm{soc}^t N = N_t$. □

Given a vertex $S_x$, we say that the vertex $S_y$ is an $n$-**predecessor** of $S_x$ if $\mathrm{Ext}^1_C(S_y, \mathrm{soc}^n E_x) \neq 0$, or equivalently, proceeding as in Lemma 3.3.3, if $S_y \subseteq \mathrm{soc}\,(E_x/\mathrm{soc}^n E_x)$.

**Convention 3.3.6.** *Throughout, given a simple comodule $S_x$, for each $n \geq 1$, we shall denote by $\{S_i\}_{i \in I_n}$ the set of simple $C$-comodules such that*

$$\mathrm{soc}\left(\frac{E_x}{\mathrm{soc}^n E_x}\right) = \bigoplus_{i \in I_n} S_i,$$

*and we will refer to it as the set of all $n$-predecessors of $S_x$. Obviously, each $n$-predecessor $S_i$ is repeated $r_i$ times, where*

$$r_i = \frac{\dim_K \mathrm{Hom}_C(S_i, E_x/\mathrm{soc}^n E_x)}{\dim_K \mathrm{End}_C(S_i)} = \frac{\dim_K \mathrm{Ext}^1_C(S_i, \mathrm{soc}^n E_x)}{\dim_K \mathrm{End}_C(S_i)}.$$

**Lemma 3.3.7.** *Let $S_x$ and $S_y$ be two simple $C$-comodules. The following assertions are equivalent:*

(*a*) *$S_y$ is a $n$-predecessor of $S_x$.*

(b) *There is a non-zero morphism* $f : \mathrm{soc}^{n+1} E_x \to E_y$ *such that* $f(\mathrm{soc}^n E_x) = 0$.

(c) *There exists a morphism* $g : E_x \to E_y$ *such that* $g(\mathrm{soc}^n E_x) = 0$ *and* $g(\mathrm{soc}^{n+1} E_x) \neq 0$

*Proof.* (a) $\Leftrightarrow$ (b). Assume that $S_y$ is an $n$-predecessor of $S_x$. Then there exists a non-zero map $h : \mathrm{soc}^{n+1} E_x/\mathrm{soc}^n E_x \to E_y$ such that $h|_{S_y} = i$, where $i : S_y \to E_y$ is the inclusion map. Then, the composition

$$\mathrm{soc}^{n+1} E_x \xrightarrow{p} \mathrm{soc}^{n+1} E_x/\mathrm{soc}^n E_x \xrightarrow{h} E_y$$

is a nonzero morphism which vanishes in $\mathrm{soc}^n E_x$.

Conversely, given such an $f$, it decomposes through a non-zero morphism $g : \bigoplus_{i \in I_n} S_i \longrightarrow E_y$. Therefore there is an $i \in I_n$ such that $g|_{S_i} : S_i \to E_y$ is non-zero. That is, $S_i = S_y$ is an $n$-predecessor of $S_x$.

(b) $\Leftrightarrow$ (c). Assume that $f$ is such a morphism. Since $E_y$ is an injective $C$-comodule, there exists a morphism $g : E_x \to E_y$ such that $g|_{\mathrm{soc}^{n+1} E_x} = f$. Obviously, $g(\mathrm{soc}^n E_x) = 0$ and $g(\mathrm{soc}^{n+1} E_x) \neq 0$.

For the converse, it is enough to consider the restriction of $g$ to the subcomodule $\mathrm{soc}^{n+1} E_x$. $\square$

**Remark 3.3.8.** *Observe that the above result also holds replacing* $E_x$ *for an arbitrary* $C$*-comodule* $M$ *and statement* (a) *by* $S_x \subseteq M/\mathrm{soc}^n M$.

We simply say that $S_y$ is a **predecessor** of $S_x$ if there exists an integer $n \geq 1$ such that $S_y$ is an $n$-predecessor of $S_x$. The following result gives a necessary and sufficient condition for the vertex $S_y$ to be a predecessor of a vertex $S_x$.

**Corollary 3.3.9.** *Let* $S_x$ *and* $S_y$ *be two simple* $C$*-comodules. Then,* $S_y$ *is a predecessor of* $S_x$ *if and only* $\mathrm{rad}_C(E_x, E_y) \neq 0$.

*Proof.* The sufficiency is proved by the former lemma. Conversely, for each $n \geq 1$, we have the short exact sequence

$$0 \longrightarrow \mathrm{soc}^n E_x \longrightarrow \mathrm{soc}^{n+1} E_x \longrightarrow \bigoplus_{i \in I_n} S_i \longrightarrow 0 .$$

If $S_y$ is not a predecessor of $S_x$ then $S_y \ncong S_i$ for all $i \in I_n$. Therefore, for any $n \geq 1$, $\mathrm{Hom}_C(\mathrm{soc}^{n+1} E_x, E_y) \cong \mathrm{Hom}_C(\mathrm{soc}^n E_x, E_y)$. Now,

$$\begin{aligned} \mathrm{Hom}_C(E_x, E_y) &\cong \mathrm{Hom}_C(\varinjlim \mathrm{soc}^n E_x, E_y) \\ &\cong \varinjlim \mathrm{Hom}_C(\mathrm{soc}^n E_x, E_y) \\ &\cong \mathrm{Hom}_C(S_x, E_y) \\ &\cong \begin{cases} 0, & \text{if } S_x \ncong S_y, \\ \mathrm{End}_C(S_x), & \text{if } S_x \cong S_y. \end{cases} \end{aligned}$$

Thus $\mathrm{rad}_C(E_x, E_y) = 0$. $\square$

**Theorem 3.3.10.** *Let $S_x$ and $S_y$ be two simple $C$-comodules. If $S_y$ is an $n$-predecessor of $S_x$ then there exists a path in $\Gamma_C$ of length $n$ from $S_y$ to $S_x$.*

*Proof.* We proceed by induction on $n$. For $n = 1$ is just Lemma 3.3.3. Let us assume that the assertion holds for $n-1$ and let $S_y$ be a $n$-predecessor of $S_x$. By Lemma 3.3.7, there exists a non-zero map $f : \mathrm{soc}^{n+1}E_x \to E_y$ such that $f(\mathrm{soc}^n E_x) = 0$. Then $f$ has a factorization $f = gp$, where $p$ is the projection of $\mathrm{soc}^{n+1}E_x$ onto the quotient $M = \mathrm{soc}^{n+1}E_x/\mathrm{soc}^{n-1}E_x$ and $g$ is a non-zero map such that $g(\mathrm{soc}^n E_x/\mathrm{soc}^{n-1}E_x) = 0$.

By Lemma 3.3.5, we have $M = \mathrm{soc}^2(E_x/\mathrm{soc}^{n-1}E_x)$ and

$$\mathrm{soc}\, M = \mathrm{soc}\,(E_x/\mathrm{soc}^{n-1}E_x) = \bigoplus_{i\in I_{n-1}} S_i.$$

Hence $S_i$ is a $(n-1)$-predecessor of $S_x$. By hypothesis of induction, for any $i \in I_{n-1}$, there is a path $\gamma_i$ of length $n-1$ from $S_i$ to $S_x$. On the other hand, $M$ is contained in $\mathrm{soc}^2(\bigoplus_{i\in I_{n-1}} E_i) = \bigoplus_{i\in I_{n-1}} \mathrm{soc}^2 E_i$. Therefore, since $E_y$ is injective, there exists a non-zero map $h : \bigoplus_{i\in I_{n-1}} \mathrm{soc}^2 E_i \to E_y$ such that $h|_M = g$. Hence there is an index $j \in I_{n-1}$ such that $h|_{\mathrm{soc}^2 E_j} \neq 0$. Lastly, since $g(\mathrm{soc}\, M) = 0$, $h(\bigoplus_{i\in I_{n-1}} S_i) = 0$ and then $h|_{S_j} = 0$. Then, by Lemma 3.3.7, there is an arrow $S_y \to S_j$ in $\Gamma_C$. The composition of this arrow and the path $\gamma_j$ gives us the required $n$-length path from $S_y$ to $S_x$. $\square$

**Remark 3.3.11.** *A result analogous to Theorem 3.3.10 is obtained independently in the recent paper [Sim06] by means of irreducible morphism between indecomposable injective comodules.*

**Remark 3.3.12.** *The reader should observe that if there is a path in $\Gamma_C$ from $S_y$ to $S_x$, then $S_y$ does not have to be a predecessor of $S_x$. For example, let $Q$ be the quiver*

$$y \xrightarrow{\alpha} z \xrightarrow{\beta} x,$$

*and $C$ be the subcoalgebra of $KQ$ generated by $\{x, y, z, \alpha, \beta\}$. Then the quiver $\Gamma_C$ is*

$$S_y \longrightarrow S_z \longrightarrow S_x.$$

*Obviously, there is a path from $S_y$ to $S_x$, but there is no non-zero morphisms*

$$f : E_x =< x, \beta >\longrightarrow E_y =< y > .$$

*On the other hand, if $C$ is the coalgebra $KQ$, the Ext-quiver of $KQ$ is also the previous quiver but, in this case, we may obtain a map*

$$f : E_x =< x, \beta, \beta\alpha >\longrightarrow E_y =< y >$$

*defined by $f(\beta\alpha) = y$ and zero otherwise.*

**Lemma 3.3.13.** *Let $S_x$ and $S_y$ be two simple $C$-comodules such that $S_y$ is an $n$-predecessor of $S_x$ and let $f : E_x \to E_y$ be a map verifying the conditions of Lemma 3.3.7(c). Then $f(\mathrm{soc}^{n+1}E_x) = S_y$ and $f(\mathrm{soc}^{n+t}E_x) \subseteq \mathrm{soc}^t E_y$ for any $t > 1$. Moreover, if $C$ is hereditary, then $f(\mathrm{soc}^{n+t}E_x) = \mathrm{soc}^t E_y$ for any $t \geq 1$ .*

*Proof.* We may factorize $f$ as a composition $gp$, where $p$ is the standard projection of $E_x$ onto $E_x/\mathrm{soc}^n E_x$. Then

$$f(\mathrm{soc}^{n+1}E_x) = gp(\mathrm{soc}^{n+1}E_x) = g\left(\frac{\mathrm{soc}^{n+1}E_x}{\mathrm{soc}^n E_x}\right)$$

is a non-zero semisimple subcomodule of $E_y$, i.e., it is $S_y$. Let us consider the chain

$$0 \subseteq f(\mathrm{soc}^n E_x) \subseteq f(\mathrm{soc}^{n+1}E_x) \subseteq \cdots \subseteq f(\mathrm{soc}^{n+t}E_x) \subseteq \cdots$$

Since each quotient $\frac{f(\mathrm{soc}^{n+i}E_x)}{f(\mathrm{soc}^{n+i-1}E_x)}$ is semisimple for any $i \geq 0$, by Lemma 3.3.4, $f(\mathrm{soc}^{n+t}E_x) \subseteq \mathrm{soc}^t E_y$ for any $t > 1$.

Suppose now that $C$ is hereditary. Then $E_x/\mathrm{soc}^n E_x \cong \bigoplus_{i \in I_n} E_i$. Since $g(\bigoplus_{i\in I_n} S_i) = S_y$, there is an index $j \in I_n$ such that $g_{|S_j} : S_j \to S_y$ is bijective. Thus $g|_{E_j}$ is an isomorphism and

$$\mathrm{soc}^t E_y = g|_{E_j}(\mathrm{soc}^t E_j) \subseteq g(\bigoplus_{i\in I_n} \mathrm{soc}^t E_i) = f(\mathrm{soc}^{n+t}E_x)$$

for any $t > 0$. □

**Corollary 3.3.14.** *Let $C$ be a hereditary coalgebra and $n$ be a natural number. The following conditions are equivalent:*

$(a)$ *There is a path in $\Gamma_C$ of length $n$ from a vertex $S_y$ to a vertex $S_x$.*

$(b)$ *$S_y$ is an $n$-predecessor of $S_x$.*

*Proof.* It is enough to prove $(a) \Rightarrow (b)$. If

$$S_y \longrightarrow S_1 \longrightarrow \cdots \longrightarrow S_{n-1} \longrightarrow S_x$$

is a path in $\Gamma_C$, there exists a sequence of (surjective) morphisms

$$E_x \xrightarrow{f_n} E_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_y$$

such that $f_i(S_i) = 0$ and $f_i(\mathrm{soc}^2 E_i) \neq 0$ for all $i = 1, \ldots, n$, where $S_n = S_x$. Then, applying repeatedly the previous lemma, we obtain that the composition $(f_1 f_2 \cdots f_n)(\mathrm{soc}^n E_x) = 0$ and $(f_1 f_2 \cdots f_n)(\mathrm{soc}^{n+1}E_x) = S_y \neq 0$. By Lemma 3.3.7, $S_y$ is an $n$-predecessor of $S_x$. □

## 3.4 Injective and simple comodules

As usual when dealing with an algebraic structure, the difficulty of studying coalgebras in a general framework has caused the appearance of several kinds of coalgebras which are investigated separately. Many of them are defined through certain properties involving their comodules or, merely, their simple or injective comodules. For instance, this is the case of cosemisimple, pure semisimple, semiperfect, quasi-co-Frobenius, (colocally) hereditary, serial or biserial coalgebras, see [Chi02], [CKQ02], [Chi], [CGT04], [DNS01], [JLMS05], [Lin75], [NTZ96] or [NS02]. Therefore it is natural to ask about the behavior of such comodules in the context of localization. That question will become of importance since, as we show below, some properties of the localization functors strongly depends on this.

### 3.4.1 The section functor

From now on we fix an idempotent element $e \in C^*$. We recall that $\mathcal{T}_e$ designates the localizing subcategory associated to $e$ and that $\{S_x\}_{x\in I_e \subset I_C}$ is the subset of simple comodules of the quotient category. Let us consider the quotient and the section functor associated to $\mathcal{T}_e$:

$$\mathcal{M}^C \underset{S=-\square_{eCe}Ce}{\overset{T=e(-)=-\square_C eC}{\rightleftarrows}} \mathcal{M}^{eCe}\ .$$

We also remind that there exists a torsion theory on $\mathcal{M}^C$ associated to the functor $T$, where a right $C$-comodule $M$ is a torsion comodule if $T(M)=0$.

We know that for a simple right $C$-comodule $S_x$, $T(S_x)=S_x$ if $x\in I_e$ and zero otherwise. Therefore there are two kinds of simple comodules: the torsion and the torsion-free simple comodules, where the second class correspond to the simple comodules of the coalgebra $eCe$. From this fact we obtain the following result. From now on we will make no distinction between torsion-free simple $C$-comodules and simple $eCe$-comodules.

**Lemma 3.4.1.** *Let $M$ be a right $C$-comodule, then $T(\operatorname{soc} M) \subseteq \operatorname{soc} T(M)$.*

*Proof.* Let us suppose that $\operatorname{soc} M = (\bigoplus_{i\in I} S_i)\bigoplus(\bigoplus_{j\in J} T_j)$, where $S_i$ and $T_j$ are simple right $C$-comodules such that $T(S_i)=S_i$ and $T(T_j)=0$ for all $i\in I$ and $j\in J$. Since $\operatorname{soc} M\subseteq M$ then we have that $T(\operatorname{soc} M)=\bigoplus_{i\in I} S_i \subseteq T(M)$. $\square$

Let us study the behavior of the injective comodules under the action of the section functor. Indeed, we shall prove that $S$ preserves indecomposable injective comodules and, consequently, injective envelopes. In what follows we denote by $\{\overline{E}_x\}_{x\in I_e}$ a complete set of pairwise non-isomorphic indecomposable injective right $eCe$-comodules, and assume that $\overline{E}_x$ is the injective envelope of the simple right $eCe$-comodule $S_x$ for each $x \in I_e$.

**Proposition 3.4.2.** *The following properties hold:*

($a$) *The functor $S$ preserves injective comodules.*

($b$) *If $N$ is a quasi-finite indecomposable right $eCe$-comodule, then $S(N)$ is indecomposable.*

($c$) *The functor $S$ preserves indecomposable injective comodules. As a consequence, $S$ preserves injective envelopes.*

($d$) *If $S_x$ is a simple $eCe$-comodule then* $\operatorname{soc} S(S_x) = S_x$.

($e$) *If $S_x$ is a simple $eCe$-comodule then $S(S_x)$ is torsion-free.*

($f$) *We have that $S(\overline{E}_x) = E_x$ for all $x \in I_e$.*

($g$) *The functor $S$ preserves quasi-finite comodules.*

($h$) *The functor $S : \mathcal{M}^{eCe} \to \mathcal{M}^C$ restricts to a fully faithful functor $S : \mathcal{M}^{eCe}_{qf} \to \mathcal{M}^C_{qf}$ between the categories of quasi-finite comodules which preserves indecomposables comodules and reflects the isomorphism classes.*

*Proof.* ($a$) The functor $T$ is exact and left adjoint of $S$ so, by [Ste75, Proposition 9.5], the result follows.

($b$) Since $N$ is quasi-finite and indecomposable then the endomorphism ring $\operatorname{End}_{eCe}(N) \cong \operatorname{End}_C(S(N))$ is a local ring, see [CKQ02, Corollary 2.2(a)]. Thus $S(N)$ is indecomposable.

($c$) It follows from ($a$) and ($b$).

($d$) Suppose that $\operatorname{soc} S(S_x) = (\bigoplus_{i\in I} S_i) \bigoplus (\bigoplus_{j\in J} T_j)$, where $S_i$ and $T_j$ are simple right $C$-comodules such that $T(S_i) = S_i$ and $T(T_j) = 0$ for all $i \in I$ and $j \in J$. By Lemma 3.4.1,

$$\bigoplus_{i\in I} S_i = T(\operatorname{soc} S(S_x)) \subseteq \operatorname{soc} TS(S_x) = \operatorname{soc} S_x = S_x.$$

Since $S$ is left exact and preserves indecomposable injective comodules, $S_x \subseteq \operatorname{soc} S(S_x) \subseteq \operatorname{soc} S(\overline{E}_x) = S_y$ for some simple comodule $S_y$. Then $S_y = S_x = \operatorname{soc} S(S_x)$.

$(e)$ If $M \subseteq S(S_x)$ is a non-zero torsion subcomodule of $S(S_x)$ then there exists a simple $C$-comodule $R$ contained in $M$ such that $T(R) = 0$. But $\operatorname{soc} S(S_x) = S_x$, so $S_x = R$ and we get a contradiction.

$(f)$ It is easy to see from $(c)$ and $(d)$.

$(g)$ Let $M$ be a quasi-finite right $eCe$-comodule. The injective envelope of $M$ is of the form $E = \bigoplus \overline{E}_x^{n_x}$, where the $n_x$'s are finite cardinal numbers. Since $S$ is left exact then $S(M)$ embeds in $S(E) = \bigoplus E_x^{n_x}$. Thus $S(M)$ is quasi-finite.

$(h)$ It follows from the above assertions and $TS \simeq 1_{\mathcal{M}^{eCe}}$.

$\square$

After proving Proposition 3.4.2, one should ask if the behavior of simple comodules is analogous to injective ones, that is, if $S$ preserves simple comodules and, consequently, in view of Proposition 3.4.2$(c)$, $S(S_x) = S_x$ for all $x \in I_e$. Unfortunately, in general, this is not true and we can only say that $S(S_x)$ is a subcomodule of $E_x$ which contains $S_x$.

**Example 3.4.3.** *This example shows that $S(S_x) \neq S_x$ for every $x \in I_e$. Let $Q$ be the quiver*

$$(y) \xrightarrow{\alpha} (x),$$

*$C = KQ$ and $e \in C^*$ the idempotent associated to the set $\{x\}$. Then, the localized coalgebra $eCe$ is $S_x$ and*

$$S(S_x) = S_x \square_{eCe} Ce = eCe \square_{eCe} Ce \cong Ce \cong < x, \alpha > \neq S_x.$$

The reader should observe that $S(S_x)$ could be an infinite dimensional right $C$-comodule. Therefore, in general, $S$ cannot be restricted to a functor between the categories of finite dimensional comodules.

**Example 3.4.4.** *Let $Q$ be the quiver*

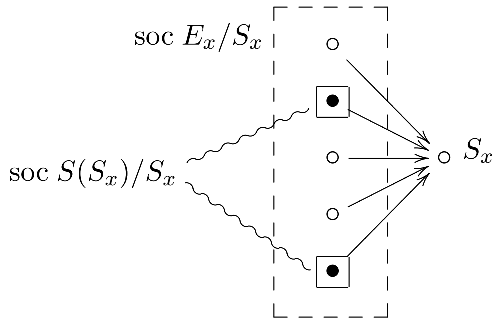
$$\cdots\cdots \xrightarrow{\alpha_n} (n) \xrightarrow{\alpha_{n-1}} \cdots\cdots \xrightarrow{\alpha_3} (3) \xrightarrow{\alpha_2} (2) \xrightarrow{\alpha_1} (1),$$

*$C = KQ$ and $e \in C^*$ the idempotent associated to the set $\{1\}$. Then the localized coalgebra $eCe$ is $S_1$ and*

$$S(S_1) = S_1 \square_{eCe} Ce = eCe \square_{eCe} Ce \cong Ce \cong < 1, \{\alpha_1 \cdots \alpha_{n-1}\alpha_n\}_{n \geq 1} > .$$

**Remark 3.4.5.** *The reader may find in the next chapter (Lemma 4.2.8) a proof of the following fact: $S$ preserves finite dimensional comodules if and only $S(S_x)$ is finite dimensional for each $x \in I_e$.*

In order to characterize the simple comodules invariant under the functor $S$ we need the following result. Observe that, when $n = 1$, it asserts that the torsion immediate predecessors of a torsion-free vertex $S_x$ in $\Gamma_C$ are the simple $C$-comodules contained in the socle of $S(S_x)/S_x$. In the following picture the torsion-free vertices are represented by white points.




**Theorem 3.4.6.** *Let $S_x$ and $S_y$ be two simple $C$-comodules such that $S_x$ is torsion-free. Then $S_y \subseteq S(S_x)/\mathrm{soc}^n S(S_x)$ if and only if $S_y$ is torsion and $S_y \subseteq E_x/\mathrm{soc}^n S(S_x)$.*

*Proof.* Consider the short exact sequence and apply $T(-)$

$$0 \longrightarrow \mathrm{soc}^n S(S_x) \longrightarrow S(S_x) \longrightarrow S(S_x)/\mathrm{soc}^n S(S_x) \longrightarrow 0$$

Since $S_x \cong TS(S_x) \cong T(\mathrm{soc}^n S(S_x))$, $S(S_x)/\mathrm{soc}^n S(S_x)$ is a torsion subcomodule of $E_x/\mathrm{soc}^n S(S_x)$. Therefore if $S_y \subseteq S(S_x)/\mathrm{soc}^n S(S_x)$ then $S_y \subseteq E_x/\mathrm{soc}^n S(S_x)$ and $T(S_y) = 0$.

Conversely, applying the functor $S$ to the exact sequence

$$0 \longrightarrow S_x \xrightarrow{\ i\ } \overline{E}_x \xrightarrow{\ p\ } \overline{E}_x/S_x \longrightarrow 0$$

we obtain the following commutative diagram:

$$\begin{tikzcd}
S(S_x) \arrow[r, "S(i)"] & E_x \arrow[rr, "S(p)"] \arrow[dr, "\mathrm{Coker}\, S(i)"'] & & S(\overline{E}_x/S_x) \arrow[r] & \mathrm{Coker}\, S(p) \\
& & E_x/S(S_x) \arrow[ur, hook] & &
\end{tikzcd}$$

Therefore we have $\mathrm{Hom}_C(S_y, E_x/S(S_x))$ is contained in the set of morphisms $\mathrm{Hom}_C(S_y, S(\overline{E}_x/S_x)) \cong \mathrm{Hom}_{eCe}(T(S_y), \overline{E}_x/S_x) = 0$. Now, applying $\mathrm{Hom}_C(S_y, -)$ to the exact sequence

$$0 \longrightarrow S(S_x) \longrightarrow E_x \longrightarrow E_x/S(S_x) \longrightarrow 0,$$

we obtain the exactness of the sequence

$$0 = \mathrm{Hom}_C(S_y, E_x) \to \mathrm{Hom}_C(S_y, E_x/S(S_x)) \to \mathrm{Ext}^1_C(S_y, S(S_x)) \to 0$$

and then $0 = \mathrm{Hom}_C(S_y, E_x/S(S_x)) \cong \mathrm{Ext}^1_C(S_y, S(S_x))$.

Applying the functor $\mathrm{Hom}_C(S_y, -)$ to the short exact sequences

$$0 \longrightarrow \mathrm{soc}^n S(S_x) \longrightarrow E_x \longrightarrow E_x/\mathrm{soc}^n S(S_x) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{soc}^n S(S_x) \longrightarrow S(S_x) \longrightarrow S(S_x)/\mathrm{soc}^n S(S_x) \longrightarrow 0$$

we obtain that $\mathrm{Hom}_C(S_y, \frac{S(S_x)}{\mathrm{soc}^n S(S_x)}) \cong \mathrm{Hom}_C(S_y, \frac{E_x}{\mathrm{soc}^n S(S_x)})$. Then the result follows. $\square$

**Corollary 3.4.7.** *Let $S_x$ be a simple $eCe$-comodule. The following assertions are equivalent:*

*(a) $E_x/S_x$ is torsion-free.*

*(b) There is no arrow in $\Gamma_C$ from a torsion vertex $S_y$ to $S_x$.*

*(c) $S(S_x) = S_x$.*

*Proof.* It is straightforward from Lemma 3.3.3 and Theorem 3.4.6 for the case $n = 1$. $\square$

**Theorem 3.4.8.** *Let $S_x$ and $S_y$ be two simple $C$-comodules such that $S_x$ is torsion-free. If $S_y \subseteq S(S_x)/\mathrm{soc}^n S(S_x)$ for some $n \geq 1$, then the following assertions hold:*

*(a) $S_y$ is a $n$-predecessor of $S_x$.*

*(b) There exists a path in $\Gamma_C$*

$$S_y = S_n \longrightarrow S_{n-1} \longrightarrow \cdots \longrightarrow S_2 \longrightarrow S_1 \longrightarrow S_x$$

*such that $S_i$ is torsion for all $i = 1, \ldots, n$.*

*Proof.* $(a)$ is obtained from the inclusion $\frac{S(S_x)}{\mathrm{soc}^n S(S_x)} \subseteq \frac{E_x}{\mathrm{soc}^n E_x}$.
$(b)$ $S_y$ is a torsion comodule by Theorem 3.4.6. In order to prove the existence of the path, we proceed by induction on $n$. The case $n = 1$ corresponds to Theorem 3.4.6. Assume now that the statement holds for $n - 1$ and that $S_y \subseteq S(S_x)/\mathrm{soc}^n S(S_x)$. Analogously to the proof of Theorem 3.3.10, we may prove that there exists an arrow from $S_y$ to a simple $C$-comodule $S_j \subseteq S(S_x)/\mathrm{soc}^{n-1} S(S_x)$. $\square$

**Theorem 3.4.9.** *Let $C$ be a hereditary coalgebra and $S_x$ and $S_y$ be two simple $C$-comodules such that $S_x$ is torsion-free. Suppose that there exists a path in $\Gamma_C$*

$$S_y = S_n \longrightarrow S_{n-1} \longrightarrow \cdots \longrightarrow S_2 \longrightarrow S_1 \longrightarrow S_x$$

*such that $S_i$ is torsion for all $i = 1, \ldots, n$. Then $S_y \subseteq S(S_x)/\mathrm{soc}^n S(S_x)$.*

*Proof.* We prove by induction that, for each $n$, the torsion simple comodules contained in $E_x/\mathrm{soc}^n S(S_x)$ are those for which there is a path as described in the hypotheses. By Corollary 3.3.14 and Theorem 3.4.6, this will imply the statement. The case $n = 1$ follows from Theorem 3.4.6. Let us assume that it holds for $n-1$. Then

$$\frac{E_x}{\mathrm{soc}^n S(S_x)} \cong \frac{\frac{E_x}{\mathrm{soc}^{n-1}S(S_x)}}{\frac{\mathrm{soc}^n S(S_x)}{\mathrm{soc}^{n-1}S(S_x)}} \cong \frac{(\oplus_{j\in J}E_j)\oplus E}{\oplus_{j\in J}S_j} \cong (\oplus_{j\in J}E_j/S_j)\oplus E,$$

where $S_j$ is torsion for all $j \in J$ and $E$ is an injective comodule whose socle is torsion-free. If there is a path from $S_y$ to $S_x$ as described above, $1 \in J$ by hypothesis and hence $S_y \subseteq E_1/S_1 \subseteq E_x/\mathrm{soc}^n S(S_x)$. Now, if $S_y \subseteq E_x/\mathrm{soc}^n S(S_x)$, there is some $j_o \in J$ such that $S_y \subseteq E_{j_0}/S_{j_0}$ and then we have an arrow $S_y \to S_{j_0}$. By hypothesis, there is a path of length $n-1$ from $S_{j_0}$ to $S_x$ whose intermediate vertices are torsion. This completes the proof. □

**Corollary 3.4.10.** *Let $Q$ be a quiver, $KQ$ the path coalgebra of $Q$ and $e \in (KQ)^*$ an idempotent element associated to the subset $X \subseteq Q_0$. For each vertex $x \in X$, the $KQ$-comodule $S(S_x)$ is generated by the set of paths*

$$\textcircled{1} \longrightarrow \textcircled{2} \longrightarrow \textcircled{3} \longrightarrow \cdots\cdots \longrightarrow \textcircled{n} \longrightarrow \textcircled{x}$$

*such that $i \notin X$ for any $i = 1, \ldots, n$*

## 3.4.2 The quotient functor

Let us now analyze the properties of the quotient functor. We start with an example which shows that, in general, $T$ does not preserve injective comodules.

**Example 3.4.11.** *Let $Q$ be the quiver*

$$\textcircled{x} \xrightarrow{\alpha} \textcircled{y} \xrightarrow{\beta} \textcircled{z},$$

*$C$ be the subcoalgebra of $KQ$ generated by $\{x, y, z, \alpha, \beta\}$ and $I_e = \{x, y\}$. The injective right $C$-comodule $E_z$ is generated by $< z, \beta >$ and $T(E_z) =< \beta >\cong S_y \neq E_y$.*

**Proposition 3.4.12.** *The following statements hold:*

*(a)* $T(E_x) = \overline{E}_x$ *for any $x \in I_e$.*

(*b*) *If $E$ is an injective torsion-free right $C$-comodule then $T(E)$ is an injective right $eCe$-comodule.*

(*c*) *If $M$ is a torsion-free right $C$-comodule then* $\operatorname{soc} M = \operatorname{soc} T(M) = T(\operatorname{soc} M)$.

(*d*) *The functor $T : \mathcal{M}^C \to \mathcal{M}^{eCe}$ restricts to a functor $T : \mathcal{M}^C_{qf} \to \mathcal{M}^{eCe}_{qf}$ and a functor $T : \mathcal{M}^C_f \to \mathcal{M}^{eCe}_f$ between the categories of quasi-finite and finite dimensional comodules, respectively.*

*Proof.* (*a*) By Proposition 3.4.2, $E_x = S(\overline{E}_x)$ for any $x \in I_e$. Then $T(E_x) = TS(\overline{E}_x) = \overline{E}_x$.

(*b*) It follows from (*a*).

(*c*) Consider the chain $\bigoplus_{x\in I} S_x = \operatorname{soc} M \subseteq M \subseteq E(M) = \bigoplus_{x\in I} E_x$. Since $M$ is torsion-free then $I \subseteq I_e$. Therefore $\operatorname{soc} M = \bigoplus_{x\in I} S_x = \bigoplus_{x\in I} T(S_x) = T(\operatorname{soc} M) \subseteq T(M) \subseteq T(E(M)) = \bigoplus_{x\in I} T(E_x) = \bigoplus_{x\in I} \overline{E}_x$ and the result follows.

(*d*) It is easy to see.

□

**Corollary 3.4.13.** *Let $E_x$ be an indecomposable injective $C$-comodule such that $S_x$ is torsion-free. Then $S(S_x) = E_x$ if and only if all predecessors of $S_x$ in $\Gamma_C$ are torsion.*

*Proof.* Assume that all predecessors of $S_x$ are torsion. Then,

$$T(\operatorname{soc}^n E_x) = T(\operatorname{soc}^{n+1} E_x) = T(\operatorname{soc} E_x) = S_x.$$

Now, by the previous proposition,

$$\overline{E}_x = T(E_x) = T(\varinjlim \operatorname{soc}^n E_x) = \varinjlim T(\operatorname{soc}^n E_x) = S_x$$

and thus, by Proposition 3.4.2(*f*), $E_x = S(\overline{E_x}) = S(S_x)$. The converse follows from Theorem 3.4.8. □

**Example 3.4.14.** *In general, the functor $T$ is not full. Let $Q$ be the quiver*

$$y \xrightarrow{\alpha} x,$$

*$C = KQ$ and $e \in C^*$ be the idempotent associated to the set $\{x\}$. Then* $\dim_K \operatorname{Hom}_C(S_x, C) = \dim_K \operatorname{End}(S_x) = 1$ *and* $\dim_K \operatorname{Hom}_{eCe}(S_x, eC) = 2$. *Therefore the map $T_{S_x,C} : \operatorname{Hom}_C(S_x, C) \to \operatorname{Hom}_{eCe}(S_x, eC)$ cannot be surjective.*

**Example 3.4.15.** *In general, the functor $T$ does not preserve indecomposable comodules. Let $KQ$ be the path coalgebra of the quiver*

$$\begin{tikzcd} x \arrow[dr, "\alpha"] & \\ & z \\ y \arrow[ur, "\beta"'] & \end{tikzcd}$$

*and $e \in C^*$ be the idempotent associated to the set $\{x, y\}$. Then $T$ maps the indecomposable injective right $C$-comodule $E_z =< z, \alpha, \beta >$ to the right $eCe$-comodule $S_x \oplus S_y$. Nevertheless, it is easy to see that $T$ preserves indecomposable torsion-free comodules.*

Since $T(S_y) = 0$ for each torsion simple $C$-comodule, one could expect the analogous property for their injective envelopes. Unfortunately, this is only true under some special conditions related to the stability of the torsion theory.

**Example 3.4.16.** *Let $KQ$ be the path coalgebra of the quiver*

$$x \xrightarrow{\alpha} y,$$

*$C = KQ$ and $e \in C^*$ be the idempotent associated to the set $\{x\}$. Then $T(E_y) = T(< y, \alpha >) \cong S_x \neq 0$.*

**Theorem 3.4.17.** *Let $E_y$ be an indecomposable injective right $C$-comodule with $y \notin I_e$. The following statements are equivalent:*

$(a)$ $T(E_y) = 0$,

$(b)$ $\mathrm{Hom}_C(E_y, E_x) = 0$ *for all* $x \in I_e$,

$(c)$ *$S_y$ has no torsion-free predecessor in $\Gamma_C$.*

*Proof.* $(a) \Rightarrow (b)$. Since $S$ is left adjoint to $T$ then we have that $\mathrm{Hom}_C(E_y, E_x) = \mathrm{Hom}_C(E_y, S(\overline{E}_x)) \cong \mathrm{Hom}_{eCe}(T(E_y), \overline{E}_x) = 0$ for all $x \in I_e$.

$(b) \Rightarrow (c)$. It is proved in Proposition 3.3.9.

$(c) \Rightarrow (a)$. For each $n \geq 1$, we have the short exact sequence

$$0 \longrightarrow \mathrm{soc}^n E_y \longrightarrow \mathrm{soc}^{n+1} E_y \longrightarrow \bigoplus_{i \in I_n} S_i \longrightarrow 0\,.$$

Since $S_y$ has no torsion-free predecessor, $T(S_i) = 0$ for all $i \in I_n$ and then $T(\mathrm{soc}^n E_y) = T(\mathrm{soc}^{n+1} E_y)$. Now,

$$T(E_y) = T(\varinjlim \mathrm{soc}^n E_y) = \varinjlim T(\mathrm{soc}^n E_y) = T(\mathrm{soc}\, E_y) = 0.$$

$\square$

Let us finish the subsection by giving an approach to the image of an indecomposable injective comodule $E_y$ with torsion socle. Firstly, from the Loewy series of $E_y$, we may obtain a chain

$$0 \subseteq T(\mathrm{soc}E_y) \subseteq T(\mathrm{soc}^2 E_y) \subseteq \cdots \subseteq T(\mathrm{soc}^n E_y) \subseteq \cdots \subseteq T(E_y)$$

such that each quotient $\frac{T(\mathrm{soc}^{n+1}E_y)}{T(\mathrm{soc}^n E_y)} \cong T\left(\frac{\mathrm{soc}^{n+1}E_y}{\mathrm{soc}^n E_y}\right)$ is the direct sum of the torsion-free $n$-predecessors of $S_y$. As a consequence, by Lemma 3.3.4, we have that $T(\mathrm{soc}^{n+1}E_y) \subseteq \mathrm{soc}^n T(E_y)$. In particular, $T(\mathrm{soc}^2 S_y)$ is the direct sum of all torsion-free immediate predecessors of $S_y$ and $T(\mathrm{soc}^2 E_y) \subseteq \mathrm{soc}\, T(E_y)$.

**Lemma 3.4.18.** *Let $S_y$ be a torsion simple right $C$-comodule. Suppose that $\{S_x, T_z\}_{x\in I, z\in J}$ is the set of all immediate predecessors of $S_y$ in $\Gamma_C$, where $S_x$ is torsion-free for all $x \in I$ and $T_z$ is torsion for all $z \in J$. Then*

$$\mathrm{soc}\, T(E_y) \subseteq \left(\bigoplus_{x\in I} S_x\right) \bigoplus \left(\bigoplus_{z\in J} \mathrm{soc}\, T(E_z)\right).$$

*If $C$ is hereditary, the opposite inclusion also holds.*

*Proof.* By Lemma 3.3.3, $\mathrm{soc}\,(E_y/S_y) = (\oplus_{x\in I} S_x) \oplus (\oplus_{z\in J} T_z)$ and, consequently, $E_y/S_y \subseteq (\oplus_{x\in I} E_x) \oplus (\oplus_{z\in J} E_z)$. Then

$$T(E_y) \cong T(E_y/S_y) \subseteq \left(\bigoplus_{x\in I} \overline{E}_x\right) \bigoplus \left(\bigoplus_{z\in J} T(E_z)\right)$$

and then

$$\mathrm{soc}\, T(E_y) \subseteq \left(\bigoplus_{x\in I} S_x\right) \bigoplus \left(\bigoplus_{z\in J} \mathrm{soc}\, T(E_z)\right).$$

Clearly, the inclusions are equalities if $C$ is hereditary. □

In a general context it is not possible to prove the equality in Lemma 3.4.18. For example, consider the quiver of Example 3.4.11, the coalgebra generated by the set $< x, y, z, \alpha, \beta >$ and $I_e = \{x\}$. Then $\mathrm{soc}\, T(E_z) = 0$ and $\mathrm{soc}\, T(E_y) = S_x$.

**Corollary 3.4.19.** *Let $E_y$ be a indecomposable injective $C$-comodule such that $S_y$ is torsion. If $S_x \subseteq \mathrm{soc}\, T(E_y)$ then*

*(a) $S_x$ is torsion-free.*

*(b) $S_x$ is a predecessor of $S_y$ in $\Gamma_C$.*

(*c*) *There exists a path in* $\Gamma_C$

$$S_x \longrightarrow S_n \longrightarrow \cdots \longrightarrow S_2 \longrightarrow S_1 \longrightarrow S_y$$

*such that* $S_i$ *is torsion for all* $i = 1, \dots, n$.

*Proof.* By Lemma 3.4.18, it is enough to prove ($b$). Now, by hypothesis, $0 \neq \mathrm{Hom}_{eCe}(T(E_y), \overline{E}_x) \cong \mathrm{Hom}_C(E_y, E_x) = \mathrm{Rad}_C(E_y, E_x)$. Corollary 3.3.9 completes the proof. □

**Remark 3.4.20.** *In case* $C$ *is hereditary we may prove a partial converse. Namely, if the conditions* ($a$) *and* ($c$) *hold, then* $S_x \subseteq \mathrm{soc}\, T(E_y)$.

### 3.4.3 The colocalizing functor

Throughout this subsection we assume that $\mathcal{T}_e$ is a colocalizing subcategory of $\mathcal{M}^C$. Therefore the localizing functor $H : \mathcal{M}^{eCe} \to \mathcal{M}^C$ exists and is left adjoint to $T$. We recall from [Tak77] that there exists such functor $H$ if and only if $eC = T(C)$ is quasi-finite as right $eCe$-comodule, i.e., since $T(C) = eCe \bigoplus (\bigoplus_{y \in I_C \setminus I_e} T(E_y))$, if and only if $\bigoplus_{y \in I_C \setminus I_e} T(E_y)$ is quasi-finite as $eCe$-comodule. According to Corollary 3.4.19, this is obtained if $\dim_K \mathrm{Ext}^1_C(S, S')$ is finite for each pair of simple comodules $S$ and $S'$, and there are finitely many paths in $\Gamma_C$

$$S_x \longrightarrow S_n \longrightarrow \cdots \longrightarrow S_3 \longrightarrow S_2 \longrightarrow S_1$$

where $S_i$ is torsion for all $i = 1, \dots, n$, for each torsion-free simple comodule $S_x$.

**Proposition 3.4.21.** *The following assertions hold:*

(*a*) $H$ *preserves projective comodules.*

(*b*) $H$ *preserves finite dimensional comodules.*

(*c*) $H$ *preserves finite dimensional indecomposable comodules.*

(*d*) *The functor* $H : \mathcal{M}^{eCe} \to \mathcal{M}^C$ *restricts to a fully faithful functor* $H : \mathcal{M}^{eCe}_f \to \mathcal{M}^C_f$ *between the categories of finite-dimensional comodules which preserves indecomposable comodules and reflects the isomorphism classes.*

*Proof.* ($a$) It is dual to the proof of Proposition 3.4.2($a$).

(*b*) Let $N$ be a finite dimensional right $eCe$-comodule. Then

$$H(N) = \mathrm{Cohom}_{eCe}(eC, N) = \mathrm{Hom}_{eCe}(N, eC)^*.$$

Now, since $eC$ is a quasi-finite right $eCe$-comodule, $\mathrm{Hom}_{eCe}(N, eC)$ has finite dimension.

(*c*) Let $N$ be a finite dimensional indecomposable right $eCe$-comodule. Since $H$ is fully faithful then $\mathrm{End}_{eCe}(N) \cong \mathrm{End}_C(H(N))$ is a local ring. Now, by (*b*), $H(N)$ is finite dimensional and then $H(N)$ is indecomposable.

(*d*) It is straightforward from (*b*), (*c*) and the equivalence $TH \simeq 1_{\mathcal{M}^{eCe}}$.

□

Analogously to the study of the section functor, let us characterize the simple comodules which are invariant under the functor $H$. For that purpose we need the following proposition:

**Proposition 3.4.22.** *Let $S_x$ be a simple $eCe$-comodule. Then $H(S_x) = S_x$ if and only if $\mathrm{Hom}_{eCe}(S_x, T(E_y)) = 0$ for all $y \notin I_e$.*

*Proof.* From the decomposition $eC = eCe \oplus eC(1-e)$ as right $eCe$-comodules, we have the following equalities:

$$\begin{aligned} \dim_K H(S_x) = & \ \dim_K \mathrm{Cohom}_{eCe}(eC, S_x) \\ = & \ \dim_K S_x + \dim_K \mathrm{Cohom}_{eCe}(eC(1-e), S_x) \\ = & \ \dim_K S_x + \textstyle\sum_{y \notin I_e} \dim_K \mathrm{Cohom}_{eCe}(T(E_y), S_x) \\ = & \ \dim_K S_x + \textstyle\sum_{y \notin I_e} \dim_K \mathrm{Hom}_{eCe}(S_x, T(E_y)). \end{aligned}$$

Therefore, $\mathrm{Hom}_{eCe}(S_x, T(E_y)) = 0$ for all $y \notin I_e$ if $H(S_x) \cong S_x$.

Conversely, there is a natural isomorphism

$$\mathrm{Hom}_C(H(S_x), S_x) \cong \mathrm{Hom}_{eCe}(S_x, S_x)$$

so there exists a non-zero (and then surjective) morphism $f : H(S_x) \to S_x$. By hypothesis, $\dim_K H(S_x) = \dim_K S_x$ and then $f$ is an isomorphism. □

**Theorem 3.4.23.** *Let $S_x$ be a simple $eCe$-comodule. Then $H(S_x) = S_x$ if and only if $\mathrm{Ext}^1_C(S_x, S_y) = 0$ for all $y \notin I_e$, i.e., there is no arrow $S_x \to S_y$ in $\Gamma_C$, where $S_y$ is a torsion simple $C$-comodule.*

*Proof.* By Proposition 3.4.22, it is enough to prove that $\mathrm{Ext}^1_C(S_x, S_y) = 0$ for all $y \notin I_e$ if and only if $\mathrm{Hom}_{eCe}(S_x, T(E_y)) = 0$ for all $y \notin I_e$.

$\Leftarrow$) Suppose that $\mathrm{Ext}^1_C(S_x, S_y) \neq 0$ for some $y \notin I_e$. By Lemma 3.3.3, $S_x \subseteq \mathrm{soc}\,(E_y/S_y)$. Then

$$S_x = T(S_x) \subseteq T(\mathrm{soc}\, E_y/S_y) \subseteq \mathrm{soc}\, T(E_y/S_y) = \mathrm{soc}\, T(E_y)$$

and therefore $\mathrm{Hom}_{eCe}(S_x, T(E_y)) \neq 0$.

$\Rightarrow$) If $\mathrm{Ext}^1_C(S_x, S_y) = 0$ for each $y \notin I_e$ then there is no path as described in Corollary 3.4.19($c$). Thus $S_x$ is not contained in $T(E_y)$ for any $y \notin I_e$. $\square$

## 3.5 Stable subcategories

The bijective correspondence between localizing subcategories of $\mathcal{M}^C$ and equivalence classes of idempotent elements in $C^*$ is interesting because we may parameterize some classes of localizing subcategories using well known classes of idempotent elements. The first interesting case appears when we consider central idempotent elements. Which localizing subcategories correspond to central idempotent elements? The answer to that question is given, for example, in [GJM99]. If $\mathcal{T}$ is a localizing subcategory such that the associated idempotent element is central, then $\mathcal{T}$ is closed under left and right links. Following these ideas, we may consider more general classes of idempotent elements and the corresponding localizing subcategories. Using the results of the last section we shall deal with semicentral idempotent elements in $C^*$ and see that they define a special, and well known, class of localizing subcategories.

Let us recall that a localizing subcategory $\mathcal{T}$ of $\mathcal{M}^C$ which is closed for essential extensions is called **stable**. The first result we consider on stable localization subcategories appears in [NT94]. There it is proved that the localizing subcategory of $\mathcal{M}^C$ defined by a coidempotent subcoalgebra $A$ of $C$ is stable if and only if $A$ is an injective right $C$-comodule. In order to characterize stable localizing subcategories of $\mathcal{M}^C$ in terms of idempotents, first we recall some definitions from the theory of idempotent elements. Following [Bir83], an idempotent $e$ of a ring $R$ is said to be **left semicentral** if $(1-e)Re = 0$. Right semicentral idempotents are defined in an analogous way. The following characterizations of semicentral idempotent are well-known and easy to prove:

$$\begin{array}{rcl} e \text{ is left semicentral in R} & \Leftrightarrow & eRe = Re \\ & \Leftrightarrow & ere = re \text{ for all } r \in R \\ & \Leftrightarrow & eR \text{ is an ideal of } R \\ & \Leftrightarrow & R(1-e) \text{ is an ideal of } R \end{array}$$

As a consequence of these equivalences, an idempotent element $e \in C^*$ is left semicentral if and only if $1-e$ is right semicentral. Let us give some extra characterizations of a left semicentral idempotent in the dual algebra $C^*$ of a coalgebra $C$.

**Lemma 3.5.1.** *Let $C$ be a coalgebra and $e$ be an idempotent in $C^*$. The following conditions are equivalent:*

*(a) $e$ is a left semicentral idempotent in $C^*$.*

*(b) $eCe = eC$.*

*(c) $C(1-e)$ is a subcoalgebra of $C$.*

*(d) $eC$ is a subcoalgebra of $C$.*

*(e) $eM$ is a subcomodule of $M$ for every right $C$-comodule $M$.*

*Proof.* $(a) \Rightarrow (b)$ For any element $x \in C$ and any $g \in C^*$, we have $g(ex) = (g*e)(x) = (e*g*e)(x) = g(exe)$. Therefore $exe = ex$ and thus $eCe = eC$.
$(b) \Rightarrow (a)$ Given $g \in C^*$, we have that for every $x \in C$, $(g*e)(x) = g(ex) = g(exe) = (e*g*e)(x)$. Therefore $e*g*e = g*e$ for every $g \in C^*$ and $e$ is left semicentral in $C^*$.
$(a) \Leftrightarrow (c)$ It is easy to see that $(C(1-e))^\perp = eC^*$, so $C(1-e)$ is a subcoalgebra of $C$ if, and only if $eC^*$ is an ideal of $C^*$ if and only if $e$ is a left semicentral idempotent in $C^*$.
$(a) \Rightarrow (e)$ Let $M$ be a right $C$-comodule and $g$ an arbitrary element in $C^*$. Then, for every $x \in eM$, we have $gx = g(ex) = (g*e)x = (e*g*e)x = e(g*ex) \in eM$. Therefore $eM$ is left $C^*$-submodule of $M$ and thus it is a right $C$-subcomodule.
$(e) \Rightarrow (d)$ It is trivial
$(d) \Leftrightarrow (a)$ As before, $(eC)^\perp = C^*(1-e)$, thus $eC$ is a subcoalgebra of $C$ if and only if $C^*(1-e)$ is an ideal of $C^*$ if and only if $e$ is a left semicentral idempotent in $C^*$. □

The following theorem is the main result of this section. In it, we describe stable subcategories from different points of view. A proof of some equivalences is given in [JMNR06, Theorem 4.3]. We recall that, for a subset $\Lambda$ of the vertex set $(\Gamma_C)_0$, we say that $\Lambda$ is **right link-closed** if it satisfies that, for each arrow $S \to T$ in $\Gamma_C$, if $S \in \Lambda$ then $T \in \Lambda$.

**Theorem 3.5.2.** *Let $C$ be a coalgebra and $\mathcal{T}_e \subseteq \mathcal{M}^C$ be a localizing subcategory associated to an idempotent $e \in C^*$. The following conditions are equivalent:*

(*a*) *$\mathcal{T}_e$ is a stable subcategory.*

(*b*) *$T(E_x) = 0$ for any $x \notin I_e$.*

(*c*) $T(E_x) = \begin{cases} \overline{E}_x & \text{if } x \in I_e, \\ 0 & \text{if } x \notin I_e. \end{cases}$

(*d*) *$\mathrm{Hom}_C(E_y, E_x) = 0$ for all $x \in I_e$ and $y \notin I_e$.*

(*e*) *$\mathcal{K} = \{S \in (\Gamma_C)_0 \mid eS = S\}$ is a right link-closed subset of $(\Gamma_C)_0$, i.e., there is no arrow $S_x \to S_y$ in $\Gamma_C$, where $T(S_x) = S_x$ and $T(S_y) = 0$.*

(*f*) *There is no path in $\Gamma_C$ from a vertex $S_x$ to a vertex $S_y$ such that $T(S_x) = S_x$ and $T(S_y) = 0$.*

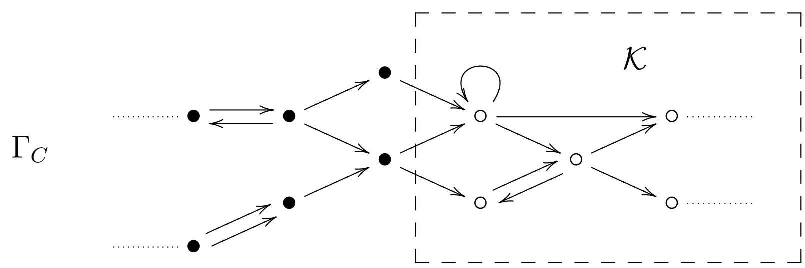
(*g*) *$e$ is a left semicentral idempotent in $C^*$.*

*If $T_e$ is a colocalizing subcategory this is also equivalent to*

(*h*) *$H(S_x) = S_x$ for any $x \in I_e$.*

*Proof.* $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f)$ follows from the definition and from Proposition 3.4.17.
$(f) \Rightarrow (e)$. Trivial.
$(e) \Rightarrow (c)$. By hypothesis, the set $P$ defined in the proof of Proposition 3.4.23 is zero. Therefore $\mathrm{soc}\, T(E_y) = 0$ for all $y \notin I_e$. Then $T(E_y) = 0$ for all $t \notin I_e$.
$(c) \Rightarrow (a)$. Let $M$ be a torsion right $C$-comodule such that its injective envelope is $\oplus_{i \in J} E_i$. Then $S_i \subseteq M$ is torsion for all $i \in J$ and, by hypothesis, $T(E_i) = 0$ for all $i \in J$. Thus $T(\oplus_{i \in J} E_i) = 0$.
$(c) \Leftrightarrow (g)$. We have $C = \oplus_{x \in I_C} E_x$ then $T(C) = \oplus_{x \in I_e} \overline{E}_x \oplus \oplus_{y \notin I_e} T(E_y)$. On the other hand, $eCe = \oplus_{x \in I_e} \overline{E}_x$. Therefore if $(c)$ holds then $eCe = T(C)$. Conversely, if $eCe = T(C)$ then $\oplus_{x \in I_e} \overline{E}_x = \oplus_{x \in I_e} \overline{E}_x \oplus \oplus_{y \notin I_e} T(E_y)$. Since $eCe$ is quasi-finite, by Krull-Remak-Schmidt-Azumaya theorem, $T(E_y) = 0$ for all $y \notin I_e$.
$(e) \Leftrightarrow (h)$. It is Proposition 3.4.23. □

Then we could say that the vertices which determine a stable localization are placed "on the right side" of the Ext-quiver. In the following picture we denote the vertices in $\mathcal{K}$ by white points.



As a direct consequence of Theorem 3.5.2, we obtain an alternative proof of the following fact:

**Corollary 3.5.3.** *[NT96, 4.6] Any stable localizing subcategory of $\mathcal{M}^C$ is a perfect colocalizing subcategory*

*Proof.* If the localizing subcategory $\mathcal{T}$ is stable, $\mathcal{T}$ is associated to a left semicentral idempotent $e \in C^*$ and hence $Ce = eCe$ is certainly injective and quasi-finite as right $eCe$-comodule. Therefore, by Corollary 3.2.10, $\mathcal{T}$ is a perfect colocalizing subcategory of $\mathcal{M}^C$. □

Let us point out the following remarks.

**Remark 3.5.4.** *It is well known that each stable localizing subcategory $\mathcal{T}$ is a TTF class, that is, the torsion class $\mathcal{T}$ is the torsionfree class for another localizing subcategory. If $\mathcal{T}_e$ is stable then $\mathcal{T}_e = \mathcal{F}_{1-e}$ the torsionfree class associated to the localizing subcategory $\mathcal{T}_{1-e}$. Indeed, using Theorem 3.5.2 and Lemma 3.5.1, for every right $C$-comodule $M$, we have that $eM$ is a subcomodule of $M$, therefore $eM$ is precisely the torsion of $M$ for the localizing subcategory $\mathcal{T}_{1-e}$. Then $M \in \mathcal{F}_{1-e}$ if and only if $eM = 0$ if and only if $M \in T_e$.*

**Remark 3.5.5.** *For an idempotent $e \in C^*$, we can consider also the localizing subcategory $\mathcal{T}'_e$ of the category ${}^C\mathcal{M}$ of left $C$-comodules, determined by $e$, that is, $\mathcal{T}'_e = \{M \in {}^C\mathcal{M} \mid Me = 0\}$. Using Theorem 3.5.2 and its left version, we obtain that the localizing subcategory $\mathcal{T}'_e$ of ${}^C\mathcal{M}$ is stable if, and only if the localizing subcategory $\mathcal{T}_{1-e}$ of $\mathcal{M}^C$ is stable.*

We may find an analogous result to Theorem 3.5.2 for right semicentral idempotents:

**Proposition 3.5.6.** *Let $C$ be a coalgebra and $\mathcal{T}_e \subseteq \mathcal{M}^C$ be a localizing subcategory associated to an idempotent $e \in C^*$. The following conditions are equivalent:*

*(a)* *$\mathcal{T}_{1-e}$ is a stable subcategory.*

*(b)* *$T(E_x) = E_x$ for any $x \in I_e$.*

*(c)* *There is no path in $\Gamma_C$ from a vertex $S_y$ to a vertex $S_x$ such that $T(S_x) = S_x$ and $T(S_y) = 0$.*

*(d)* *$\mathcal{K} = \{S \in (\Gamma_C)_0 \mid eS = S\}$ is a left link-closed subset of $(\Gamma_C)_0$, i.e., there is no arrow $S_y \to S_x$ in $\Gamma_C$, where $T(S_x) = S_x$ and $T(S_y) = 0$.*

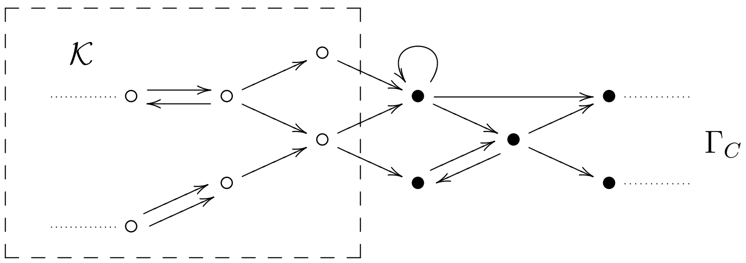
(*e*) *e is a right semicentral idempotent in* $C^*$.

(*f*) *The torsion subcomodule of a right* $C$*-comodule* $M$ *is* $(1-e)M$

(*g*) $S(S_x) = S_x$ *for all* $x \in I_e$.

*Proof.* By the above remarks and Proposition 3.5.2, it is easy to prove $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (f)$. $(d) \Leftrightarrow (g)$ is Corollary 3.4.7. □

And now, we could say that the vertices of the localization are placed "on the left side" of the Ext-quiver.




As a consequence, for a non necessarily indecomposable coalgebra $C$, we get the following immediate results:

**Corollary 3.5.7.** *The following are equivalent:*

(*a*) *e is a central idempotent.*

(*b*) *For each arrow* $S_x \to S_y$ *in* $\Gamma_C$, $T(S_x) = 0$ *if and only if* $T(S_y) = 0$.

(*c*) *For each connected component of* $\Gamma_C$*, either all vertices are torsion or all vertices are torsion-free.*

(*d*) $T(E_x) = 0$ *for any* $x \notin I_e$ *and* $S(S_x) = S_x$ *for all* $x \in I_e$.

**Corollary 3.5.8.** *Let e be a central idempotent in* $C^*$ *and* $\{\Sigma_C^t, \Delta_C^s\}_{t,s}$ *be the connected components of* $\Gamma_C$*, where the vertices of each* $\Sigma_C^t$ *are torsion-free and the vertices of each* $\Delta_C^s$ *are torsion. Then* $\{\Sigma_C^t\}_t$ *are the connected components of* $\Gamma_{eCe}$.

## 3.6 Localization in pointed coalgebras

We finish the chapter detailing the localization in pointed coalgebras. We recall that, by Theorem 1.6.6, every pointed coalgebra $C$ is isomorphic to a subcoalgebra of a path coalgebra $KQ$. Then localization is somehow combinatoric since it is done by keeping and removing suitable vertices and arrows of $Q$ according to the associated idempotent. This technique have some applications to Representation Theory which we present in the next chapter.

Assume then that $C$ is an admissible subcoalgebra of a path coalgebras $KQ$. Thus, by Corollary 2.3.5, it is a relation subcoalgebra in the sense of Simson [Sim05], that is, $C = \bigoplus_{a,b \in Q_0} C_{ab}$, where $C_{ab} = C \cap KQ(a,b)$ and $Q(a,b)$ is the set of all paths in $Q$ from $a$ to $b$. Hence, $C$ has a basis in which every basis element is a linear combination of paths with common source and common sink.

**Lemma 3.6.1.** *Let $C$ be an admissible subcoalgebra of a path coalgebra $KQ$ of a quiver $Q$. There exists a bijective correspondence between localizing subcategories of $\mathcal{M}^C$ and subsets of vertices of $Q$.*

*Proof.* The set of simple $C$-comodules is $\{Kx\}_{x \in Q_0}$ and therefore there is a bijection between the subsets of simple comodules and the subsets of vertices of $Q$. By the arguments of Section 2, the result follows. □

Let $X$ be a subset of vertices of $Q$. We denote by $\mathcal{T}_X$ the localizing subcategory of $\mathcal{M}^C$ associated to $X$, and by $e_X$ the corresponding idempotent of $C^*$.

We may specify the aforementioned bijection as follows. For any idempotent element $e$ in $C^*$ and any vertex $x$ in $Q$, we have that either $e(x) = 0$ or $e(x) = 1$. Hence two idempotent elements $e$, $f \in C^*$ are equivalent if and only if $e_{|Q_0} = f_{|Q_0}$. In this way, we obtain that every localizing subcategory of $\mathcal{M}^C$ is associated to an idempotent element $e \in C^*$ such that $e(p) = 0$ for any non stationary path $p$. Therefore, for an idempotent $e \in C^*$, we may consider the subset of vertices $X_e = \{x \in Q_0 \text{ such that } e(x) = 1\}$ and, conversely, for a subset of vertices $X$, we may attach to it the idempotent $e_X \in C^*$ such that $e_X(x) = 1$ if $x \in X$, and zero otherwise. In what follows, by the idempotent associated to a subset of vertices, we shall mean the idempotent described above.

For the convenience of the reader we introduce the following notation. Let $Q$ be a quiver and $p = \alpha_n \alpha_{n-1} \cdots \alpha_1$ be a non-stationary path in $Q$. We denote by $I_p$ the set of vertices $\{s(\alpha_1), t(\alpha_1), t(\alpha_2), \ldots, t(\alpha_n)\}$. Given a subset of vertices $X \subseteq Q_0$, we say that $p$ is a **cell** in $Q$ relative to $X$ (shortly a cell) if $I_p \cap X = \{s(p), t(p)\}$ and $t(\alpha_i) \notin X$ for all $i = 1, \ldots, n-1$. Given $x, y \in X$, we denote by $\mathcal{C}ell_X^Q(x,y)$ the set of all cells from $x$ to $y$. We denote the set of all cells in $Q$ relative to $X$ by $\mathcal{C}ell_X^Q$.

**Lemma 3.6.2.** *Let $Q$ be a quiver and $X \subseteq Q_0$ be a subset of vertices. Given a path $p$ in $Q$ such that $s(p)$ and $t(p)$ are in $X$, then $p$ has a unique decomposition $p = q_r \cdots q_1$, where each $q_i$ is a cell in $Q$ relative to $X$.*

*Proof.* It is straightforward. □

Let $p$ be a non-stationary path in $Q$ which starts and ends at vertices in $X \subseteq Q_0$. We call the **cellular decomposition** of $p$ relative to $X$ to the decomposition given in the above lemma.

Fix an idempotent $e \in C^*$. Then there exists a subset $X$ of $Q_0$ such that the localized coalgebra $eCe$ has a decomposition $eCe = \bigoplus_{a,b \in X} C_{ab}$, that is, the elements of $eCe$ are linear combinations of paths with source and sink at vertices in $X$. It follows that $eCe$ is a pointed coalgebra so, by Theorem 1.6.6 again, there exists a quiver $Q^e$ such that $eCe$ is an admissible subcoalgebra of $KQ^e$. The quiver $Q^e$ is described as follows:

*Vertices.* We know that $Q_0$ equals the set of group-like elements $\mathcal{G}(C)$ of $C$, therefore $(Q^e)_0 = \mathcal{G}(eCe) = e\mathcal{G}(C)e = eQ_0e = X$.

*Arrows.* Let $x$ and $y$ be vertices in $X$. An element $p \in eCe$ is a non-trivial $(x,y)$-primitive element if and only if $p \notin KX$ and $\Delta_{eCe}(p) = y \otimes p + p \otimes x$. Without loss of generality we may assume that $p = \sum_{i=1}^{n} \lambda_i p_i$ is an element in $eCe$ such that each path $p_i$ is not stationary, and $p_i = \alpha^i_{n_i} \cdots \alpha^i_2 \alpha^i_1$ and $p_i = q^i_{r_i} \cdots q^i_1$ are the decomposition of $p_i$ in arrows of $Q$ and the cellular decomposition of $p_i$ relative to $X$, respectively, for all $i = 1, \ldots, n$. Then

$$\Delta_C(p) = \sum_{i=1}^{n} \lambda_i p_i \otimes s(p_i) + \sum_{i=1}^{n} \lambda_i t(p_i) \otimes p_i + \sum_{i=1}^{n} \lambda_i \sum_{j=2}^{n_i} \alpha^i_{n_i} \cdots \alpha^i_j \otimes \alpha^i_{j-1} \cdots \alpha^i_1$$

and therefore,

$$\begin{aligned}\Delta_{eCe}(p) = \sum_{i=1}^{n} \lambda_i (e\, p_i\, e \otimes e\, s(p_i)\, e) + \sum_{i=1}^{n} \lambda_i (e\, t(p_i)\, e \otimes e\, p_i\, e) + \\ + \sum_{i=1}^{n} \lambda_i \sum_{j=2}^{n_i} e(\alpha^i_{n_i} \cdots \alpha^i_j)e \otimes e(\alpha^i_{j-1} \cdots \alpha^i_1)e.\end{aligned}$$

It follows that, for each path $q$ in $Q$, $eqe = q$ if $q$ starts and ends at vertices in $X$, and zero otherwise. Thus,

$$\begin{aligned}\Delta_{eCe}(p) = \sum_{i=1}^{n} \lambda_i (p_i \otimes s(p_i)) + \sum_{i=1}^{n} \lambda_i (t(p_i) \otimes p_i) + \\ + \sum_{i=1}^{n} \lambda_i \sum_{j=2}^{r_i} q^i_{r_i} \cdots q^i_j \otimes q^i_{j-1} \cdots q^i_1.\end{aligned}$$

Now, this is a linear combination of linearly independent vectors of the vector space $eCe \otimes eCe$, so $\Delta_{eCe}(p) = y \otimes p + p \otimes x$ if and only if we have

$(a)$ $s(p_i) = x$ for all $i = 1, \dots, n$;

$(b)$ $t(p_i) = y$ for all $i = 1, \dots, n$;

$(c)$ $\sum_{i=1}^{n} \lambda_i \sum_{j=2}^{r_i} q_{r_i}^i \cdots q_j^i \otimes q_{j-1}^i \cdots q_1^i = 0.$

Condition $(c)$ is satisfied if and only if $r_i = 1$ for all $i = 1, \dots, n$. Therefore $\Delta_{eCe}(p) = y \otimes p + p \otimes x$ if and only if $p_i$ is a cell from $x$ to $y$ for all $i = 1, \dots, n$. Thus the vector space of all non-trivial $(x, y)$-primitive elements is $K\mathcal{C}ell_X^Q(x, y) \cap C$.

**Proposition 3.6.3.** *Let $C$ be an admissible subcoalgebra of a path coalgebra $KQ$ of a quiver $Q$. Let $e_X$ be the idempotent of $C^*$ associated to a subset of vertices $X$. Then the localized coalgebra $e_X C e_X$ is an admissible subcoalgebra of the path coalgebra $KQ^{ex}$, where $Q^{ex}$ is the quiver whose set of vertices is $(Q^{ex})_0 = X$ and the number of arrows from $x$ to $y$ is $\dim_K K\mathcal{C}ell_X^Q(x, y) \cap C$ for all $x, y \in X$.*

**Corollary 3.6.4.** *Let $Q$ be a quiver and $e_X$ be the idempotent of $(KQ)^*$ associated to a subset of vertices $X$. Then the localized coalgebra $e_X(KQ)e_X$ is an admissible subcoalgebra of the path coalgebra $KQ^{ex}$, where $Q^{ex} = (X, \mathcal{C}ell_X^Q)$.*

*Proof.* By [Gab72], the global dimension of $e_X C e_X$ is less or equal than the global dimension of $C$. Then $e_X C e_X$ is hereditary, i.e., it is the path coalgebra of its Gabriel quiver, that is isomorphic to the quiver $Q^{ex}$, by Proposition 3.6.3. □

**Example 3.6.5.** *Let $Q$ be the quiver*

$$\begin{tikzcd} & 2 \arrow[dr, "\alpha_2"] & \\ 1 \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_3"'] & & 4 \\ & 3 \arrow[ur, "\alpha_4"'] & \end{tikzcd}$$

*and $C$ be the admissible subcoalgebra generated by $\alpha_2\alpha_1 + \alpha_4\alpha_3$. Let $X = \{1, 3, 4\}$. Then $e_X C e_X$ is the path coalgebra of the quiver $Q^{ex}$*

$$\begin{tikzcd} 1 \arrow[r, "\overline{\alpha_3}"] & 3 \arrow[r, "\overline{\alpha_4}"] & 4 \end{tikzcd}$$

*Here, the element $\alpha_2\alpha_1 + \alpha_4\alpha_3$ corresponds to the composition of the arrows $\overline{\alpha_3}$ and $\overline{\alpha_4}$ of $Q^{ex}$.*

*On the other hand, if we consider the path coalgebra $KQ$, the quiver associated to $e_X(KQ)e_X$, say $Q_{e_X}$, is the following:*

$$\begin{tikzcd} ① \arrow[r, "\overline{\alpha_3}"'] \arrow[rr, bend left, "\beta\equiv\alpha_2\alpha_1"] & ③ \arrow[r, "\overline{\alpha_4}"'] & ④ \end{tikzcd}$$

*and $\alpha_2\alpha_1 + \alpha_4\alpha_3$ corresponds to the element $\beta + \overline{\alpha_4}\,\overline{\alpha_3}$.*

**Remark 3.6.6.** *As in the previous example, it is worth pointing out that if $C$ is a proper admissible subcoalgebra of a path coalgebra $KQ$, then we may consider two quivers: the quiver $Q^e$ defined above and the quiver $Q_e$ such that $e(KQ)e \cong KQ_e$. Clearly, $Q^e$ is a subquiver of $Q_e$ and the differences appear in the set of arrows.*

Let us now restrict our attention to the colocalizing subcategories of $\mathcal{M}^C$. For the convenience of the reader we introduce the following notation. We say $p = \alpha_n \cdots \alpha_2\alpha_1$ is a $s(p)$**-tail** in $Q$ relative to $X$ if $I_p \cap X = \{s(p)\}$ and $t(\alpha_i) \notin X$ for all $i = 1, \ldots, n$. If there is no confusion we simply say that $p$ is a tail. Given a vertex $x \in X$ we denote by $\mathcal{T}ail_X^Q(x)$ the set of all $x$-tails in $Q$ relative to $X$.

**Lemma 3.6.7.** *Let $Q$ be a quiver and $X \subseteq Q_0$ be a subset of vertices. Given a path $p$ in $Q$ such that $s(p) \in X$ and $t(p) \notin X$, then $p$ has a unique decomposition $p = cq_r \cdots q_1$, where $c, q_1, \ldots, q_r$ are subpaths of $p$ such that $c \in \mathcal{T}ail_X^Q(s(c))$ and $q_i \in \mathcal{C}ell_X^Q$ for all $i = 1, \ldots, r$.*

*Proof.* It is straightforward. □

Let $p$ be a path in $Q$ such that $s(p) \in X \subseteq Q_0$ and $t(p) \notin X$, we call the **tail decomposition** of $p$ relative to $X$ to the decomposition given in the above lemma. We say that $c$ is the tail of $p$ relative to $X$ if $p = cq_r \cdots q_1$ is the tail decomposition of $p$ relative to $X$.

Assume that $C$ is an admissible subcoalgebra of a path coalgebra $KQ$ and let $\{S_x\}_{x\in Q_0}$ be a complete set of pairwise non isomorphic indecomposable simple right $C$-comodules. We recall that a right $C$-comodule $M$ is **quasi-finite** if and only if $\mathrm{Hom}_C(S_x, M)$ has finite dimension for all $x \in Q_0$. Let $x \in Q_0$ and $f$ be a linear map in $\mathrm{Hom}_C(S_x, M)$. Then $\rho_M \circ f = (f \otimes I) \circ \rho_{S_x}$, where $\rho_M$ and $\rho_{S_x}$ are the structure maps of $M$ and $S_x$ as right $C$-comodules, respectively. In order to describe $f$, since $S_x = Kx$, it is enough to choose the image for $x$. Suppose that $f(x) = m \in M$. Since $(\rho_M f)(x) = ((f \otimes I)\rho_{S_x})(x)$, we obtain that $\rho_M(m) = m \otimes x$. Therefore

$$M_x := \mathrm{Hom}_C(S_x, M) \cong \{m \in M \text{ such that } \rho_M(m) = m \otimes x\},$$

as $K$-vector spaces, and $M$ is quasifinite if and only if $M_x$ has finite dimension for all $x \in Q_0$.

Our aim is to establish when a localizing subcategory $\mathcal{T}_e$ is colocalizing, or equivalently, by Proposition 3.2.9, when $eC$ is a quasifinite right $eCe$-comodule. We recall that the structure of $eC$ as right $eCe$-comodule is given by $\rho_{eC}(p) = \sum_{(p)} ep_{(1)} \otimes ep_{(2)}e$ if $\Delta_{KQ}(p) = \sum_{(p)} p_{(1)} \otimes p_{(2)}$, for all $p \in eC$. It is easy to see that $eC$ has a decomposition $eC = \bigoplus_{a \in X, b \in Q_0} C_{ab}$, as vector space, that is, the elements of $eC$ are linear combinations of paths which start at vertices in $X$.

**Proposition 3.6.8.** *Let $C$ be an admissible subcoalgebra of a path coalgebra $KQ$. Let $e_X$ be an idempotent in $C^*$ associated to a subset of vertices $X$. The following conditions are equivalent:*

$(a)$ *The localizing subcategory $\mathcal{T}_X$ of $\mathcal{M}^C$ is colocalizing.*

$(b)$ *$e_X C$ is a quasifinite right $e_X C e_X$-comodule.*

$(c)$ $\dim_K K\mathcal{T}ail_X^Q(x) \cap C$ *is finite for all $x \in X$.*

*Proof.* By the arguments mentioned above, it is enough to prove that $(e_X C)_x = K\mathcal{T}ail_X^Q(x) \cap C$. For simplicity we write $e$ instead of $e_X$.

Let $p = \sum_{i=1}^n \lambda_i c_i \in C$ be a $K$-linear combination of $x$-tail such that $c_i = \alpha_{r_i}^i \cdots \alpha_1^i$ ends at $y_i$ for all $i = 1, \dots n$. Then,

$$\Delta_{KQ}(p) = p \otimes x + \sum_{i=1}^n y_i \otimes \lambda_i c_i + \sum_{i=1}^n \lambda_i \sum_{j=2}^{r_i} \alpha_{r_i}^i \cdots \alpha_j^i \otimes \alpha_{j-1}^i \cdots \alpha_1^i,$$

and then,

$$\begin{aligned}\rho_{eC}(p) = e\,p \otimes e\,x\,e + \sum_{i=1}^n \lambda_i e\,y_i \otimes e\,c_i\,e + \\ + \sum_{i=1}^n \lambda_i \sum_{j=2}^{r_i} e(\alpha_{r_i}^i \cdots \alpha_j^i) \otimes e(\alpha_{j-1}^i \cdots \alpha_1^i)e = p \otimes x\end{aligned}$$

because $\alpha_j^i$ ends at a point not in $X$ for all $j = 1, \dots, r_i$ and $i = 1, \dots n$. Thus $p \in (eC)_x$.

Conversely, consider an element $p = \sum_{i=1}^n \lambda_i p_i + \sum_{k=1}^m \mu_k q_k \in (eC)_x$, where $t(p_i) \in X$ for all $i = 1, \dots, n$, and $t(q_k) \notin X$ for all $k = 1, \dots, m$. Moreover, let us suppose that $p_i = \overline{p}_{r_i}^i \cdots \overline{p}_1^i$ is the cellular decomposition of $p_i$ relative to $X$ for all $i = 1, \ldots, n$, and $q_k = c_k \overline{q}^k_{s_k} \cdots \overline{q}^k_1$ is the tail decomposition of $q_k$ relative to $X$ for all $k = 1, \ldots, m$. Then,

$$\rho_{eC}(p) = \sum_{i=1}^{n} \lambda_i \sum_{j=2}^{r_i} \overline{p}^i_{r_i} \cdots \overline{p}^i_j \otimes \overline{p}^i_{j-1} \ldots \overline{p}^i_1 + \sum_{i=1}^{n} \lambda_i t(p_i) \otimes p_i + \sum_{i=1}^{n} \lambda_i p_i \otimes s(p_i) +$$
$$+ \sum_{k=1}^{m} \mu_k c_k \otimes q_k + \sum_{k=1}^{m} \mu_k \sum_{l=2}^{s_k} c_k \overline{q}^k_{s_k} \cdots \overline{q}^k_l \otimes \overline{q}^k_{l-1} \ldots \overline{q}^k_1 + \sum_{k=1}^{m} \mu_k q_k \otimes s(q_k).$$

A straightforward calculation shows that if $\rho_{eC}(p) = p \otimes x$ then $n = 0$, $s(q_k) = x$ and $s_k = 0$ for all $k = 1, \ldots, m$. Therefore $p \in K\mathcal{T}ail^Q_X(x) \cap C$ and the proof is finished. $\square$

**Corollary 3.6.9.** *Let $Q$ be a quiver and $X$ be a subset of $Q_0$. Then the following conditions are equivalent:*

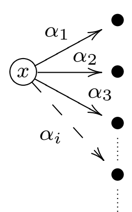
($a$) *The localizing subcategory $\mathcal{T}_X$ of $\mathcal{M}^C$ is colocalizing.*

($b$) *The localizing subcategory $\mathcal{T}_X$ of $\mathcal{M}^C$ is perfect colocalizing.*

($c$) *$\mathcal{T}ail^Q_X(x)$ is a finite set for all $x \in X$. That is, there is at most a finite number of paths starting at the same point whose only vertex in $X$ is the first one.*

*Proof.* Since $\mathcal{T}ail^Q_X(x)$ is a basis of the vector space $K\mathcal{T}ail^Q_X(x) \cap C$, applying Proposition 3.6.8, we get that ($a$) $\Leftrightarrow$ ($c$). The statements ($a$) and ($b$) are equivalent by [NT96, Theorem 4.2]. $\square$

**Example 3.6.10.** *Consider the quiver $Q$*




*where $i \in \mathbb{N}$ and the subset $X = \{x\}$. Then $\mathcal{T}ail^Q_X(x) = \{\alpha_i\}_{i \in \mathbb{N}}$ is an infinite set and the localizing subcategory $\mathcal{T}_X$ is not colocalizing.*

**Remark 3.6.11.** *If $C$ is finite dimensional or, more generally, if the set $Q_0 \backslash X$ is finite and $Q$ is acyclic, every localizing subcategory is colocalizing.*

**Remark 3.6.12.** *The reader should observe that all results and proofs stated in this section also remains valid for an arbitrary field $K$.*

# Chapter 4

# Applications to representation theory

In this chapter we continue looking into the question raised in Section 2.5. As it is shown there, there is a class of coalgebras which cannot be written as path coalgebras of quivers with relations in the sense of Simson (cf. [Sim01]). Therefore we cannot expect to find a full Gabriel's theorem for coalgebras using this notion. Nevertheless, for all these counterexamples, the category of finitely generated comodules has very bad properties. This bad behavior is similar to the notion of wildness given for the finitely generated module categories of finite dimensional algebras, meaning that this category is so big that it is not realistic aiming to give an explicit description of it. Therefore it is natural to reformulate the problem as follows: *any basic tame coalgebra $C$, over an algebraically closed field, is isomorphic to the path coalgebra $C(Q,\Omega)$ of a quiver with relations $(Q,\Omega)$.* In order to study this question we make use of the theory of localization developed in Chapter 3. This is hardly surprising since we follow the categorical approach of Gabriel (cf. [Gab62]). Therefore we relate the wildness and tameness (in the sense of [Sim01] and [Sim05]) of a comodule category and its quotient categories. As a consequence, we may prove the above problem under the further assumption that the quiver $Q$ is acyclic, that is, $Q$ has no oriented cycles. This extends [Sim05, Theorem 3.14(c)] from the case in which $Q$ is intervally finite and $C \subseteq KQ$ is an arbitrary admissible subcoalgebra to the case in which $Q$ is acyclic and $C \subseteq KQ$ is tame.

## 4.1 Comodule types of coalgebras

Throughout this chapter we assume that $K$ is an algebraically closed field. It is well-known that the category of finite dimensional $K$-algebras is the disjoint union of two classes: tame and wild algebras. This is known as the **tame-wild dichotomy**, see [Dro79] or [Sim92]. The idea of such classes is that the category of finite dimensional modules over a wild algebra is so big that it contains (via an exact representation embedding) the category of all finite dimensional representations of the noncommutative polynomial algebra $K\langle x, y\rangle$. As it is well-known, the category of finite dimensional modules over $K\langle x, y\rangle$ contains (again via an exact representation embedding) the category of all finitely generated representations for any other finite dimensional algebra, and thus it is not realistic aiming to give an explicit description of this category (or, by extension, of any wild algebra). In opposition, a tame algebra is one whose indecomposable modules of finite dimension are parameterized by a finite number of one-parameter families for each dimension vector. We refer the reader to [Sim92] for basic definitions and properties about module type of algebras. In this section we recall from [Sim01] and [Sim05] the analogous concepts for a basic (pointed) coalgebra.

Let $C$ be a basic coalgebra such that $C_0 = \oplus_{i\in I_C} S_i$. For every finite dimensional right $C$-comodule $M$ we consider the **length vector** of $M$, $\underline{\text{length}}\, M = (m_i)_{i\in I_C} \in \mathbb{Z}^{(I_C)}$, where $m_i \in \mathbb{N}$ is the number of simple composition factors of $M$ isomorphic to $S_i$. In [Sim01] it is proved that the map $M \mapsto \underline{\text{length}}\, M$ extends to a group isomorphism $K_0(C) \longrightarrow \mathbb{Z}^{(I_C)}$, where $K_0(C)$ is the **Grothendieck group** of $C$. We recall that the Grothendieck group of a coalgebra (or of the category $\mathcal{M}_f^C$) is the quotient of the free abelian group generated by the set of isomorphism classes $[M]$ of modules $M$ in $\mathcal{M}_f^C$ modulo the subgroup generated by the elements $[M] - [N] - [L]$ corresponding to all exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{M}_f^C$.

Let $R$ be a $K$-algebra. By a $R$-$C$-bimodule we mean a $K$-vector space $L$ endowed with a left $R$-module structure $\cdot : R\otimes L \to L$ and a right $C$-comodule structure $\rho : L \to L \otimes C$ such that $\rho_L(r\cdot x) = r\cdot \rho_L(x)$, i.e., the following diagram is commutative

$$\begin{tikzcd} R\otimes L \arrow[r, "\cdot"] \arrow[d, "I\otimes\rho_L"'] & L \arrow[d, "\rho_L"] \\ R\otimes L\otimes C \arrow[r, "\cdot\otimes I"] & L\otimes C \end{tikzcd}$$

We denote by ${}_R\mathcal{M}^C$ the category of $R$-$C$-bimodules.

**Lemma 4.1.1.** *Let $L$ be a $R$-$C$-bimodule and $e \in C^*$ be an idempotent. Then $eL$ is a $R$-$eCe$-bimodule and we have a functor*

$$T = e(-) : {}_R\mathcal{M}^C \longrightarrow {}_R\mathcal{M}^{eCe}.$$

*Proof.* The above compatibility property yields the following equalities:

$$\sum_{(r\cdot x)} (r \cdot x)_{(0)} \otimes (r \cdot x)_{(1)} = \rho_L(r \cdot x) = r \cdot \rho_L(x) = \sum_{(x)} r \cdot x_{(0)} \otimes x_{(1)}, \quad (4.1)$$

for each element $r \in R$ and $x \in L$. Now, we have

$$\begin{aligned} r \cdot (e \cdot x) &= r \cdot (\textstyle\sum_{(x)} x_{(0)} e(x_{(1)})) \\ &= \textstyle\sum_{(x)} r \cdot x_{(0)} e(x_{(1)}) \\ &= (I \otimes e)(\textstyle\sum_{(x)} r \cdot x_{(0)} \otimes x_{(1)}) \\ &\overset{(4.1)}{=} (I \otimes e)\rho_L(r \cdot x) \\ &= e \cdot (r \cdot x). \end{aligned}$$

Then $eL$ has a structure of left $R$-module and right $eCe$-comodule, and we have the compatibility property.

$$\begin{aligned} r \cdot \rho_{eL}(e \cdot x) &= r \cdot (\textstyle\sum_{(x)} e \cdot x_{(0)} \otimes e \cdot x_{(1)} \cdot e) \\ &= \textstyle\sum_{(x)} r \cdot (e \cdot x_{(0)}) \otimes e \cdot x_{(1)} \cdot e \\ &= \textstyle\sum_{(x)} e \cdot (r \cdot x_{(0)}) \otimes e \cdot x_{(1)} \cdot e \\ &\overset{(4.1)}{=} \textstyle\sum_{(r\cdot x)} e \cdot (r \cdot x)_{(0)} \otimes e \cdot (r \cdot x)_{(1)} \cdot e \\ &= \rho_{eL}(e \cdot (r \cdot x)) \\ &= \rho_{eL}(r \cdot (e \cdot x)). \end{aligned}$$

□

We recall from [Sim01] and [Sim05] that a coalgebra $C$ over an algebraically closed field $K$ is said to be of **tame comodule type** (tame for short) if for every $v \in K_0(C)$ there exist $K[t]$-$C$-bimodules $L^{(1)}, \ldots, L^{(r_v)}$, which are finitely generated free $K[t]$-modules, such that all but finitely many indecomposable right $C$-comodules $M$ with $\underline{\text{length}}\, M = v$ are of the form $M \cong K^1_\lambda \otimes_{K[t]} L^{(s)}$, where $s \leq r_v$, $K^1_\lambda = K[t]/(t - \lambda)$ and $\lambda \in K$. If there is a common bound for the numbers $r_v$ for all $v \in K_0(C)$, then $C$ is called **domestic**.

If $C$ is a tame coalgebra then there exists a **growth function** $\mu_C^1 : K_0(C) \to \mathbb{N}$ defined as $\mu_C^1(v)$ to be the minimal number $r_v$ of $K[t]$-$C$-bimodules $L^{(1)}, \ldots, L^{(r_v)}$ satisfying the above conditions, for each $v \in K_0(C)$. $C$ is said to be of **polynomial growth** if there exists a formal power series

$$G(t) = \sum_{m=1}^{\infty} \sum_{j_1,\ldots,j_m \in I_C} g_{j_1,\ldots,j_m} t_{j_1} \ldots t_{j_m}$$

with $t = (t_j)_{j \in I_C}$ and non-negative coefficients $g_{j_1,\ldots,j_m} \in \mathbb{Z}$ such that $\mu_C^1(v) \leq G(v)$ for all $v = (v(j))_{j \in I_C} \in K_0(C) \cong \mathbb{Z}^{(I_C)}$ such that $\|v\| := \sum_{j \in I_C} v(j) \geq 2$. If $G(t) = \sum_{j \in I_C} g_j t_j$, where $g_j \in \mathbb{N}$, then $C$ is called of **linear growth**. If $\mu_C^1$ is zero we say that $C$ is of **discrete comodule type**.

Let $Q$ be the quiver $\circ \Rrightarrow \circ$ and $KQ$ be the path algebra of the quiver $Q$. Let us denote by $\mathcal{M}_{KQ}^f$ the category of finite dimensional right $KQ$-modules. A $K$-coalgebra $C$ is of **wild comodule type** (wild for short) if there exists an exact and faithful $K$-linear functor $F : \mathcal{M}_{KQ}^f \to \mathcal{M}_f^C$ that respects isomorphism classes and carries indecomposable right $KQ$-modules to indecomposable right $C$-comodules. If, in addition, the functor $F$ is fully faithful, we say that $C$ is of **fully wild comodule type**.

Let us collect from [Sim01] and [Sim05] some properties about wild and tame comodule type.

**Proposition 4.1.2.** $(a)$ *The tame, polynomial growth, linear growth, discrete, domestic and wild comodule type are invariant under Morita-Takeuchi equivalence of coalgebras.*

$(b)$ *The notion of wild comodule type is left-right symmetric.*

$(c)$ *If there exist a pair $S$, $S'$ of simple right $C$-comodules such that the integer* $\dim_K \mathrm{Ext}_C^1(S, S') \geq 3$ *then $C$ is of wild comodule type.*

$(d)$ *The following conditions are equivalent:*

- $(i)$ *$C$ is of wild comodule type.*
- $(ii)$ *There exists a finite dimensional subcoalgebra $H$ of $C$ of wild comodule type.*
- $(iii)$ *The coalgebra $C$ is a direct union of finite dimensional subcoalgebras of wild comodule type.*

$(e)$ *If $C$ is tame then each finite dimensional subcoalgebra of $C$ is also tame.*

**Corollary 4.1.3.** *[Sim05] Let $C$ be a coalgebra and $D$ be a subcoalgebra of $C$ of wild comodule type. Then $C$ is of wild comodule type.*

As a consequence of the former proposition, Simson proves in [Sim05] the *weak tame-wild dichotomy for coalgebras.*

**Corollary 4.1.4.** *Every coalgebra of tame comodule type is not of wild comodule type.*

We hope that the following tame-wild dichotomy holds.

**Conjecture 4.1.5.** *[Sim05][*Tame-wild dichotomy for coalgebras*] Every coalgebra is either of tame comodule type, or of wild comodule type, and these types are mutually exclusive.*

## 4.2 Localization and tame comodule type

This section and the subsequent are devoted to study relations between the comodule type of a coalgebra and its localized coalgebras. Let $e$ be an idempotent in the dual algebra $C^*$. Recall that we denote by $I_e = \{i \in I_C \mid eS_i = S_i\}$ and $\mathcal{K} = \{S_i\}_{i \in I_e}$. Let us analyze the behavior of the length vector under the action of the quotient functor.

**Lemma 4.2.1.** *Let $C$ be a coalgebra and $e \in C^*$ be an idempotent.*

*(a) If $L$ is a finite dimensional right $C$-comodule, then we have $(\underline{\mathrm{length}}\, L)_i = (\underline{\mathrm{length}}\, eL)_i$ for all $i \in I_e$.*

*(b) The following diagram is commutative*

$$\begin{tikzcd} \mathcal{M}_f^C \arrow[r, "e(-)"] \arrow[d, "\underline{\mathrm{length}}"'] & \mathcal{M}_f^{eCe} \arrow[d, "\underline{\mathrm{length}}"] \\ K_0(C) \arrow[r, "f"] & K_0(eCe) \end{tikzcd}$$

*where $f$ is the projection from $K_0(C) \cong \mathbb{Z}^{(I_C)}$ onto $K_0(eCe) \cong \mathbb{Z}^{(I_e)}$.*

*Proof.* $(a)$ Let $\mathcal{K} = \{S_i\}_{i \in I_e}$. Let us consider $0 \subset L_1 \subset L_2 \subset \cdots \subset L_{n-1} \subset L_n$ a composition series for $L$. Then, we obtain the inclusions $0 \subseteq eL_1 \subseteq eL_2 \subseteq \cdots \subseteq eL_{n-1} \subseteq eL_n$. Since $e(-)$ is an exact functor, $eL_j/eL_{j-1} \cong e(L_j/L_{j-1}) = eS_j$, where $S_j$ is a simple $C$-comodule for all $j = 1, \ldots, n$. But $eS_j = S_j$ if $S_j \in \mathcal{K}$ and zero otherwise. Thus $(\underline{\mathrm{length}}\, L)_i = (\underline{\mathrm{length}}\, eL)_i$ for all $i \in I_e$. It is easy to see that $(b)$ follows from $(a)$.

□

**Corollary 4.2.2.** *For any finite dimensional right $C$-comodule $M$, we have* length $(eM) \leq$ length $(M)$.

*Proof.* By Lemma 4.2.1,

$$\text{length }(eM) = \sum_{i \in I_e} (\underline{\text{length}}\, eM)_i \leq \sum_{i \in I_C} (\underline{\text{length}}\, M)_i = \text{length }(M).$$

□

Let us now consider the converse problem, that is, take a right $eCe$-comodule $N$ whose length vector is known, which is the length vector of $S(N)$? Since, in general, the functor $S$ does not preserve finite dimensional comodules, we have to impose some extra conditions. We start with a simple case.

**Lemma 4.2.3.** *Let $N$ be a finite dimensional right $eCe$-comodule such that $\underline{\text{length}}\, N = (v_i)_{i \in I_e}$. Suppose that $S(S_i) = S_i$ for all $i \in I_e$ such that $v_i \neq 0$. Then*

$$\left(\underline{\text{length}}\, S(N)\right)_i = \begin{cases} v_i, & \text{if } i \in I_e \\ 0, & \text{if } i \in I_C \backslash I_e \end{cases}$$

*Proof.* Let $0 \subset N_1 \subset N_2 \subset \cdots \subset N_{n-1} \subset N_n = N$ be a composition series for $N$. Since $S$ is left exact, we have the chain of right $C$-comodules

$$0 \subset S(N_1) \subset S(N_2) \subset \cdots \subset S(N_{n-1}) \subset S(N_n) = S(N).$$

Now, for each $j = 0, \ldots, n-1$, we consider the short exact sequence

$$0 \longrightarrow N_j \xrightarrow{i} N_{j+1} \xrightarrow{p} S_{j+1} \longrightarrow 0$$

and applying the functor $S$ we have an exact sequence

$$0 \longrightarrow S(N_j) \xrightarrow{S(i)} S(N_{j+1}) \xrightarrow{S(p)} S(S_{j+1}) = S_{j+1}$$

This sequence is exact since $S(p)$ is non-zero (otherwise $S(i)$ is bijective and then so is $i$). Thus $S(N_{j+1})/S(N_j) \cong S_{j+1}$ and the chain is a composition series of $S(N)$. □

**Lemma 4.2.4.** *Let $C$ be a $K$-coalgebra and $R$ be a $K$-algebra. Suppose that $N$ is a $R$-$C$-bimodule, $M$ is a right $R$-module and $f$ is an idempotent in $C^*$. Then $f(M \otimes_R N) \simeq M \otimes_R fN$.*

*Proof.* Let us suppose that the right $C$-comodule structure of $N$ is the map $\rho_N$. Then $M \otimes_R N$ is endowed with a structure of right $C$-comodule given by the map $I \otimes_R \rho_N : M \otimes_R N \to M \otimes_R N \otimes C$ defined by

$$m \otimes_R n \mapsto m \otimes_R \left( \sum_{(n)} n_{(0)} \otimes n_{(1)} \right) = \sum_{(n)} (m \otimes_R n_{(0)} \otimes n_{(1)}),$$

for all $m \in M$ and $n \in N$.

Therefore $f \cdot (m \otimes_R n) = \sum_{(n)} m \otimes_R n_{(0)} \otimes f(n_{(1)}) = \sum_{(n)} m \otimes_R n_{(0)} f(n_{(1)}) = m \otimes_R \sum_{(n)} n_{(0)} f(n_{(1)}) = m \otimes_R f \cdot n$ for all $m \in M$ and $n \in N$. Thus $f(M \otimes_R N) \simeq M \otimes_R fN$. $\square$

**Proposition 4.2.5.** *Let $v = (v_i)_{i \in I_e} \in K_0(eCe)$ such that $S(S_i) = S_i$ for all $i \in I_e$ with $v_i \neq 0$. If $C$ satisfies the tameness condition for $\overline{v} \in K_0(C)$ given by*

$$(\overline{v})_i = \begin{cases} v_i, & \text{if } i \in I_e \\ 0, & \text{if } i \in I_C \backslash I_e \end{cases}$$

*then $eCe$ satisfies the tameness condition for $v$, and $\mu^1_{eCe}(v) \leq \mu^1_C(\overline{v})$.*

*Proof.* By hypothesis, there exist $K[t]$-$C$-bimodules $L^{(1)}, L^{(2)}, \ldots, L^{(r_{\overline{v}})}$, which are finitely generated free $K[t]$-modules, such that all but finitely many indecomposable right $C$-comodules $M$ with $\underline{\text{length}}\, M = \overline{v}$ are of the form $M \cong K^1_\lambda \otimes_{K[t]} L^{(s)}$, where $s \leq r_{\overline{v}}$, $K^1_\lambda = K[t]/(t-\lambda)$ and $\lambda \in K$. Consider the $K[t]$-$eCe$-bimodules $eL^{(1)}, \ldots, eL^{(r_{\overline{v}})}$. Obviously, they are finitely generated free as left $K[t]$-modules. Let now $N$ be a right $eCe$ comodule with $\underline{\text{length}}\, N = v$. By Lemma 4.2.3, $\underline{\text{length}}\, S(N) = \overline{v}$ and therefore $S(N) \cong K^1_\lambda \otimes_{K[t]} L^{(s)}$ for some $s \leq r_{\overline{v}}$ and some $\lambda \in K$ (since $S$ is an embedding, there are only finitely many $eCe$-comodules $N$ such that $S(N)$ is not of the above form). Then, by the previous lemma, $N \cong eS(N) \cong K^1_\lambda \otimes_{K[t]} eL^{(s)}$. Thus $eCe$ satisfies the tameness condition for $v$ and, furthermore, $\mu^1_{eCe}(v) \leq \mu^1_C(\overline{v})$. $\square$

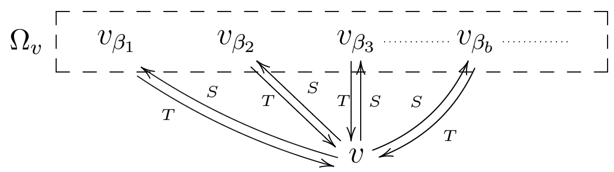
**Corollary 4.2.6.** *Let $C$ be a coalgebra and $e \in C^*$ be a right semicentral idempotent. If $C$ is tame (of polynomial growth, of linear growth, domestic, discrete) then $eCe$ is tame (of polynomial growth, of linear growth, domestic, discrete).*

*Proof.* By Proposition 3.5.6, $e$ is right semicentral if and only of $S(S_i) = S_i$ for all $i \in I_e$. Then the statement follows from the former result. $\square$

The underlying idea of the proof of Proposition 4.2.5 goes as follows: if we control the $C$-comodules whose length vector is obtained as $\underline{\text{length}}\, S(T)$

for some $eCe$-comodule $T$ such that $\underline{\text{length}}\, T = v$ (under the assumption of Proposition 4.2.5 the unique length vector obtained in such a way is exactly $\overline{v}$), then we may control the $eCe$-comodules of length $v$. In the following result we generalize Proposition 4.2.5 applying the aforementioned method.

For the convenience of the reader we introduce the following notation. To any vector $v \in K_0(eCe) \cong \mathbb{Z}^{(I_e)}$ we associate the set $\Omega_v = \{v^{(\beta)}\}_{\beta \in B}$ of all vectors in $K_0(C) \cong \mathbb{Z}^{(I_C)}$ such that each $v^{(\beta)} = \underline{\text{length}}\, S(N)$ for some $eCe$-comodule $N$ such that $\underline{\text{length}}\, N = v$.




**Proposition 4.2.7.** *Let $v \in K_0(eCe)$ such that $\Omega_v$ is finite. If $C$ satisfies the tameness condition for each vector $v^{(\beta)} \in \Omega_v$ then $eCe$ satisfies the tameness condition for $v$.*

*Proof.* Consider the set of all $K[t]$-$C$-bimodules associated to all vectors $v^{(\beta)} \in \Omega_v$, namely $\mathcal{L} = \{L_\beta^{(j)}\}_{\beta \in \Omega_v, j=1,\ldots,r_\beta}$. By hypothesis, this is a finite set and then so is $T(\mathcal{L})$. We proceed analogously to Proposition 4.2.5 and the result follows. $\square$

Given two vectors $v = (v_i)_{i \in I_C}, w = (w_i)_{i \in I_C} \in K_0(C)$, we say that $v \leq w$ if $v_i \leq w_i$ for all $i \in I_C$.

**Lemma 4.2.8.** *Let $v = (v_i)_{i \in I_e} \in K_0(eCe)$ such that $S(S_i)$ is a finite dimensional right $C$-comodule for all $i \in I_e$ with $v_i \neq 0$. Then $\Omega_v$ is a finite set.*

*Proof.* Let $N$ be a right $eCe$-comodule such that $\underline{\text{length}}\, N = v$. Consider a composition series for $N$, $0 \subset N_1 \subset N_2 \subset \cdots \subset N_{n-1} \subset N_n = N$. Since $S$ is left exact, we have the chain of right $C$-comodules

$$0 \subset S(N_1) \subset S(N_2) \subset \cdots \subset S(N_{n-1}) \subset S(N_n) = S(N).$$

Then, for each $j = 1, \ldots, n$, we have an exact sequence

$$0 \longrightarrow S(N_{j-1}) \longrightarrow S(N_j) \longrightarrow S(S_j)\ ,$$

where $S_j$ is a simple $eCe$-comodule.

By Proposition 3.4.2, soc $(S(S_j)) = S_j$ and hence $S(S_j)$ has a composition series

$$0 \subset S_j \subset S(S_j)_2 \subset \cdots \subset S(S_j)_{r-1} \subset S(S_j).$$

Then we can complete the following commutative diagram taking the pull-back $P_i$ for $i = 1, \ldots, r-1$.

$$\begin{tikzcd}
S(N_j) \arrow[r, tail] \arrow[d, equal] & P_1 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\blacksquare", pos=0.2] & S_j \arrow[d, tail] & \\
S(N_j) \arrow[r, tail] \arrow[d, equal] & P_2 \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\blacksquare", pos=0.2] & S(S_j)_2 \arrow[r] \arrow[d, tail] & \mathrm{Coker} \\
\vdots \arrow[d, equal] & \vdots \arrow[d, tail] & \vdots \arrow[d, tail] & \vdots \\
S(N_j) \arrow[r, tail] \arrow[d, equal] & P_{r-1} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\blacksquare", pos=0.2] & S(S_j)_{r-1} \arrow[r] \arrow[d, tail] & \mathrm{Coker} \\
S(N_j) \arrow[r, tail] & S(N_{j+1}) \arrow[r] & S(S_j) \arrow[r] & \mathrm{Coker}
\end{tikzcd}$$

Consider two consecutive rows and their quotient sequence

$$\begin{tikzcd}
S(N_j) \arrow[r, tail] \arrow[dd, equal] & P_t \arrow[rrr, "g_t"] \arrow[dr] \arrow[dd, tail, "i"] \arrow[dr, phantom, "\blacksquare", pos=0.15] & & & S(S_j)_t \arrow[r] \arrow[dd, tail, "i"] & \mathrm{Coker} \\
& & \mathrm{Im}\, g_t \arrow[urr] \arrow[r] \arrow[dd, tail, "i"] & 0 & & \\
S(N_j) \arrow[r, tail] \arrow[dd] & P_{t+1} \arrow[rrr, "g_{t+1}"] \arrow[dr] \arrow[dd, two heads, "p"] & & & S(S_j)_{t+1} \arrow[r] \arrow[dd, two heads, "p"] & \mathrm{Coker} \\
& & \mathrm{Im}\, g_{t+1} \arrow[urr] \arrow[r] \arrow[dd, two heads, "p"] & 0 & & \\
0 \arrow[r] & P_{t+1}/P_t \arrow[rrr, "\overline{g}"] \arrow[dr] & & & S_k \cong S(S_j)_{t+1}/S(S_j)_t & \\
& & \mathrm{Im}\, g_t/\mathrm{Im}\, g_{t+1} \cong \mathrm{Im}\, \overline{g} \arrow[urr] \arrow[r] & 0 & &
\end{tikzcd}$$

Suppose that $P_{t+1} \neq P_t$, then $P_{t+1}/P_t \cong \mathrm{Im}\, \overline{g} \hookrightarrow S_k$, and thus $P_{t+1}/P_t \cong S_k$. Hence we have obtained a chain

$$0 \subset P_1^1 \subseteq \cdots \subseteq P_{r_1}^1 = S(N_1) \subseteq \cdots \subseteq S(N_{n-1}) \subseteq P_1^n \subseteq \cdots \subseteq P_{r_n}^n = S(N),$$

where the quotient of two consecutive comodules is zero or a simple comodule. Therefore $\underline{\text{length}}\, S(N) \leq \sum_{j=1}^{n} \underline{\text{length}}\, S(S_j)$ for any right $eCe$-comodule $N$ whose $\underline{\text{length}}\, N = v$. Thus $\Omega_v$ is a finite set. $\square$

As a consequence of the former results we may state the following theorem:

**Theorem 4.2.9.** *Let $C$ be a coalgebra and $e \in C^*$ be an idempotent such that $S(S_i)$ is a finite dimensional right $C$-comodule for all $i \in I_e$. If $C$ is of tame (discrete) comodule type then $eCe$ is of tame (discrete) comodule type.*

*Proof.* By Lemma 4.2.8, $\Omega_v$ is a finite set for each $v \in K_0(eCe)$. Then, by Proposition 4.2.7, $eCe$ satisfies the tameness condition for $v$. $\square$

**Remark 4.2.10.** *The reader should observe that the proof of Lemma 4.2.8 also shows that the section functor $S$ preserves finite dimensional comodules if and only if $S(S_i)$ is finite dimensional for all $i \in I_e$.*

In particular, the conditions of Theorem 4.2.9 are satisfied for any idempotent if $C$ is left semiperfect. A coalgebra is said to be **left semiperfect** if every finite dimensional left comodule has a finite-dimensional projective cover, or equivalently, if any indecomposable injective right comodule is finite dimensional.

**Corollary 4.2.11.** *Let $C$ be a left semiperfect coalgebra and $e \in C^*$ be an idempotent. If $C$ is of tame (discrete) comodule type then $eCe$ is of tame (discrete) comodule type.*

*Proof.* By Proposition 3.4.2, for any $i \in I_e$, $S(S_i)$ is a subcomodule of an indecomposable injective right $C$-comodule. Therefore, if $C$ is left semiperfect, $S(S_i)$ is finite dimensional for any $i \in I_e$. Thus the result follows from Theorem 4.2.9. $\square$

The following problem remains still open.

**Problem 4.2.12.** *Assume that $C$ is a coalgebra of tame comodule type and $e$ is an idempotent in $C^*$. Is the coalgebra $eCe$ of tame comodule type?*

It would be interesting to know if the localization process preserves polynomial growth, linear growth, discrete comodule type or domesticity. It is clear that the converse result is not true as the following example shows.

**Example 4.2.13.** *Let us consider the quiver*

$$Q: \quad \begin{tikzcd} & & \circ \arrow[d] \\ \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \end{tikzcd}$$

*Since its underlying graph is neither a Dynkin diagram nor an Euclidean graph, $KQ$ is wild, see [Sim05, Theorem 9.4]. But it is easy to see that $eCe$ is of tame comodule type for each non-trivial idempotent $e \in C^*$.*

## 4.3 Split idempotents

Let us study the wildness of a coalgebra and its localized coalgebras. Directly from the definition we may prove the following proposition.

**Proposition 4.3.1.** *Let $C$ be a coalgebra and $e \in C^*$ be an idempotent which defines a perfect colocalization. If $eCe$ is wild then $C$ is wild.*

*Proof.* By hypothesis, there is an exact and faithful functor $F : \mathcal{M}^f_{KQ} \to \mathcal{M}^{eCe}_f$, where $Q$ is the quiver $\circ \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \circ$, which respects isomorphism classes and preserves indecomposables. By Proposition 3.4.21, the colocalizing functor $H$ restricts to a functor $H : \mathcal{M}^{eCe}_f \to \mathcal{M}^C_f$ that preserves indecomposables and respects isomorphism classes. $H$ is also exact by hypothesis. Therefore the composition $HF : \mathcal{M}^f_{KQ} \to \mathcal{M}^C_f$ is an exact and faithful functor that preserves indecomposables and respects isomorphism classes. Thus $C$ is wild. □

A similar result may be obtained using the section functor if $S$ preserves finite dimensional comodules. For example, if $C$ is left semiperfect.

**Proposition 4.3.2.** *Let $C$ be a left semiperfect coalgebra and $e \in C^*$ be an idempotent which defines a perfect localization. If $eCe$ is wild then $C$ is wild.*

*Proof.* It is analogous to the proof of the former proposition. It is enough to show that if $C$ is left semiperfect then $S$ preserves finite dimensional comodules. Let $M$ be a finite dimensional right $eCe$-comodule. Then $\operatorname{soc} M \subseteq M$ is finite dimensional. Suppose that $\operatorname{soc} M = S_1 \oplus \cdots \oplus S_n$, then $\overline{E}_1 \oplus \cdots \oplus \overline{E}_n$ is the injective envelope of $M$. Therefore $E_1 \oplus \cdots \oplus E_n = E$ is the injective envelope of $S(M)$. By hypothesis, $E$ is finite dimensional and thus $S(M)$ so is. □

Let us now consider the following question: when is the coalgebra $eCe$ a subcoalgebra of $C$? This is interesting for us because, by by Corollary 4.1.3, we obtain the following result:

**Proposition 4.3.3.** *Let $C$ be a coalgebra and $e \in C^*$ be an idempotent such that $eCe$ is a subcoalgebra of $C$. If $eCe$ is wild then $C$ is wild.*

In general, we always have the inclusion $eCe \subseteq C$, nevertheless the coalgebra structures may be different. This is not the case if, for instance, $e$ is a left semicentral idempotent. In that case, by [JMNR06], $eC = eCe$ is a subcoalgebra of $C$. The same result holds if $e$ is a right semicentral or a central idempotent.

An idempotent $e \in C^*$ is said to be **split** if in the decomposition $C^* = eC^*e \oplus eC^*f \oplus fC^*e \oplus fC^*f$, where $e + f = 1$, the direct summand $H_e := eC^*f \oplus fC^*e \oplus fC^*f$ is a twosided ideal of $C^*$. These elements were used by Lam in [Lam06]. The main result there, see [Lam06, Theorem 4.5], assures that the following statements are equivalent:

$(a)$ $H_e$ is a twosided ideal of $C^*$.

$(b)$ $e(C^*fC^*)e = 0$.

$(c)$ $exeye = exye$ for any $x$, $y \in C^*$.

As a consequence, every left or right semicentral idempotent in $C^*$ is split. Let us characterize when $eCe$ is a subcoalgebra of $C$.

**Theorem 4.3.4.** *Let $e$ be an idempotent in $C^*$. Then the following statements are equivalent.*

$(a)$ *$e$ is a split idempotent in $C^*$.*

$(b)$ *$eCe$ is a subcoalgebra of $C$.*

*Proof.* Let us denote $f = 1 - e$. By Proposition 2.1.2, for any subspace $V \subseteq C$, $V$ is a subcoalgebra of $C$ if and only if $V^{\perp}$ is a twosided ideal of $C^*$. Then we proceed as follows in order to compute the orthogonal of $eCe$.

$$\begin{aligned}
(eCe)^{\perp} &= (eC \cap Ce)^{\perp} \\
&= (eC)^{\perp} + (Ce)^{\perp} \\
&= C^*f + fC^* \\
&= eC^*f + fC^*f + fC^*e + fC^*f \\
&= eC^*f + fC^*e + fC^*f \\
&= H_e
\end{aligned}$$

Thus $eCe$ is a subcoalgebra of $C$ if and only if $H_e$ is a twosided ideal of $C^*$ if and only if $e$ is a split idempotent in $C^*$. $\square$

Let us give a description of the split idempotents. Suppose that $C$ is a pointed coalgebra, that is, $C$ is an admissible subcoalgebra of a path coalgebra. We recall from Theorem 3.5.2 and Proposition 3.5.6 that left (right) semicentral idempotents can be described as follows.

**Proposition 4.3.5.** *Let $C$ be an admissible subcoalgebra of a path coalgebra $KQ$ and $e_X$ be the idempotent in $(KQ)^*$ associated to a subset $X \subseteq Q_0$. Then:*

(a) *$e_X$ is left semicentral if and only if there is no arrow $y \to x$ in $Q$ such that $y \notin X$ and $x \in X$.*

(b) *$e_X$ is right semicentral if and only if there is no arrow $x \to y$ in $Q$ such that $y \notin X$ and $x \in X$.*

We want to give a geometric description of the split idempotents in a similar way. In order to do this, we start giving an approach by means of path coalgebras.

**Lemma 4.3.6.** *Let $Q$ be a quiver and $e_X \in (KQ)^*$ be the idempotent associated to a subset of vertices $X$. Then $e_X$ is split in $(KQ)^*$ if and only if $I_p \subseteq X$ for any path $p$ in $e_X(KQ)e_X$, i.e., there is no cell in $Q$ relative to $X$ of length greater than one.*

*Proof.* Note that $e_X(KQ)e_X$ is a subcoalgebra of $KQ$ if and only if $\Delta(p) \in e_X(KQ)e_X \otimes e_X(KQ)e_X$ for any path $p$ in $e_X(KQ)e_X$.

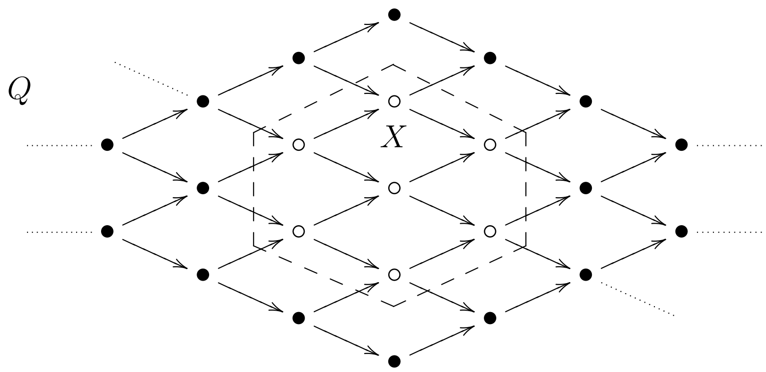
Let $p = \alpha_n \cdots \alpha_1 \in e_X(KQ)e_X$, $\Delta(p) \in e_X(KQ)e_X \otimes e_X(KQ)e_X$ if and only if

$$\sum_{j=2}^{n} \alpha_n \cdots \alpha_j \otimes \alpha_{j-1} \cdots \alpha_1 \in e_X(KQ)e_X \otimes e_X(KQ)e_X.$$

Since all summands are linearly independent, this happens if and only if $s(\alpha_i) \in X$ for all $i = 1, \ldots n-1$. That is, if and only if $I_p \subseteq X$. $\square$

Note that Lemma 4.3.6 asserts that the vertices associated to a split idempotent is a convex set of vertices in the quiver $Q$. Then it is really easy to decide whether or not $e_X(KQ)e_X$ is a subcoalgebra of $KQ$.

**Example 4.3.7.** *Let $Q$ be the following quiver:*




*Then the idempotent associated to the set of white vertices $X$ is a split idempotent.*

The proof of Lemma 4.3.6 easily extends to pointed coalgebras. Recall that we denote by $\mathcal{Q}$ the set of all paths in $Q$.

**Lemma 4.3.8.** *Let $Q$ be a quiver and $C$ be an admissible subcoalgebra of $KQ$. Let $e_X \in C^*$ be the idempotent associated to a subset of vertices $X$. Then $e_X$ is split in $C^*$ if and only if $I_p \subseteq X$ for any path $p$ in* $\mathrm{PSupp}(e_X C e_X)$.

**Example 4.3.9.** *Let $Q$ be the quiver*

$$\textcircled{1} \xrightarrow{\alpha} \textcircled{2} \xrightarrow{\beta} \textcircled{3}$$

*and $C$ be the admissible subcoalgebra of $KQ$ generated by $\{1,2,3,\alpha,\beta\}$. Then $e \equiv \{1,3\}$ is a split idempotent because $eCe = S_1 \oplus S_3$ is a subcoalgebra of $C$.*

Let us finish the section with an open problem for further development of representation theory of coalgebras.

**Problem 4.3.10.** *Let $C$ be a coalgebra and $e \in C^*$ be an idempotent. If $eCe$ is of wild comodule type then $C$ is of wild comodule type.*

Obviously, Problem 4.2.12 and Problem 4.3.10 are equivalent if the *tame-wild dichotomy* for coalgebras, conjectured by Simson in [Sim05], is true.

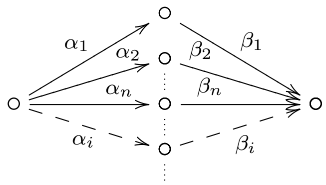
## 4.4 A theorem of Gabriel for coalgebras

In Chapter 2 we intent to prove that every basic coalgebra, over an algebraically closed field, is the path coalgebra $C(Q,\Omega)$ of a quiver $Q$ with a set of relations $\Omega$. This is an analogue for coalgebras of Gabriel's theorem. The result is proven in [Sim05] for the family of coalgebras $C$ such that the Gabriel quiver $Q_C$ of $C$ is intervally finite. Unfortunately, that proof does not hold for arbitrary coalgebras, as a class of counterexamples given in Section 2.5 shows.

Nevertheless, it is worth noting that these counterexamples are of wild comodule type in the sense described in Section 4.1. Since, according to Theorem 2.5.11, the coalgebras $C$ which are not the path coalgebra of quiver with relations are close to be wild, we may reformulate the problem stated in [Sim01, Section 8] as follows: *any basic tame coalgebra $C$, over an algebraically closed field, is isomorphic to the path coalgebra $C(Q,\Omega)$ of a quiver with relations* $(Q,\Omega)$. This section is devoted to solve this problem when the Gabriel quiver $Q_C$ of $C$ is assumed to be acyclic.

Now we show that the coalgebra $H$ of Example 2.5.5 is of wild comodule type.

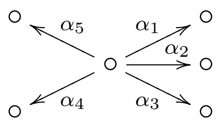
**Example 4.4.1.** *Let $Q$ be the quiver*




$\gamma_i = \beta_i\alpha_i$ *for all* $i \in \mathbb{N}$

*and suppose that $H$ is the admissible subcoalgebra of $KQ$ generated by the set $\Sigma = \{\gamma_i - \gamma_{i+1}\}_{i\in\mathbb{N}}$. It is proved in Example 2.5.5 that $H$ is not the path coalgebra of a quiver with relations. Nevertheless, $H$ is of wild comodule type because it contains the path coalgebra of the quiver*

$\Gamma \equiv$

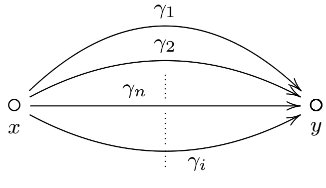



*Since $K\Gamma$ is a finite dimensional coalgebra then we have an algebra isomorphism $(K\Gamma)^* \cong A$, where $A$ is the path algebra of the quiver $\Gamma$, and there exists an equivalence between the category of finite dimensional right $A$-modules and the category of finite dimensional right $K\Gamma$-comodules. But it is well-known that $A$ is a wild algebra and hence $K\Gamma$ is a wild coalgebra. By Corollary 4.1.3, it follows that $H$ is wild.*

**Theorem 4.4.2.** *Let $K$ be an algebraically closed field, $Q$ be an acyclic quiver and $C$ be an admissible subcoalgebra of $KQ$ which is not the path coalgebra of a quiver with relations. Then $C$ is of wild comodule type.*

*Proof.* By Corollary 2.5.11, since $C$ is not the path coalgebra of a quiver with relations, there exists an infinite number of paths $\{\gamma_i\}_{n\in\mathbb{N}}$ in $Q$ between two vertices $x$ and $y$ such that:

- None of them is in $C$.
- $C$ contains a set $\Sigma = \{\Sigma_n\}_{n\in\mathbb{N}}$ such that $\Sigma_n = \gamma_n + \sum_{j>n} a_j^n \gamma_j$, where $a_j^n \in K$ for all $j, n \in \mathbb{N}$.




Consider $\mathrm{PSupp}(\Sigma_1 \cup \Sigma_2 \cup \Sigma_3) = \{\gamma_1, \gamma_2, \ldots, \gamma_t\}$ and $\Gamma$ the finite subquiver of $Q$ formed by the paths $\gamma_i$ for $i = 1, \ldots, t$.

Then $D = K\Gamma \cap C$ is a finite dimensional subcoalgebra of $C$ (and an admissible subcoalgebra of $K\Gamma$) which contains the elements $\Sigma_1$, $\Sigma_2$ and $\Sigma_3$. It is enough to prove that $D$ is wild.

Consider the idempotent element $e \in D^*$ such that $e(x) = e(y) = 1$ and zero otherwise, i.e., its associated subset of vertices is $X = \{x, y\}$. Then, by Lemma 3.6.3, each $\Sigma_i$ corresponds to an arrow from $x$ to $y$ in the quiver $\Gamma^e$, that is, $\Gamma^e$ contains the subquiver $\circ \mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}} \circ$ and then $\dim_K \mathrm{Ext}^1_{eDe}(S_x, S_y) \geq 3$. Thus $eDe = K\Gamma^e$ is wild by [Sim05, Corollary 5.5]. Note also that the quiver $\Gamma^e$ is of the form

$$\begin{tikzcd} \underset{x}{\circ} \arrow[r, shift left=3, "\alpha_1"] \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\alpha_n"'] & \underset{y}{\circ} \end{tikzcd}$$

an then the simple right $eDe$-comodule $S_x$ is injective.

Let us prove that the localizing subcategory $\mathcal{T}_e$ of $\mathcal{M}^D$ is perfect colocalizing. Since $\Gamma$ is finite and and acyclic then $\dim_K K\mathcal{T}ail^{\Gamma}_X(x)$ is finite and $\dim_K K\mathcal{T}ail^{\Gamma}_X(y) = 0$ so, by Proposition 3.6.8, the subcategory $\mathcal{T}_e$ is colocalizing. Let now $g$ be an element in $eC(1-e)$. Then $g$ is a linear combination of tails starting at $x$ and then $\rho_{eC(1-e)}(g) = g \otimes x$ (see the proof of Proposition 3.6.8). Therefore $< g > \cong S_x$ as right $eCe$-comodules. Suppose that $m = \dim_K eC(1-e)$. Hence $eC = eCe \oplus eC(1-e) = eCe \oplus S_x^m$ and $eC$ is an injective right $eCe$-comodule. Thus the colocalization is perfect and, by Corollary 4.3.1, $D$ is wild. $\square$

Now we are able to extend [Sim05, Theorem 3.14 (c)], from the case $Q$ is an intervally finite quiver and $C' \subseteq KQ$ is an arbitrary admissible subcoalgebra to the case $Q$ is acyclic and $C' \subseteq KQ$ is tame, as follows.

**Corollary 4.4.3** (Acyclic Gabriel's theorem for coalgebras)**.** *Let $Q$ be an acyclic quiver and let $K$ be an algebraically closed field.*

(*a*) *Any tame admissible subcoalgebra $C'$ of the path coalgebra $KQ$ is isomorphic to the path coalgebra $C(Q, \Omega)$ of a quiver with relation $(Q, \Omega)$.*

(*b*) *The map $\Omega \mapsto C(Q, \Omega)$ defines a one-to-one correspondence between the set of relation ideals $\Omega$ of the path $K$-algebra $KQ$ and the set of admissible subcoalgebras $H$ of the path coalgebra $KQ$. The inverse map is given by $H \mapsto H^{\perp}$.*

*Proof.* Apply Theorem 4.4.2 and the weak tame-wild dichotomy proved by Simson in [Sim05]. $\square$

**Remark 4.4.4.** *It follows from the proof of Theorem 4.4.2 that if $Q$ is acyclic, a basis of a tame admissible subcoalgebra $C$ cannot contain three linearly independent combinations of paths with common source and common* *sink and then* $C = KQ_0 \bigoplus \bigoplus_{x \neq y \in Q_0} C_{xy}$ *with* $\dim_K C_{xy} \leq 2$ *for all* $x, y \in Q_0$. *Nevertheless, this fact does not imply that the quiver is intervally finite (the number of paths between two vertices is finite). It is enough to consider the quiver*

$$\begin{tikzcd}
\cdot \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & {} \arrow[r, dotted, dash] \arrow[d, dotted] & {} \\
\cdot & \cdot \arrow[l] & \cdot \arrow[l] & \cdot \arrow[l] & {} \arrow[l] \arrow[r, dotted, dash] & {}
\end{tikzcd}$$

*and the admissible coalgebra* $C = C(Q, \Omega)$, *where* $\Omega$ *is the ideal*

$$\Omega = KQ_2 \oplus KQ_3 \oplus \cdots \oplus KQ_n \oplus \cdots .$$

$C$ *is a string coalgebra and then it is tame (see [Sim05, Section 6]).*

# Chapter 5

# Some examples

In this final chapter we look into certain kind of coalgebras by applying the topics developed all along this work. Mainly, we care for hereditary and serial coalgebras. Hereditariness is introduced in [NTZ96] following the standard categorical notion, namely, a coalgebra is hereditary if its comodules have global dimension less or equal than one. Therefore, from a homological point of view, this is the simplest case (we may omit semisimple coalgebras since its representation theory is trivial). Since, in view of the previous chapters, pointed coalgebras are of importance, then path coalgebras are treated extensively. With respect to serial coalgebras, a systematic study of them is initiated in [CGT04], where, in particular, it is shown that any serial indecomposable coalgebra over an algebraically closed field is Morita-Takeuchi equivalent to a subcoalgebra of a path coalgebra of a quiver which is either a cycle or a chain (finite or infinite). In this chapter we take advantage of the valued Gabriel quivers associated to a coalgebra to characterize indecomposable serial coalgebras over any field. Also we would like to highlight a version for coalgebras of the theorem of Eisenbud and Griffith which states that any proper quotient of a hereditary noetherian prime ring is serial.

## 5.1 Hereditary coalgebras

We recall from [NTZ96] that a coalgebra $C$ is said to be **right hereditary** if, for each subcomodule $N$ of an injective right $C$-comodule $E$, the quotient $E/N$ is an injective right $C$-comodule. We collect here some known characterizations of a right hereditary coalgebra, see [Chi02] and [JLMS05].

**Theorem 5.1.1.** *Let $C$ be a coalgebra. The following conditions are equivalent:*

(*a*) *$C$ is right hereditary.*

(*b*) *The global dimension of $C$ is less or equal than one.*

(*c*) *The injective dimension of any simple right $C$-comodule is less or equal than one.*

(*d*) *$C/N$ is an injective right $C$-comodule for each right coideal $N$.*

(*e*) *$C/S$ is an injective right $C$-comodule for any simple right $C$-comodule $S$.*

(*f*) *$C$ is left hereditary.*

*If the coradical $C_0$ of $C$ is coseparable, these conditions are also equivalent to*

(*g*) *$C$ is formally smooth.*

(*h*) *The global dimension of the enveloping coalgebra $C \otimes C^{cop}$ is less or equal than 1.*

(*i*) Coker $\Delta$ *is an injective $(C, C)$-bicomodule, where $\Delta$ is the comultiplication of $C$.*

(*j*) *$C$ is isomorphic to the tensor coalgebra $T_{C_0}(N)$, where $N$ is the injective $(C, C)$-bicomodule $\frac{C_0 \wedge C_0}{C_0}$.*

*Furthermore, if $C$ is pointed then these conditions are equivalent to*

(*k*) *$C$ is isomorphic to the path coalgebra $KQ$ of a quiver $Q$.*

*Proof.* $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (f)$ can be found in [NTZ96]. $(a) \Leftrightarrow (g) \Leftrightarrow (h) \Leftrightarrow (i) \Leftrightarrow (j)$ is proved in [JLMS05]. Finally, $(a) \Leftrightarrow (k)$ appears in [Chi02]. □

**Corollary 5.1.2.** *The notion of hereditary coalgebra is left-right symmetric.*

Let $C$ be a coalgebra. A right $C$-comodule $M$ is called **colocal** if soc $M$ is a simple right $C$-comodule.

**Proposition 5.1.3.** *Let $C$ be a coalgebra. The following conditions are equivalent:*

(*a*) *$C/D$ is an injective right $C$-comodule for any colocal right coideal $D$ of $C$.*

(*b*) *Every quotient of an indecomposable injective right $C$-comodule is injective.*

(c) *Every quotient of an injective right $C$-comodule by a colocal subcomodule is injective.*

(d) *Every quotient of an indecomposable injective right $C$-comodule by its socle is injective.*

(e) *$C/S$ is an injective right $C$-comodule for any simple right $C$-comodule $S$.*

(f) *$C$ is right hereditary.*

*Proof.* $(a) \Rightarrow (b)$. Let $E_i$ be an indecomposable injective right $C$-comodule and $N$ be a subcomodule of $E_i$. Since $\operatorname{soc} N \subseteq \operatorname{soc} E_i = S_i$ then $N$ is a colocal right coideal of $C$. Therefore $C/N \cong E_i/N \oplus (\oplus_{k \neq i} E_k)$ is injective and thus so is $E_i/N$.

$(b) \Rightarrow (c)$. Let $E$ be an injective right $C$-comodule and $N$ be a colocal subcomodule of $E$. Then $\operatorname{soc} N = S_i$ and $N$ have the same injective envelope, say $E_i$, and there exists a monomorphism $f : E_i \to E$. Now, the exact sequence $E_i \xrightarrow{f} E \xrightarrow{p} E/E_i = E'$ splits so $E = E_i \oplus E'$ and $E'$ is injective. Thus $E/N \cong E_i/N \oplus E'$ is injective.

$(c) \Rightarrow (d)$. Trivial.

$(d) \Rightarrow (e)$. Let $S_i$ be a simple $C$-comodule and $E_i$ be its injective envelope, that is, $E_i$ is an indecomposable injective right $C$-comodule and $\operatorname{soc} E_i = S_i$. Then $C/S_i \cong E_i/\operatorname{soc} E_i \oplus (\oplus_{j \neq i} E_j)$ and thus $C/S_i$ is a direct sum of injective right $C$-comodules.

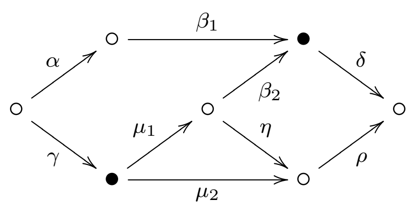
$(e) \Rightarrow (f)$. It is proved in Proposition 5.1.1.

$(f) \Rightarrow (a)$. Trivial □

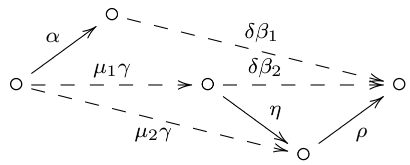
### 5.1.1 Localization in path coalgebras

Let us now assume $C$ is a pointed coalgebra. Then, by Corollary 1.5.6 and Theorem 5.1.1, $C$ is the path coalgebra of a quiver $Q$. We recall from Chapter 3 that, for any idempotent $e_X$ in $(KQ)^*$ associated to a subset of vertices $X \subset Q_0$, the localized coalgebra $e_X(KQ)e_X$ is the path coalgebra of the quiver $Q^{e_X} = (X, \mathcal{C}ell_X^Q)$. Here we present the following examples.

**Example 5.1.4.** *Let $KQ$ the path coalgebra of the quiver $Q$ given by*

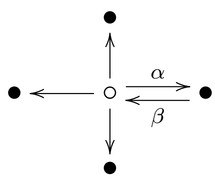


*and $X$ be the subset of vertices formed by the white points. Then, the set of cells is $\{\alpha, \eta, \rho, \delta\beta_1, \delta\beta_2, \mu_1\gamma, \mu_2\gamma\}$. Therefore the quiver $Q^{e_X}$ is the following:*

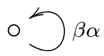



*where the dashed arrows are the cells of length greater than one.*

**Example 5.1.5.** *Let $KQ$ be the path coalgebra of the quiver $Q$*




*and $X$ be the set of vertices formed by the only white point. Then the set of cells is $\{\beta\alpha\}$, that is, the quiver $Q^{e_X}$ is*




*and $e_X(KQ)e_X \cong K[\beta\alpha]$.*

We apply this idea in order to obtain a description of the quotient functor $T : \mathcal{M}^{KQ} \to \mathcal{M}^{KQ^e}$. Recall that the category of right $KQ$-comodules is equivalent to the category $\mathrm{Rep}_K^{\mathrm{lnlf}}(Q)$ of locally nilpotent representations of finite length of the quiver $Q$.

**Proposition 5.1.6.** *Let $KQ$ be a path coalgebra and $e_X \in (KQ)^*$ be the idempotent associated to a subset of vertices $X$. Then, the functor $T$ : $\mathrm{Rep}^{\mathrm{lnlf}}(Q) \longrightarrow \mathrm{Rep}^{\mathrm{lnlf}}(Q^{e_X})$ maps the representation $(V_x, \varphi_\alpha)_{x\in Q_0, \alpha\in Q_1}$ of $Q$ into the representation $(\bar{V}_y, \bar{\varphi}_\beta)_{y\in X, \beta\in \mathcal{C}ell_X^Q}$ of $Q^{e_X}$ given by:*

- $\bar{V}_x = V_x$ *for every $x \in X$.*
- $\bar{\varphi}_\beta = \varphi_{\alpha_n} \cdots \varphi_{\alpha_1}$ *for each $\beta \in \mathcal{C}ell_X^Q$ such that $\beta = \alpha_n\alpha_{n-1} \cdots \alpha_1$ in $Q$.*

In Chapter 3 we prove that, for a subset of vertices $X$ of a quiver $Q$, the localizing subcategory $\mathcal{T}_X$ is colocalizing if and and if $\mathcal{T}ail_X^Q(x)$ is a finite set for each $x \in X$. Let us now prove that, under this conditions, the colocalizing subcategories are also perfect colocalizing.

**Theorem 5.1.7.** *Let $Q$ be a quiver and $e_X$ be the idempotent in $(KQ)^*$ associated to a subset $X \subseteq Q_0$. Then*

$$e_X(KQ) \cong \bigoplus_{x \in X} \overline{E}_x^{\mathrm{Card}(\mathcal{T}ail_X^Q(x))+1}$$

*as right $KQ^{e_X}$-comodules, where $\{\overline{E}_x\}_{x \in X}$ is a complete set of pairwise non-isomorphic indecomposable injective right $KQ^{e_X}$-comodules.*

*Proof.* The right $KQ^{e_X}$-comodule $e_X(KQ)$ may be decomposed as $e_X(KQ) = e_X(KQ)e_X \oplus e_X(KQ)(1 - e_X)$. Since there are isomorphisms $e_X(KQ)e_X \cong KQ^{eX} \cong \bigoplus_{x \in X} \overline{E}_x$, it is enough to prove that

$$e_X(KQ)(1 - e_X) \cong \bigoplus_{x \in X} E_x^{\mathrm{Card}(\mathcal{T}ail_X^Q(x))}$$

as right $KQ^{e_X}$-comodules.

Let us assume that, for each $x \in X$, we have

$$\mathcal{T}ail_X^Q(x) = \{\tau_x^i\}_{i \in J_x}.$$

The $K$-vector space $e_X(KQ)(1 - e_X)$ is generated by the set of all paths starting at vertices in $X$ and ending at vertices which do not belong to $X$. Then, for any path $p \in e_X(KQ)(1 - e_X)$, there exists a unique tail decomposition $p = \tau_x^i p_n \cdots p_1$ for some $\tau_x^i \in \mathcal{T}ail_X^Q(x)$, where $x = t(p_n) \in X$.

We consider the linear map

$$f : e_X(KQ)(1 - e_X) \longrightarrow \bigoplus_{x \in X}(\bigoplus_{i \in J_x} \overline{E}_{x,i})$$

defined by $f(\tau_x^i p_n \cdots p_1) = p_n \cdots p_1 \in \overline{E}_{x,i}$ for all $p = \tau_x^i p_n \cdots p_1 \in e_X(KQ)$. Clearly $f$ is well defined and it is a $e_X(KQ)e_X$-comodule map. Since $\overline{E}_x$ is generated by the set of all paths in $Q^{e_X}$ which end at $x$, $f$ is bijective. $\square$

**Corollary 5.1.8.** *Let $Q$ be a quiver and $e_X$ be the idempotent in $(KQ)^*$ associated to a subset $X \subseteq Q_0$. Then $e_X(KQ)$ is an injective right $KQ^{e_X}$-comodule.*

**Theorem 5.1.9.** *Let $Q$ be a quiver and $e_X$ be the idempotent in $(KQ)^*$ associated to a subset $X \subseteq Q_0$. The following conditions are equivalent:*

(*a*) *The localizing subcategory $\mathcal{T}_X$ of $\mathcal{M}^{KQ}$ is colocalizing.*

(*b*) *The localizing subcategory $\mathcal{T}_X$ of $\mathcal{M}^{KQ}$ is perfect colocalizing.*

(c) *$\mathcal{T}ail_X^Q(x)$ is a finite set for all $x \in X$. That is, there are at most a finite number of paths starting at the same point whose only vertex in $X$ is the first one.*

We remark that since any path coalgebra is hereditary, the equivalence between (a) and (b) in the previous Theorem can be obtained from [NT96]. Here we show that this property does not hold for an arbitrary pointed coalgebra.

**Example 5.1.10.** *Let us consider the quiver $Q$ and the coalgebra $C$ defined on Example 3.6.5. Let $X$ be the subset of vertices $X = \{1, 2, 3\}$. Then, $e_X C e_X$ is the path coalgebra of the quiver $Q^{e_X}$*

$$2 \xleftarrow{\alpha_1} 1 \xrightarrow{\alpha_3} 3$$

*and then, the indecomposable injective right $e_X C e_X$-comodules are $E_1 = K\langle 1\rangle$, $E_2 = K\langle 2, \alpha_1\rangle$ and $E_3 = K\langle 3, \alpha_3\rangle$. If $e_X C(1-e_X) = K\langle \alpha_2, \alpha_4, \alpha_2\alpha_1 + \alpha_4\alpha_3\rangle$ were injective then it would be a sum of indecomposable injective right $e_X C e_X$-comodules. Since $e_X C(1-e_X)$ has dimension 3, thus it would be isomorphic to $E_1 \oplus E_1 \oplus E_1$, or $E_1 \oplus E_2$ or $E_1 \oplus E_3$. A straightforward calculation shows that this is not possible.*

### 5.1.2 Tame and wild path coalgebras

By [Gab62, Chapter III, Proposition 7], any localizing subcategory of a category of comodules over a path coalgebra is perfect localizing. Then, from the above results and the ones obtained in Chapter 4, we have the following:

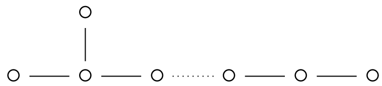
**Proposition 5.1.11.** *Let $Q$ be a quiver and $e$ be an idempotent of $(KQ)^*$ such that $\mathcal{T}_e$ is a colocalizing subcategory of $\mathcal{M}^{KQ}$. If $KQ^e$ is wild then $KQ$ is wild.*

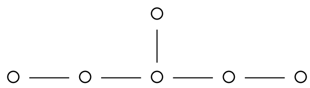
**Proposition 5.1.12.** *Let $Q$ be a quiver and $e$ be an idempotent of $(KQ)^*$ such that the section functor $S : \mathcal{M}^{KQ^e} \to \mathcal{M}^{KQ}$ preserves finite dimensional comodules. If $KQ^e$ is wild then $KQ$ is wild.*

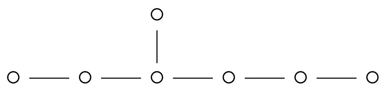
Following [Sim01], we finish the section giving a complete description of all tame path coalgebras. First we show the list the of all Dynkin diagrams, Euclidean graphs and infinite locally Dynkin diagrams:

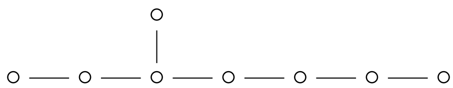
**Dynkin diagrams**

$\mathbb{A}_n$ :  $n$ vertices, $n \geq 1$

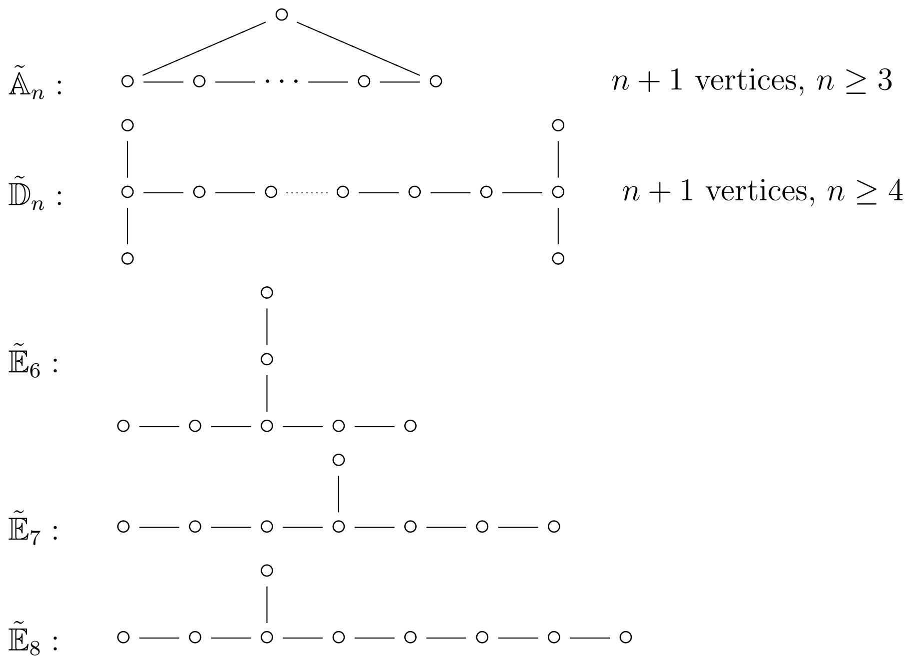
$\mathbb{D}_n$ :  $n$ vertices, $n \geq 4$

$\mathbb{E}_6$ : 

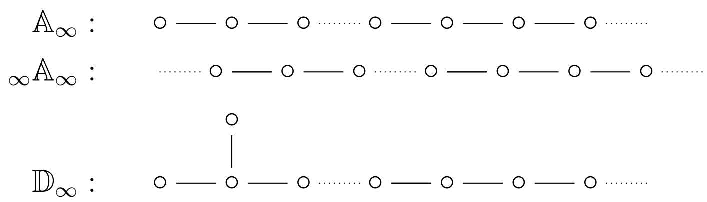
$\mathbb{E}_7$ : 

$\mathbb{E}_8$ : 

**Euclidean graphs**




**Infinite locally Dynkin diagrams**




**Theorem 5.1.13.** *[Sim01] Let $Q$ be a quiver and $KQ$ be the path coalgebra of $Q$. The following conditions are equivalent:*

$(a)$ *$KQ$ is of tame comodule type.*

$(b)$ *$KQ$ is domestic of tame comodule type.*

$(c)$ *The underlying graph of $Q$ is a Dynkin diagram, or a Euclidean graph or a infinite locally Dynkin diagram.*

$(d)$ *$KQ$ is not of wild comodule type.*

Therefore the tame-wild dichotomy holds for this kind of coalgebras.

## 5.2 Colocally hereditary coalgebras

The aim of this section is to introduce a generalization of hereditariness: colocally hereditariness. This notion comes from the one for algebras introduced by Simson in the Representation Theory seminar of the Nikołaja Kopenika University of Toruń in 1977. An algebra is said to be right **l-hereditary**, or right **locally hereditary**, if any local right ideal is projective, see [Les04] for a description of all tame locally hereditary algebras of finite dimension. We list some equivalent conditions for finite dimensional algebras.

**Theorem 5.2.1.** *[Les04] Let $A$ be a finite dimensional algebra. The following conditions are equivalent:*

($a$) *$A$ is right l-hereditary.*

($b$) *Any local submodule of a projective right $A$-module is projective.*

($c$) *Any nonzero morphism between indecomposable projective right $A$-modules is a monomorphism.*

($d$) *$A$ is left l-hereditary.*

($e$) *$A$ is isomorphic to*

$$A' = \begin{pmatrix} F_1 & & & \\ {}_2M_1 & F_2 & 0 & \\ \cdots & \cdots & & \\ {}_nM_1 & {}_nM_2 & \cdots & F_n \end{pmatrix},$$

*where $(F_i, {}_iM_j)$ is a $K$-species such that there exist $F_i$-$F_j$-homomorphisms $c_{ijk} : {}_iM_j \otimes_{F_j} {}_jM_k \to {}_iM_k$ verifying that $c_{ijk}(x \otimes y) = 0$ if and only if $x = 0$ or $y = 0$ for all $i, j, k = 1, \ldots, n$.*

When dealing with coalgebras, we may give a dual notion of l-hereditariness as follows. A coalgebra $C$ is said to be right **cl-hereditary** or right **colocally hereditary** if, for any right coideal $N$ such that $C/N$ is colocal, the quotient $C/N$ is injective. The following lemma is obvious.

**Lemma 5.2.2.** *Any hereditary coalgebra is right and left cl-hereditary.*

Let us give some characterizations of a cl-hereditary coalgebra following the style of Theorem 5.2.1.

**Proposition 5.2.3.** *Let $C$ be a coalgebra. The following conditions are equivalent:*

(a) *For any subcomodule $N$ of a injective right $C$-comodule $E$ such that $E/N$ is colocal, the quotient $E/N$ is an injective right $C$-comodule.*

(b) *For any nonzero morphism $f : E \to F$, where $E$ and $F$ are right $C$-comodules such that $E$ is injective and $F$ is colocal,* $\operatorname{Im} f$ *is an injective right $C$-comodule.*

(c) *Every nonzero morphism between indecomposable injective right $C$-comodules is surjective.*

(d) *$C$ is right cl-hereditary.*

(e) *For any subcomodule $N$ of an indecomposable injective right $C$-comodule $E_i$ such that $E_i/N$ is colocal, the quotient $E_i/N$ is an injective right $C$-comodule.*

*Proof.* $(a) \Rightarrow (b)$. We have $E/\operatorname{Ker} f \cong \operatorname{Im} f$ and $\operatorname{soc}(\operatorname{Im} f) \subseteq \operatorname{soc} F$. Therefore $\operatorname{soc}(\operatorname{Im} f)$ is simple and, by hypothesis, $\operatorname{Im} f$ is injective.

$(b) \Rightarrow (c)$. Let $f : E_i \to E_j$ be a nonzero morphism between indecomposable injective right $C$-comodules. Since $\operatorname{soc} E_j$ is simple, $\operatorname{Im} f$ is injective. Therefore, the short exact sequence

$$0 \longrightarrow \operatorname{Im} f \longrightarrow E_j \longrightarrow E_j/\operatorname{Im} f \longrightarrow 0$$

splits and $E_j = \operatorname{Im} f \oplus E_j/\operatorname{Im} f$. Since $E_j$ is indecomposable and $f$ is nonzero, we deduce $E_j/\operatorname{Im} f = 0$.

$(c) \Rightarrow (d)$. Let $N$ be a right coideal such that $\operatorname{soc}(C/N) = S$ is simple. Let $E$ be the injective envelope of $S$. Then $f : C/N \to E$ is also the injective envelope of $C/N$. Therefore, there exists an index $k \in I_C$ such that the composition $E_k \xhookrightarrow{i} C = \oplus_{i\in I_C} E_i \xrightarrow{p} C/N$ is nonzero, where $i$ is the inclusion and $p$ is the projection. Then $fpi$ is surjective and so is $f$.

$(d) \Rightarrow (e)$. Let $N \leq E_i$ such that $\operatorname{soc}(E_i/N)$ is simple. Let us consider the right coideal $N' = N \oplus (\oplus_{j\neq i} E_j)$. Then $C/N' \cong E_i/N$ has simple socle and thus $E_i/N$ is injective.

$(c) \Rightarrow (a)$. It is similar to the proof of $(c) \Rightarrow (d)$.

$(e) \Rightarrow (c)$. Let $f : E_i \to E_j$ be a nonzero morphism. We have $\operatorname{Im} f \cong E_i/\operatorname{Ker} f$. Since $\operatorname{Im} f$ has simple socle, by hypothesis, it is injective and the result follows as in $(b) \Rightarrow (c)$. □

**Problem 5.2.4.** *Is the notion of cl-hereditariness left-right symmetric?*

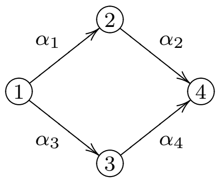
Let us show that cl-hereditary coalgebras are a generalization of finite dimensional l-hereditary algebras.

**Lemma 5.2.5.** *If $R$ is a finite dimensional right l-hereditary algebra then $R^*$ is a right cl-hereditary coalgebra. Conversely, if $C$ is a right cl-hereditary coalgebra of finite dimension, $C^*$ is a l-hereditary algebra.*

*Proof.* Let $f : E_i \to E_j$ be a nonzero morphism between indecomposable injective right $R^*$-comodules. The dual morphism $f^* : E_j^* \to E_i^*$ is a non-zero morphism between indecomposable projective right $R$-modules. By hypothesis, $f^*$ is a monomorphism and then $f$ is surjective. The converse result is similar. □

The following example shows that there exist cl-hereditary coalgebras which are not hereditary coalgebras.

**Example 5.2.6.** *Let $Q$ be the quiver*



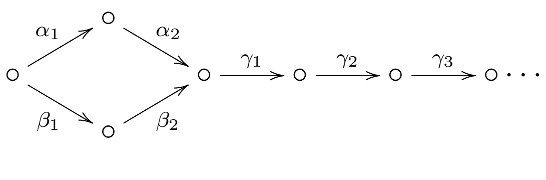

*and $C$ be the coalgebra generated by the set*

$$\{1, 2, 3, 4, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_2\alpha_1 + \alpha_4\alpha_3\}.$$

*The indecomposable injective right $C$-comodules are $E_1 = \langle 1\rangle$, $E_2 = \langle 2, \alpha_1\rangle$, $E_3 = \langle 3, \alpha_3\rangle$ and $E_4 = \langle 4, \alpha_2, \alpha_4, \alpha_2\alpha_1 + \alpha_4\alpha_3\rangle$. We consider the subcomodule $A = \langle 4\rangle$ of $E_4$ and therefore $E_4/A = \langle \overline{\alpha_2}, \overline{\alpha_4}, \overline{\alpha_2\alpha_1 + \alpha_4\alpha_3}\rangle$. It is easy to see that $\mathrm{soc}\,(E_4/A) = S_2 \oplus S_3$ and then the injective envelope $E(E_4/A) = E_2 \oplus E_3 \neq E_4/A$. Thus $E_4/A$ is not injective and $C$ is not hereditary. Nevertheless, a straightforward calculation shows that $C$ is cl-hereditary. We sum it up in the following table:*

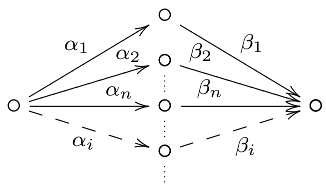
| | *subcomodules with colocal quotient* | *quotient* |
|---|---|---|
| $E_1$ | $\emptyset$ | — |
| $E_2$ | $< x_2 >$ | $E_1$ |
| $E_3$ | $< x_3 >$ | $E_1$ |
| $E_4$ | $< 4, \alpha_2 >$, $< 4, \alpha_4 >$, $< 4, \alpha_2, \alpha_4 >$ | $E_3$, $E_2$, $E_1$ |

**Example 5.2.7.** *The last example may be extended to an infinite dimensional coalgebra. Let $Q$ be the quiver*



*and $C$ be the subcoalgebra of $KQ$ generated by the set of vertices, the set of arrows and $\{\gamma_n \cdots \gamma_1(\alpha_2\alpha_1+\beta_2\beta_1)\}_{n\geq 0}$ $\{\gamma_n \cdots \gamma_1\alpha_2\}_{n\geq 1}$, $\{\gamma_n \cdots \gamma_1\beta_2\}_{n\geq 1}$ and $\{\gamma_i \cdots \gamma_j\}_{i>j\geq 1}$. Proceeding as above, we may prove that $C$ is a cl-hereditary coalgebra. On the other hand, since $C$ is not a path coalgebra, $C$ is not hereditary (see [JLMS05]).*

**Example 5.2.8.** *Consider the quiver $Q$*




$\gamma_i = \beta_i\alpha_i$ *for all* $i \in \mathbb{N}$

*and let $H$ be the admissible subcoalgebra of $KQ$ generated by the set $\Sigma = \{\gamma_i - \gamma_{i+1}\}_{i\geq 1}$. Then $H$ is a cl-hereditary non-hereditary coalgebra.*

**Remark 5.2.9.** *It follows from the above example that not every cl-hereditary coalgebra is the path coalgebra of a quiver with relations. Unlike it happens with hereditary coalgebras.*

As we show in Section 3.3.2, the paths in the Ext-quiver (in general, in the valued Gabriel quiver) are closely related to the morphism between indecomposable injective comodules. Moreover, that connection is rather good if we deal with hereditary coalgebras (cf. Corollary 3.3.14). For cl-hereditary coalgebras we may prove a partial result:

**Lemma 5.2.10.** *Let $C$ be a cl-hereditary coalgebra and $S$ and $S'$ two simple comodules. If there is a $n$-path in $\Gamma_C$ from $S$ to $S'$ then $S$ is a $t$-predecessor of $S'$ for some integer $t \geq n$. As a consequence, there is a path in $\Gamma_C$ from $S$ to $S'$ if and only if $S$ is a predecessor of $S'$.*

*Proof.* Let us assume that

$$S \longrightarrow S_1 \longrightarrow \cdots \longrightarrow S_{n-1} \longrightarrow S'$$

is a path in $\Gamma_C$. Then there exists a sequence of (irreducible) morphisms

$$E' \xrightarrow{f_n} E_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} E_1 \xrightarrow{f_1} E$$

such that $f_i(S_i) = 0$ and $f_i(\mathrm{soc}^2 E_i) \neq 0$ for all $i = 1, \ldots, n$, where $E_n = E'$ and $S_n = S'$. Since $C$ is cl-hereditary, any of these morphisms is surjective and then the composition $f_1 f_2 \cdots f_n$ so is. Thus there exists an integer $t$ such that $(f_1 f_2 \cdots f_n)(\mathrm{soc}^t E') \neq 0$. Finally, by Lemma 3.3.13, $(f_1 f_2 \cdots f_n)(\mathrm{soc}^n E') = 0$ and therefore $S$ is a $t$-predecessor of $S'$ for some $t \geq n$ $\square$

The following example shows that we cannot state the equality $t = n$ in Lemma 5.2.10 for an arbitrary cl-hereditary coalgebra, unlike hereditary coalgebras.

**Example 5.2.11.** *Let $Q$ be the quiver*

$$\begin{tikzcd}
 & \textcircled{2} \arrow[rr, "\alpha_2"] & & \textcircled{3} \arrow[dr, "\alpha_3"] & \\
\textcircled{1} \arrow[ur, "\alpha_1"] \arrow[drr, "\beta_1"'] & & & & \textcircled{5} \\
 & & \textcircled{4} \arrow[urr, "\beta_2"'] & &
\end{tikzcd}$$

*and let $C$ be the subcoalgebra of $KQ$ generated (as vector space) by the set*

$$\{1, 2, 3, 4, 5, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \alpha_2\alpha_1, \alpha_3\alpha_2, \alpha_3\alpha_2\alpha_1 + \beta_2\beta_1\}.$$

*Let us now $E_5$ be the indecomposable injective associated to the vertex 5,*

$$E_5 = \langle 5, \alpha_3, \beta_2, \alpha_3\alpha_2, \alpha_3\alpha_2\alpha_1 + \beta_2\beta_1\rangle.$$

*Then the socle filtration of $E_5$ is the following*

$$S_5 = \langle 5\rangle \subset \langle 5, \alpha_3, \beta_2\rangle \subset \langle 5, \alpha_3, \beta_2, \alpha_3\alpha_2\rangle \subset E_5,$$

*and the factors are $\mathrm{soc}^2 E_5/\mathrm{soc}\, E_5 \cong S_3 \oplus S_4$, $\mathrm{soc}^3 E_5/\mathrm{soc}^2 E_5 \cong S_2$ and $E_5/\mathrm{soc}^3 E_5 \cong S_1$. Therefore $S_3$ and $S_4$ are the 1-predecessors of $S_5$, $S_2$ is the unique 2-predecessor of $S_5$, and $S_1$ is the unique 3-predecessor of $S_5$. Nevertheless, there is a two-length path in $Q = \Gamma_C$ from $S_1$ to $S_5$.*

**Remark 5.2.12.** *Observe that Example 5.2.6 shows that there exist cl-hereditary (non hereditary) coalgebras verifying the property stated in Corollary 3.3.14, and therefore it does not characterize hereditariness.*

Let now $e \in C^*$ be an idempotent. Then we may consider the functors associated to the localization

$$\mathcal{M}^C \underset{S=-\square_{eCe}Ce}{\overset{T=e(-)=-\square_C eC}{\rightleftarrows}} \mathcal{M}^{eCe}.$$

As we show above, the localization process preserves hereditariness; actually, one could prove that the global dimension of $\mathcal{M}^{eCe}$ is less or equal than the global dimension of $\mathcal{M}^C$ (cf. [Gab62]). A converse result may be obtained using the localization on certain injective comodules. For the convenience we introduce the following notation. Let us define inductively several subsets of the set of indexes $I_C$ of the coalgebra $C$. Given some element $x \in I_C$, we denote the sets

$$I^0(x) \subseteq I^1(x) \subseteq \cdots \subseteq I^n(x) \subseteq \cdots \subseteq I^\infty(x)$$

as follows. We set $I^0(x) = \{x\}$ and, for any $n > 0$,

$$I^n(x) = \{y \in I_C \mid \text{there is } S_y \to S_z \text{ in } \Gamma_C \text{ for some } z \in I^{n-1}(x)\}.$$

Finally, we set $I^\infty(x) = \cup_n I^n(x)$. For any $x \in I_C$, we denote by $e_x$ the idempotent in $C^*$ associated to the set $I^\infty(x)$. We recall from [Sim07] that the quotient and the section functors define an equivalence of categories between $\mathcal{M}^{eCe}$ and the category $\mathcal{M}^C_E$ of $E$-**copresented** right $C$-comodules, where $E = Ce$, that is, the right $C$-comodules $M$ which admit an exact sequence

$$0 \longrightarrow M \longrightarrow E_0 \longrightarrow E_1 ,$$

where $E_0$ and $E_1$ are direct sums of direct summands of the comodule $E$.

**Lemma 5.2.13.** *Let $C$ be a coalgebra. $C$ is hereditary if and only if $e_xCe_x$ is hereditary for any $x \in I_C$.*

*Proof.* Assume that the coalgebra $e_xCe_x$ is hereditary for any $x \in I_C$. For any indecomposable injective comodule $E_x$, we consider the quotient $E_x/S_x$. It is clear that $E_x/S_x$ is $E$-copresented and then, by [Sim07], there exists an $e_xCe_x$-comodule $M$ such that $S(M) = E_x/S_x$. Moreover, $M = \overline{E}_x/S_x$ and, by hypothesis, $M$ is injective. Since $S$ preserves injective comodules, $E_x/S_x$ is an injective $C$-comodule. This proves that $C$ is hereditary. □

**Theorem 5.2.14.** *If the coalgebra $C$ is a right cl-hereditary then $eCe$ is right cl-hereditary for any idempotent $e \in C^*$. Moreover, if $e_xCe_x$ is cl-hereditary for any $x \in I_C$ then $C$ is cl-hereditary.*

*Proof.* Let $f : \overline{E}_i \to \overline{E}_j$ be a nonzero morphism between indecomposable injective right $eCe$-comodules. Then $S(f) : E_i \to E_j$ is a nonzero morphism between indecomposable injective right $C$-comodules. By hypothesis, $S(f)$ is surjective and since $T$ is exact, $TS(f) = f$ is surjective. The proof of the second statement is similar to the one of the above lemma. □

## 5.3 Serial coalgebras

Following [CGT04], a right comodule is called **uniserial** if its Loewy series is a composition series. This property can be characterized through the socle filtration:

**Lemma 5.3.1.** *[CGT04] The following statements are equivalent:*

*(a) $M$ is uniserial.*

*(b) The Loewy series is a composition series.*

*(c) Each finite dimensional subcomodule of $M$ is uniserial.*

A coalgebra $C$ is said to be **right (left) serial** if any indecomposable injective right (left) $C$-comodule is uniserial. $C$ is called **serial** if it is both right and left serial.

### 5.3.1 The valued Gabriel quiver

Let us see that right serial coalgebras are easy to distinguish from its valued Gabriel quiver. The following lemma appears in [CGT04, Proposition 1.7]. For the convenience of the reader, we give a new proof only by means of "coalgebraic" arguments.

**Lemma 5.3.2.** *A basic coalgebra $C$ is right serial if and only if the right $C$-comodule $\mathrm{soc}^2 E/\mathrm{soc}\, E$ is zero or simple for each indecomposable injective right $C$-comodule $E$.*

*Proof.* Let $E$ be an indecomposable injective right $C$-comodule. Let us prove that the quotient $\mathrm{soc}^i E/\mathrm{soc}^{i-1} E$ is simple or zero for any $i \geq 2$. We proceed by induction on $i$. The case $i = 2$ is a consequence of the hypothesis. Let us now assume that the statement holds for some integer $k \geq 2$, that is,

$$\mathrm{soc}\left(\frac{E}{\mathrm{soc}^{k-1} E}\right) = \frac{\mathrm{soc}^k E}{\mathrm{soc}^{k-1} E}$$

is a simple comodule (if it was zero, then $E = \mathrm{soc}^k E$ and the result would follow) and hence the right injective envelope of $E/\mathrm{soc}^{k-1} E$ is an indecomposable injective right $C$-comodule $E'$. Therefore, by Lemma 3.3.5,

$$\frac{\mathrm{soc}^{k+1} E}{\mathrm{soc}^k E} \cong \frac{\frac{\mathrm{soc}^{k+1} E}{\mathrm{soc}^{k-1} E}}{\frac{\mathrm{soc}^k E}{\mathrm{soc}^{k-1} E}} \cong \frac{\mathrm{soc}^2\left(\frac{E}{\mathrm{soc}^{k-1} E}\right)}{\mathrm{soc}\left(\frac{E}{\mathrm{soc}^{k-1} E}\right)} \leq \frac{\mathrm{soc}^2 E'}{\mathrm{soc}\, E'}$$

which is simple or zero by hypothesis. The converse implication is trivial. $\square$

**Proposition 5.3.3.** *A basic coalgebra $C$ is right serial if and only if each vertex $S_i$ of the right valued Gabriel quiver $(Q_C, d_C)$ is at most the sink of one arrow and, if such an arrow exists, it is of the form*

$$S_j \xrightarrow{(1,d)} S_i \ ,$$

*for some vertex $S_j$ and some positive integer $d$. In particular, if $C$ is pointed, $C$ is right serial if and only if each vertex in the (non-valued) Gabriel quiver of $C$ is the sink of at most one arrow.*

*Proof.* Recall that, by Lemma 3.3.3, for any simple right $C$-comodule $S_i$,

$$\mathrm{Ext}^1_C(S_j, S_i) \cong \mathrm{Hom}_C(S_j, E_i/S_i)$$

as right $G_j$-modules for all simple right $C$-comodule $S_j$, where $G_j = \mathrm{End}_C(S_j)$, see also [KS05] and [Nav08, Lemma 1.2].

Assume now that $C$ is right serial and $E_i/S_i \neq 0$ (otherwise there is no arrow ending at $S_i$) then $E_i/S_i$ is a subcomodule of an indecomposable injective right comodule $E_j$, and then

$$\mathrm{Ext}^1_C(S, S_i) \cong \mathrm{Hom}_C(S, E_i/S_i) \cong \begin{cases} G_j, & \text{if } S_j = S; \\ 0, & \text{otherwise.} \end{cases}$$

as right $\mathrm{End}_C(S)$-modules. Hence, there is a unique arrow ending at $S_i$ of the form

$$S_j \xrightarrow{(1,d)} S_i \ .$$

Conversely, the immediate predecessors of $S_i$ correspond to the simple comodules contained in $\mathrm{soc}\,(E_i/S_i)$. Since, by hypothesis, there is only one arrow ending at $S_i$, $\mathrm{soc}\,(E_i/S_i) = (S_j)^t$ for some simple right comodule $S_j$ and some positive integer $t$. Now, since $t$ is the first component of the label of the arrow, $\mathrm{soc}^2 E_i/\mathrm{soc}\, E_i = \mathrm{soc}\,(E_i/S_i) = S_j$ is a simple comodule. By the previous lemma, $C$ is right serial. $\square$

Symmetrically, we prove that $C$ is left serial if and only if each vertex $S$ of the left valued Gabriel quiver $({}_CQ, {}_Cd)$ is at most the sink of one arrow and, if such an arrow exists, it is of the form

$$S' \xrightarrow{(1,d)} S \ ,$$

for some vertex $S'$ and some positive integer $d$.

Let us now prove a generalization of [CGT04, Theorem 2.10]. In what follows we denote the labeled arrows $\circ \xrightarrow{(1,1)} \circ$ simply by $\circ \longrightarrow \circ$. As well we denote a valued quiver $(Q, d)$ simply by $Q$ if $(d^1_{ij}, d^2_{ij}) = (1, 1)$ for any $i$ and $j$.

**Theorem 5.3.4.** *Let $C$ be a indecomposable basic coalgebra over an arbitrary field $K$. Then $C$ is serial if and only if the right (and then also the left) valued Gabriel quiver of $C$ is one of the following valued quivers:*

$$
\begin{array}{lll}
(a) & {}_{\infty}\mathbb{A}_{\infty} : & \cdots\cdots \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \cdots\cdots \\
(b) & \mathbb{A}_{\infty} : & \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \cdots\cdots \\
(c) & {}_{\infty}\mathbb{A} : & \cdots\cdots \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \\
(d) & \mathbb{A}_n : & \circ \longrightarrow \circ \longrightarrow \circ \cdots\cdots \circ \longrightarrow \circ \longrightarrow \circ \qquad n \textit{ vertices, } n \geq 1
\end{array}
$$

$(e)$ $\widetilde{\mathbb{A}}_n$ :

$$
\begin{tikzcd}
 & \circ \arrow[dl] & \circ \arrow[l] & \circ \arrow[l] & \\
\circ \arrow[dr] & & & & \circ \arrow[ul] \\
 & \circ \arrow[r] & \circ \arrow[r, dotted, no head] & \circ \arrow[ur] &
\end{tikzcd}
\qquad n \textit{ vertices, } n \geq 1
$$

*Proof.* Assume that $C$ is serial and let $S$ be a simple right (and left) $C$-comodule. Since $C$ is right serial, there exists at most one arrow in $(Q_C, d_C)$ ending at $S$. Analogously, since $C$ is left serial there exists at most one arrow in $({}_CQ, {}_Cd)$ ending at $S$ and then, by Proposition 3.3.2, we may deduce that there is at most one arrow in $(Q_C, d_C)$ starting at $S$. Also, by Proposition 5.3.3 and its left-hand version and Proposition 3.3.2, any of these possible arrows are labelled by $(1, 1)$.

Taking into account the above discussion, let $S_0$ be a simple right comodule. If there is no arrow neither ending nor starting at $S_0$, i.e., $S_0$ is an isolated vertex, since $C$ is indecomposable (and therefore $Q_C$ is connected, cf. [Sim06]) then $Q_C = \mathbb{A}_1$. Similarly, if there is a loop at $S_0$, then $Q_C = \widetilde{\mathbb{A}}_1$. Therefore we may assume that there is no loop in $Q_C$ and then $S_0$ is inside a (maybe infinite) path

$$\cdots\cdots \longrightarrow S_{-n} \longrightarrow \cdots\cdots \longrightarrow S_{-1} \longrightarrow S_0 \longrightarrow S_1 \longrightarrow \cdots\cdots \longrightarrow S_m \longrightarrow \cdots\cdots$$

If there exist two non-negative integers $n$ and $m$ such that $S_{-n} = S_m$, then $Q_C$ must be a crown, i.e., $Q_C = \widetilde{\mathbb{A}}_p$ for some integer $p$. If not, $Q_C$ must be a line, that is, it is a quiver as showed in $(a)$ ,$(b)$, $(c)$ or $(d)$ depending on the finiteness of the two branches. Clearly, the converse holds. $\square$

**Corollary 5.3.5.** *A basic coalgebra $C$ is serial if and only if each of the connected component of its right (or left) valued Gabriel quiver is either ${}_{\infty}\mathbb{A}_{\infty}$, or $\mathbb{A}_{\infty}$, or ${}_{\infty}\mathbb{A}$; or $\mathbb{A}_n$ or $\widetilde{\mathbb{A}}_n$ for some $n \geq 1$.*

### 5.3.2 Finite dimensional comodules

This subsection is devoted to give a complete list of all indecomposable finite-dimensional right comodules over a serial coalgebra and a description of the Auslander-Reiten quiver of the category $\mathcal{M}^C_f$. We recall from [CGT04] that any finite dimensional indecomposable comodule $M$ over a serial coalgebra $C$ is uniserial, and then, there exists an integer $t \geq 1$ such that

$\mathrm{soc}^t M = M$. Thus $M$ is a right $D$-comodule, where $D$ is the subcoalgebra of $C$, $\mathrm{soc}^t C = \oplus_{i \in I_C} \mathrm{soc}^t E_i$ (and then it is serial, cf. [CGT04]). We refer the reader to [CKQ02], [KS05], [NS02] and [Sim01] for definitions and terminology concerning almost split sequence and the Auslander-Reiten quiver of a coalgebra, see also [ASS05].

**Theorem 5.3.6.** *Let $C$ be a serial coalgebra. The following statements hold:*

($a$) *Each finite dimensional indecomposable right $C$-comodule is isomorphic to $\mathrm{soc}^k E$ for some positive integer $k$ and some indecomposable injective right $C$-comodule.*

($b$) *The category of finite dimensional right $C$-comodules has almost split sequences. Furthermore, for each indecomposable non-injective right $C$-comodule $\mathrm{soc}^k E$, the almost split sequence starting on this comodule is*

$$0 \longrightarrow \mathrm{soc}^k E \xrightarrow{\binom{i}{p}} \mathrm{soc}^{k+1} E \oplus \frac{\mathrm{soc}^k E}{\mathrm{soc}\ E} \xrightarrow{(q\ -j)} \frac{\mathrm{soc}^{k+1} E}{\mathrm{soc}\ E} \longrightarrow 0\ ,$$

*where $i$ and $j$ are the standard inclusions and $p$ and $q$ are the standard projections.*

*Proof.* ($a$). Let $M$ be a finite dimensional indecomposable comodule, by [CGT04], $M$ is uniserial and then $M = \mathrm{soc}^t M$ for some $t > 0$ (we may consider the minimal one). Since $M$ has simple socle, its injective envelope is an indecomposable injective comodule $E$. Let us prove that the Loewy series of $M$ and $E$ are the same until the step $t$. We know that $\mathrm{soc}\ M = \mathrm{soc}\ E = S$ is a simple comodule. Let us now assume that $\mathrm{soc}^k M = \mathrm{soc}^k E$ for each $k \leq t-1$, and also $\mathrm{soc}^{t-1} M \neq \mathrm{soc}^t M$. Then

$$0 \neq \frac{\mathrm{soc}^t M}{\mathrm{soc}^{t-1} M} \leq \frac{\mathrm{soc}^t E}{\mathrm{soc}^{t-1} E} \cong S_t,$$

where $S_t$ is a simple comodule. That is, $\mathrm{soc}^t M / \mathrm{soc}^{t-1} M \cong S_t$. Thus $M$ is uniserial and, by its definition, $\mathrm{soc}^t M = \mathrm{soc}^t E$. Then the result follows.

($b$). Here we essentially follow the proof of [ASS05, Theorem 4.1]. The given sequence is easily seen to be exact. By Krull-Remak-Schmidt-Azumaya Theorem, it is not split and, since $C$ is right serial, it has indecomposable end terms. Let us prove that the homomorphism $f = \binom{i}{p}$ is left almost split. It is clear that f is not a section. Let $N$ be an indecomposable finite dimensional $C$-comodule, and $h : \mathrm{soc}^k E \to N$ be a non-isomorphism. We have two cases. If $h$ is not injective, it decomposes through $\mathrm{soc}^k E / \mathrm{soc}\ E$, namely $h = h'p$. But then the homomorphism $g = (0\ h')$ satisfies that $gf = h$. If, on the other hand, $h$ is injective, since it is not an isomorphism, by (a), $N \cong \mathrm{soc}^t E$ with $t > k$. Then $N$ is injective as right $\mathrm{soc}^t C$-comodule and there exists a morphism $h' : \mathrm{soc}^{k+1} E \to N$ such that $h = h'i$. Hence $(h'\ 0)f = h$. Since the left term and the right term are indecomposable comodules and $f$ is a left almost split morphism, the sequence is almost split in the category of finite dimensional $C$-comodules, see [ASS05, Chapter IV, Theorem 1.13]. $\square$

By applying Theorem 5.3.6, we can easily calculate the Auslander-Reiten quiver of a serial coalgebra. For example, we do it for hereditary serial path coalgebras.

**Type** ${}_\infty\mathbb{A}_\infty$. Let $Q$ be the quiver

$$\cdots \xrightarrow{\alpha_{-2}} \text{-2} \xrightarrow{\alpha_{-1}} \text{-1} \xrightarrow{\alpha_0} 0 \xrightarrow{\alpha_1} 1 \xrightarrow{\alpha_2} 2 \xrightarrow{\alpha_3} \cdots$$

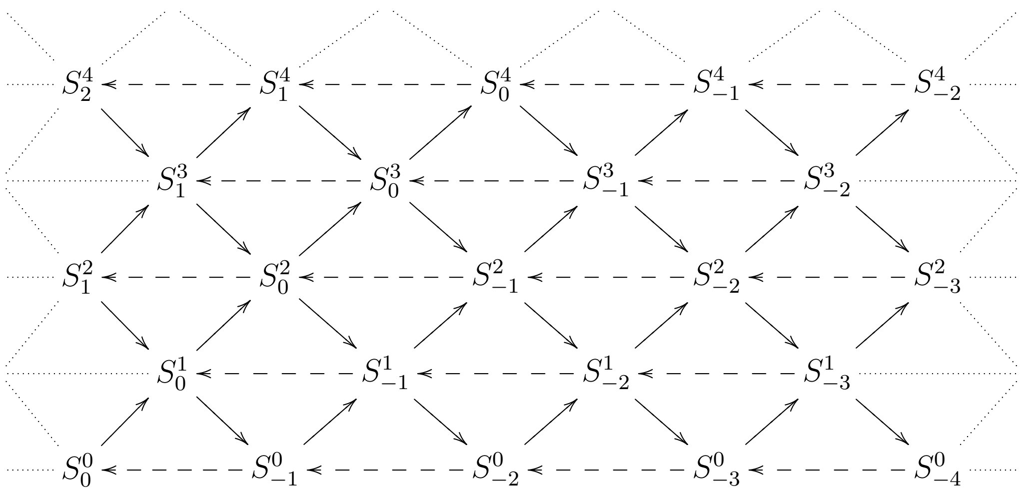
and let $C = KQ$ be the path coalgebra of $Q$. By Theorem 5.3.4, $C$ is serial. Let $E_i$ be the indecomposable injective right $C$-comodule associated to the vertex $i$, that is, $E_i = Ke_i \bigoplus \left(\bigoplus_{t\geq 0} K\alpha_i \cdots \alpha_{i-t}\right)$, where $e_i$ is the stationary path at $i$. Let

$$S_i^k = \mathrm{soc}^k E_i = Ke_i \bigoplus \left( \bigoplus_{t=0}^{k-1} K\alpha_i \cdots \alpha_{i-t} \right).$$

Now, since $\mathrm{soc}^k E_i / \mathrm{soc}\, E_i \cong \mathrm{soc}^{k-1} E_{i-1}$ for any $k$ and $i$, the almost split sequences are the following:

$$0 \longrightarrow S_i^k \longrightarrow S_i^{k+1} \oplus S_{i-1}^{k-1} \longrightarrow S_{i-1}^k \longrightarrow 0\,,$$

for each $k \geq 0$ and each $i \in \mathbb{Z}$. Therefore, the Auslander-Reiten quiver of $C$ is the following.



where each dashed arrow $Y \dashleftarrow X$ means that $Y = \tau(X)$, where $\tau = \mathrm{DTr}$ is the Auslander-Reiten translation, see [CKQ02] for definitions and details.

**Type $_\infty\mathbb{A}$.** Let $Q$ be the quiver

$$\cdots \xrightarrow{\alpha_4} ④ \xrightarrow{\alpha_3} ③ \xrightarrow{\alpha_2} ② \xrightarrow{\alpha_1} ① \xrightarrow{\alpha_0} ⓪$$

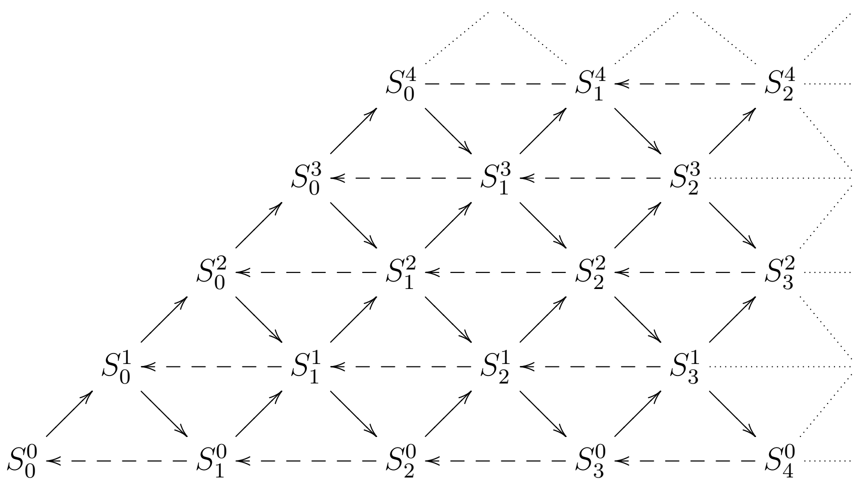
and let $C = KQ$ be the path coalgebra of $Q$. Again, by Theorem 5.3.4, $C$ is serial. Let $E_i$ be the indecomposable injective right $C$-comodule associated to the vertex $i$, that is, $E_i = Ke_i \bigoplus \left(\bigoplus_{t\geq 0} K\alpha_i \cdots \alpha_{i-t}\right)$, where $e_i$ is the stationary path at $i$. Let

$$S_i^k = \mathrm{soc}^k E_i = Ke_i \bigoplus \left( \bigoplus_{t=0}^{k-1} K\alpha_i \cdots \alpha_{i-t} \right).$$

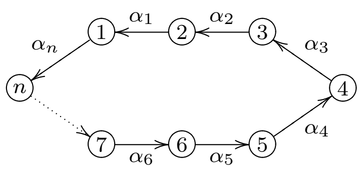
Since $\mathrm{soc}^k E_i/\mathrm{soc}\, E_i \cong \mathrm{soc}^{k-1} E_{i+1}$ for any $k$ and $i$, the almost split sequences are the following:

$$0 \longrightarrow S_i^k \longrightarrow S_i^{k+1} \oplus S_{i+1}^{k-1} \longrightarrow S_{i+1}^k \longrightarrow 0\,,$$

for each $i, k \geq 0$. Therefore, the Auslander-Reiten quiver of $C$ is the following.




**Type $\widetilde{\mathbb{A}}_n$,** $n \geq 1$. Let $Q$ be the quiver



and let $C = KQ$ be the path coalgebra of $Q$. Clearly, $C$ is serial. Let $E_i$ be the indecomposable injective right $C$-comodule associated to the vertex $i$, that is, $E_i = Ke_i \bigoplus \left(\bigoplus_{t\geq 0} K\alpha_{[i]}\cdots\alpha_{[i-t]}\right)$ where $[p] \equiv p \pmod n$ for any $p > 0$. Let

$$S_i^k = \mathrm{soc}^k E_i = Ke_i \bigoplus \left( \bigoplus_{t=0}^{k-1} K\alpha_{[i]}\cdots\alpha_{[i-t]} \right).$$

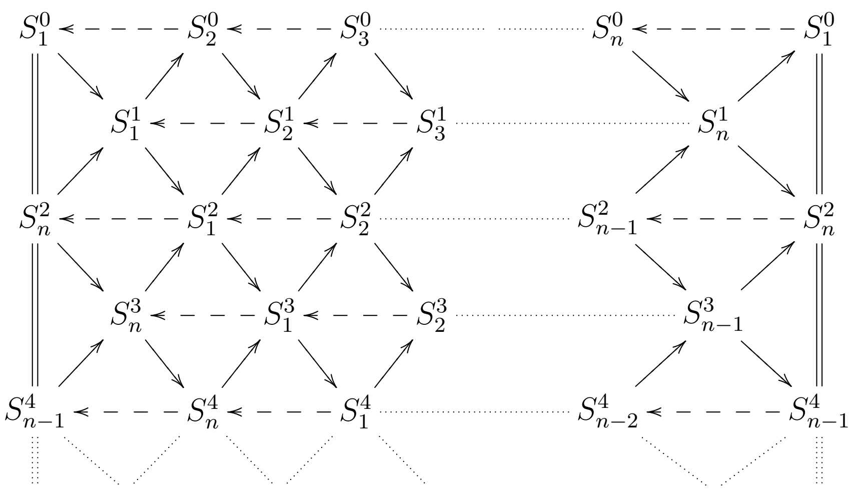
Now, since $\mathrm{soc}^k E_n/\mathrm{soc}\, E_n \cong \mathrm{soc}^{k-1}E_1$ and $\mathrm{soc}^k E_i/\mathrm{soc}\, E_i \cong \mathrm{soc}^{k-1}E_{i+1}$ for each $i = 1, \ldots, n-1$, for any $k \geq 0$, the almost split sequences are the following:

$$0 \longrightarrow S_i^k \longrightarrow S_i^{k+1} \oplus S_{i+1}^{k-1} \longrightarrow S_{i+1}^k \longrightarrow 0\ ,$$

for each $i = 1, \ldots, n-1$ and $k \geq 0$; and

$$0 \longrightarrow S_n^k \longrightarrow S_n^{k+1} \oplus S_1^{k-1} \longrightarrow S_1^k \longrightarrow 0\ ,$$

for any $k \geq 0$. Therefore, the Auslander-Reiten quiver of $C$ is the following.



This structure is called a stable tube of rank $n$ since the shape of the quiver obtained is a tube if we identify the vertical double lines.

The Auslander-Reiten quiver of the remaining serial path coalgebras are described in [NS02].

**Remark 5.3.7.** *Observe that, for each indecomposable finite-dimensional comodule* $M = \mathrm{soc}^n E$, $\tau(M) = \mathrm{soc}^{k+1}E/\mathrm{soc}\, E$, *and then* $\underline{\mathrm{length}}\, M = \underline{\mathrm{length}}\, \tau(M)$. *That is, the comodules lying in the same* $\tau$*-orbit has the same length.*

**Corollary 5.3.8.** *Any serial coalgebra over an algebraically closed field is of tame comodule type.*

### 5.3.3 Localization in serial coalgebras

Let us now apply the localization techniques developed in Chapter 3 to (right) serial coalgebras. In particular, we give a characterization of serial coalgebras by means of its "local structure", that is, by means of its localized coalgebras.

The following proposition shows that the localization process preserves the uniseriality of comodules and the seriality of coalgebras.

**Proposition 5.3.9.** *Let $E = Ce$ be a quasi-finite injective right $C$-comodule and $M$ a uniserial right $C$-comodule. Then $T(M) = \mathrm{Cohom}_C(E, M) = eM$ is a uniserial right $eCe$-comodule.*

*Proof.* Let us consider the (composition) Loewy series of $M$ as right $C$-comodule,

$$\mathrm{soc}\, M = S_1 \subset \mathrm{soc}^2 M \subset \mathrm{soc}^3 M \subset \cdots \subset M$$

whose composition factors are $S_1$ and $S_k = M[k] = \mathrm{soc}^k M/\mathrm{soc}^{k-1} M$ for $k \geq 2$. Since a simple $C$-comodule is either torsion or torsion-free, let us suppose that $S_{i_1}$, $S_{i_2}$, $S_{i_3}, \ldots$ are the torsion-free composition factors of $M$, where $i_1 < i_2 < i_3 < \cdots$.

For each $k < i_1$, $T(M[k]) = T(S_k) = 0$ and then $T(\mathrm{soc}^k M) = T(S_1) = 0$. As a consequence, by Remark 3.2.8, $T(\mathrm{soc}^{i_1} M) = T(S_{i_1}) = S_{i_1}$. Moreover, since $M[i_1] = \mathrm{soc}(M/\mathrm{soc}^{i_1 - 1} M) = S_{i_1}$, then $M/\mathrm{soc}^{i_1-1} M$ is torsion-free and, by Proposition 3.4.12,

$$S_{i_1} = T(S_{i_1}) = T\left(\mathrm{soc}\left(\frac{M}{\mathrm{soc}^{i_1-1} M}\right)\right) = \mathrm{soc}\left(T\left(\frac{M}{\mathrm{soc}^{i_1-1} M}\right)\right) = \mathrm{soc}\, T(M).$$

Applying the same arguments, we may obtain that $T(\mathrm{soc}^k M) = S_{i_1}$ for each $i_1 \leq k < i_2$, and $M/\mathrm{soc}^{i_2-1} M$ is a torsion-free right $C$-comodule. Then

$$\begin{aligned} T(M)[2] = \frac{\mathrm{soc}^2 T(M)}{\mathrm{soc}\, T(M)} &= \mathrm{soc}\left(\frac{T(M)}{\mathrm{soc}\, T(M)}\right) = \mathrm{soc}\left(\frac{T(M)}{T(\mathrm{soc}^{i_1} M)}\right) \\ &= \mathrm{soc}\left(\frac{T(M)}{T(\mathrm{soc}^{i_2-1} M)}\right) = T\left(\mathrm{soc}\left(\frac{M}{\mathrm{soc}^{i_2-1} M}\right)\right) = S_{i_2} \end{aligned}$$

Thus $\mathrm{soc}^2 T(M) = T(\mathrm{soc}^{i_2} M)$. If we continue in this fashion, we may prove that

$$T(\mathrm{soc}^{i_1} M) \subset T(\mathrm{soc}^{i_2} M) \subset T(\mathrm{soc}^{i_3} M) \subset \cdots \subset T(M)$$

is the Loewy series of $T(M)$. Hence $T(M)$ is uniserial as a right $eCe$-comodule. □

**Corollary 5.3.10.** *Let $C$ be a right (left) serial coalgebra and $e \in C^*$ an idempotent. Then the localized coalgebra $eCe$ is right (left) serial.*

*Proof.* Let $\overline{E}_i$ be an indecomposable injective right $eCe$-comodule. By Proposition 3.4.12, $T(E_i) = \overline{E}_i$, where $E_i$ is the indecomposable injective right $C$-comodule such that $\operatorname{soc} E_i = \operatorname{soc} \overline{E}_i$. Since $E_i$ is uniserial, by Proposition 5.3.9, so is $\overline{E}_i$. $\square$

**Lemma 5.3.11.** *Let $C$ be a coalgebra. If the localized coalgebra $eCe$ is right (left) serial for each idempotent $e \in C^*$ associated to a subset of simple comodules with cardinal less or equal than three, then $C$ is right (left) serial.*

*Proof.* Let us suppose that $C$ is not right serial. By Lemma 5.3.2 there exists an indecomposable injective right $C$-comodule $E$ such that $S_1 \oplus S_2 \subseteq \operatorname{soc}(E/S)$, where $S_1$ and $S_2$ are simple comodules. Consider the idempotent $e \in C^*$ associated to the set $\{S, S_1, S_2\}$. Then, by Lemma 3.4.1,

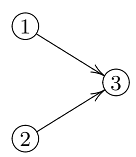
$$T(S_1 \oplus S_2) = S_1 \oplus S_2 \subseteq T(\operatorname{soc}(E/S)) \subseteq \operatorname{soc}(T(E/S)) = \operatorname{soc}(\overline{E}/S),$$

where $T(E) = \overline{E}$ is an indecomposable injective $eCe$-comodule. Thus $eCe$ is not right serial. $\square$

**Proposition 5.3.12.** *Let $C$ be a coalgebra. $C$ is right (left) serial if and only if each socle-finite localized coalgebra of $C$ is right (left) serial.*

*Proof.* Apply Corollary 5.3.10 and Lemma 5.3.11. $\square$

**Remark 5.3.13.** *The subsets of simple comodules mentioned in Lemma 5.3.11 cannot have cardinal bounded by less than three. For instance, if $C$ is the path coalgebra $KQ$ of the quiver $Q$*




*then each localized coalgebra of $C$ at subsets of two or one simple comodules is right (and left) serial but clearly $C$ is not.*

The former results are quite surprising since, as the following proposition shows, the localization process increases the label of an arrow (if exists) between two torsion-free vertices.

**Proposition 5.3.14.** *Let $C$ be a coalgebra and $e \in C^*$ idempotent. Let $S_1$ and $S_2$ be two torsion-free simple $C$-comodules in the torsion theory associated to the localizing subcategory $\mathcal{T}_e$. If there exists an arrow $S_1 \to S_2$ in $(Q_C, d_C)$ labelled by $(d'_{12}, d''_{12})$, then there exists an arrow $S_1 \to S_2$ in $(Q_{eCe}, d_{eCe})$ labelled by $(t'_{12}, t''_{12})$, where $t'_{12} \geq d'_{12}$ and $t''_{12} \geq d''_{12}$.*

*Proof.* Let us suppose that $\mathrm{soc}(E_2/S_2) = \oplus_{i\in I_C} S_i^{n_i}$ for some non-negative integers $n_i$. Then there exists an arrow $S_i \to S_2$ if and only if $n_i \neq 0$ and, furthermore, in such a case, it is labelled by $(n_i, m_i)$ for some positive integer $m_i$. Now, if there exists an arrow $S_1 \to S_2$ in $(Q_C, d_C)$ labelled by $(d'_{12}, d''_{12})$, then

$$S_1^{n_1} \subseteq \oplus_{i\in I_e} S_i^{n_i} = T_e(\mathrm{soc}(E_2/S_2)) \subseteq \mathrm{soc}T(E_2/S_2) \subseteq \overline{E}_2/S_2,$$

since $S_1$ and $S_2$ are torsion-free. Hence there is an arrow

$$S_1 \xrightarrow{(t'_{12},t''_{12})} S_2$$

in $(Q_{eCe}, d_{eCe})$ and $t'_{12} = \dim_{G_1}\mathrm{Hom}_{eCe}(S_1, \overline{E}_2/S_2) \geq n_1 = d'_{12}$. By the left-hand version of this reasoning and Proposition 3.3.2, also $t''_{12} \geq d''_{12}$. $\square$

**Remark 5.3.15.** *It is not possible to prove the equalities on the components of the labels in the statement of Proposition 5.3.14. The reader only have to consider the path coalgebra $KQ$ of the quiver $Q$*

$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr] & & 3 \end{tikzcd}$$

*and the idempotent $e \in (KQ)^*$ associated to the subset* $\{1, 3\}$.

### 5.3.4 A theorem of Eisenbud and Griffith for coalgebras

We finish the section by a version of the theorem of Eisenbud and Griffith [EG71, Corollary 3.2] for coalgebras. We recall that this theorem asserts that every proper quotient of a hereditary noetherian prime ring is serial. Obviously, first we need a translation of the concepts from ring terminology to the notions used in coalgebra theory. About hereditariness, the concept is recalled above and it is not needed any explication. The "coalgebraic" version of noetherianess is the so-called co-noetherianess (cf. [GTNT07]). We recall that a comodule $M$ is said to be co-noetherian if every quotient of $M$ is embedded in a finite direct sum of copies of $C$. Nevertheless, we shall use a weaker concept: strictly quasi-finiteness [GTNT07], namely, $M$ is strictly quasi-finite if every quotient of $M$ is quasi-finite. This is due to fact that we may reduce the problem to socle-finite coalgebras and then, under this condition, both classes of comodules coincide [GTNT07, Proposition 1.6]. Finally, following [JMR], a coalgebra is called prime if for any subcoalgebras $A, B \subseteq C$ such that $A \wedge B = C$, then $A = C$ or $B = C$. For the convenience of the reader we present the following example:

**Example 5.3.16.** *Let $C$ be a hereditary colocal coalgebra such that $C/S \cong C \oplus C$, where $S$ is the unique simple comodule (or subcoalgebra). We prove that $C$ is not co-noetherian.*

*Let us consider the subcomodule $N_2$ of $C$ which yields the following commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r] \arrow[d, equal] & N_2 \arrow[r] \arrow[d] & S \arrow[r] \arrow[d, "(0,i)"] & 0 \\ 0 \arrow[r] & S \arrow[r] & C \arrow[r] & C \oplus C \arrow[r] & 0 \end{tikzcd}$$

*Then, $N_2 \subseteq \mathrm{soc}^2 C$ and $N_2/S \cong S$. Now,*

$$C/N_2 \cong (C/S)/(N_2/S) \cong (C \oplus C)/S \cong (C \oplus C \oplus C) = C^3.$$

*Analogously, let $N_3$ be the subcomodule of $C$ which yields the commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & N_2 \arrow[r] \arrow[d, equal] & N_3 \arrow[r] \arrow[d] & S \arrow[r] \arrow[d, "(0,0,i)"] & 0 . \\ 0 \arrow[r] & N_2 \arrow[r] & C \arrow[r] & C \oplus C \oplus C \arrow[r] & 0 \end{tikzcd}$$

*Again, $N_3/N_2 \cong S$ and*

$$C/N_3 \cong (C/N_2)/(N_3/N_2) \cong (C \oplus C \oplus C)/S \cong (C \oplus C \oplus C \oplus C) = C^4.$$

*If we continue in this way, we obtain a increasing family of subcomodules $\{N_t\}_{t \geq 1}$, where $N_1 = S$, such that $C/N_t \cong C^{t+1}$ for any $t \geq 1$. Let us consider the uniserial subcomodule $N = \cup_{t \geq 1} N_t$ of $C$ whose composition series (or Loewy series) is given by*

$$0 \subset S \subset N_2 \subset N_3 \subset \cdots \subset N.$$

*The comodule $C/N$ has infinite dimensional socle. To see this, for each $i \geq 0$, consider the short exact sequence*

$$0 \longrightarrow N/N_i \longrightarrow C/N_i \longrightarrow C/N \longrightarrow 0$$

*which yields the exact sequence*

$$0 \longrightarrow \mathrm{soc}\,(N/N_i) \longrightarrow \mathrm{soc}\,(C/N_i) \longrightarrow \mathrm{soc}\,(C/N)\ ,$$

*where $\mathrm{soc}\,(N/N_i) \cong S$ and $\mathrm{soc}\,(C/N_i) \cong S^{i+1}$. Thus $\dim_K \mathrm{soc}\,(C/N) \geq i \cdot \dim_K S$ for any $i \geq 1$. Consequently, $C$ is not co-noetherian.*

**Theorem 5.3.17.** *Let $C$ be a basic socle-finite coalgebra over an arbitrary field. If $C$ is coprime, hereditary and co-noetherian then $C$ is serial.*

*Proof.* If $C$ is colocal, then the valued Gabriel quiver of $C$ is either a single point or a vertex with a loop labeled by a pair $(d', d'')$. Now, if $d' \geq 2$ or $d'' \geq 2$, proceeding as in Example 5.3.16, $C$ is not co-noetherian. Thus $d' = d'' = 1$ and, by Theorem 5.3.4, $C$ is serial. Assume then that $C$ is not colocal. Let us first develop some properties about the valued Gabriel quiver of a localized coalgebra of $C$. These are inspired by the ones obtained in Section 3.6 for path coalgebras. Let us suppose that $S_x$, $S_y$ and $S_z$ are three simple $C$-comodules (where $S_x$ could equals $S_z$) such that there is path in $(Q_C, d_C)$

$$S_x \xrightarrow{(d_1,d_2)} S_y \xrightarrow{(c_1,c_2)} S_z \ .$$

Let $e \in C^*$ be an idempotent and $eCe$ the localized coalgebra associated to $e$ whose quotient functor we denote by $T$. Assume that $S_x$ and $S_z$ are torsion-free and $S_y$ is torsion. Since $C$ is hereditary, $E_z/S_z \cong \bigoplus_{j\in J} E_j^{r_j} \bigoplus \bigoplus_{t\in T} E_t^{r_t}$, where $S_j$ is torsion for all $j \in J$ and $S_t$ is torsion-free for all $t \in T$, and $r_\alpha$ is a positive integer for any $\alpha \in J \cup T$. Now, since there is an arrow from $S_y$ to $S_z$, $y \in J$ and $E_y^{r_y} \subseteq E_z/S_z$, where $r_y = c_1$. Then $T(E_y^{c_1}) \subseteq T(E_z/S_z)$. Finally, since $S_x^{d_1} \subseteq T(E_y) \cong T(E_y/S_y)$ then $S_x^{d_1c_1} \subseteq T(E_z/S_z) = \overline{E}_z/S_z$. That is, there exists an arrow

$$S_x \xrightarrow{(h_1,h_2)} S_z$$

in $(Q_{eCe}, d_{eCe})$ such that $h_1 \geq d_1c_1$. By Proposition 3.3.2, it is easy to see that $h_2 \geq d_2c_2$. Note that the hereditariness is a left-right symmetric property.

By an easy induction one may prove that if there is a path

$$S_x \xrightarrow{(a_0,b_0)} S_1 \xrightarrow{(a_1,b_1)} \cdots \longrightarrow S_{n-1} \xrightarrow{(a_{n-1},b_{n-1})} S_n \xrightarrow{(a_n,b_n)} S_z \tag{5.1}$$

such that $S_i$ is torsion, for all $i = 1, \ldots, n$. Then there is an arrow

$$S_x \xrightarrow{(h_1,h_2)} S_z$$

in $(Q_{eCe}, d_{eCe})$ such that $h_1 \geq a_0a_1\cdots a_n$ and $h_2 \geq b_0b_1\cdots b_n$. Furthermore, following this procedure, one may prove that if $\mathfrak{P} = \{p^l\}_{l\in\Lambda}$ is non empty, where $\mathfrak{P}$ is the set of all possible paths $p^l$ in $(Q_C, d_C)$ as described in (5.1), i.e., starting at $S_x$, ending at $S_y$ and whose intermediate vertices are torsion, then there is an arrow

$$S_x \xrightarrow{(h_1,h_2)} S_z$$

in $(Q_{eCe}, d_{eCe})$ such that $h_1 = \sum_l a_0^l a_1^l \cdots a_{n_l}^l$ and $h_2 = \sum_l b_0^l b_1^l \cdots b_{n_l}^l$. Here we have denoted by $a_0^l, \ldots, a_{n_l}^l$ and by $b_0^l, \ldots, b_{n_l}^l$ the first and the second component, respectively, of the labels of the arrows whose composition build the path $p^l$.

Now we consider a primitive orthogonal idempotent $e_x \in C^*$ and $e_xCe_x$ the localized coalgebra of $C$ associated to $e_x$. By [GTNT07, Proposition 1.8] and [Gab62, p. 376, Corollary 5], $e_xCe_x$ is co-neotherian and hereditary, respectively. Therefore, following the colocal case, the valued Gabriel quiver of $e_xCe_x$ must be a single point or a vertex with a loop labeled by $(1,1)$. As a consequence, by the above considerations, each vertex of the valued Gabriel quiver of $C$ is inside of at most one cycle and, if exists, the arrows of that cycle are labeled by $(1,1)$.

Finally, we prove that for each pair of vertices of $Q_C$ there is a cycle passing through these two vertices. This yields the statement of the theorem since, together with the above conditions, the only possible quiver is $(Q_C, d_C) = \widetilde{\mathbb{A}}_n$ for some $n \geq 1$ and then $C$ is serial.

Fix two different simple comodules $S_x$ and $S_y$. Let $e_x$ and $e_y$ be the primitive orthogonal idempotents in $C^*$ associated to $S_x$ and $S_y$, respectively. We set $e = e_x + e_y$. By [JMR, Proposition 4.1], $eCe$ is prime. First, let us suppose that there is no path in $Q_C$ from $S_x$ to $S_y$ nor vice versa. Then $eCe$ has two connected components and, by [Sim06, Corollary 2.4(b)], $eCe$ is not indecomposable. Thus $eCe$ is not prime (cf. [JMR, Lemma 1.4]). Now, suppose that there is a path from $S_x$ to $S_y$ but there is no path from $S_y$ to $S_x$. Then the valued Gabriel quiver of $eCe$ is a subquiver of the following quiver:

$$\begin{tikzcd} S_x \arrow[loop above] \arrow[r, "{(a,b)}"] & S_y \arrow[loop above] \end{tikzcd} .$$

By Corollary 3.4.19, $e_xCe_y = T_x(E_y) \neq 0$ and $e_yCe_x = T_y(E_x) = 0$. Therefore, there is a vector space direct sum decomposition $eCe = e_xCe_x \oplus e_xCe_y \oplus e_yCe_y$. A straightforward calculation shows that the linear map

$$\Psi : D = \begin{pmatrix} e_yCe_y & e_xCe_y \\ 0 & e_xCe_x \end{pmatrix} \longrightarrow eCe,$$

between $eCe$ and the bipartite coalgebra $D$ (in the sense of [KSb]), given by

$$\Psi \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = a + b + c$$

is an isomorphism of coalgebras, see [CGT02] for definitions and details about the structures of the spaces $e_xCe_x$, $e_yCe_y$ and $e_xCe_y$. We recall that the coalgebra structure of the bipartite coalgebra $D$ is given by the formulae:

- $\Delta(a + b + c) = \Delta_y(a) + \rho_y(b) + \rho_x(b) + \Delta_x(c)$, where $\rho_y$ and $\rho_x$ are the $e_yCe_y$-$e_xCe_x$-bicomodule structure maps of $e_xCe_y$; and $\Delta_x$ and $\Delta_y$ are the comultiplication of the coalgebras $e_yCe_y$ and $e_xCe_x$, respectively.

- $\epsilon(a+b+c) = \epsilon_y(a) + \epsilon_x(b)$, where $\epsilon_y$ and $\epsilon_x$ are the counit of the coalgebras $e_yCe_y$ and $e_xCe_x$, respectively.

Then $eCe = e_yCe_y \wedge e_xCe_x$ and therefore $eCe$ is not prime. $\square$

Now we prove the Eisenbud-Griffith Theorem for coalgebras.

**Corollary 5.3.18.** *If $C$ is a subcoalgebra of a prime, hereditary and strictly quasi-finite (left and right) coalgebra over an arbitrary field, then $C$ is serial.*

*Proof.* By [CGT04, Proposition 1.5], we may assume that $C$ is prime, hereditary and strictly quasi-finite itself. Let $e \in C^*$ be an idempotent such that the localized coalgebra $eCe$ is socle-finite. By [Gab72, p. 376, Corollary 5], [JMR, Proposition 4.1] and [GTNT07, Proposition 1.8], $eCe$ is hereditary, prime and right strictly quasi-finite, respectively. Moreover, by [GTNT07, Proposition 1.5], $eCe$ is co-noetherian. Therefore, by Theorem 5.3.17, $eCe$ is serial. Thus the result follows from Proposition 5.3.12. $\square$

## 5.4 Other examples

Following [Sim05], a **string coalgebra** is a path coalgebra $C = C(Q, \Omega)$ of a quiver with relations $(Q, \Omega)$ which satisfies the following properties:

$(a)$ Each vertex of $Q$ is the source of at most two arrows and the sink of at most two arrows.

$(b)$ The ideal $\Omega$ is generated by a set of paths.

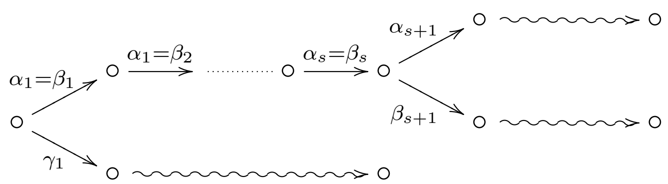
$(c)$ Given an arrow $i \xrightarrow{\beta} j$ in $Q$, there is at most one arrow $j \xrightarrow{\alpha} k$ in $Q$ and at most one arrow $l \xrightarrow{\gamma} i$ in $Q$ such that $\alpha\beta \in C$ and $\beta\gamma \in C$.

In [Sim05], it is proved that every string coalgebra is of tame comodule type. See also [Chi] for a relation with special biserial coalgebras and its representation theory. Let us show that the localization process preserves string coalgebras.

**Theorem 5.4.1.** *Let $C = C(Q, \Omega)$ be a string coalgebra and $e \in C^*$ be an idempotent element. Then, the localized coalgebra $eCe$ is the string coalgebra $C(Q^e, \Omega^e)$, where $\Omega^e = e\Omega e \cap KQ^e$.*

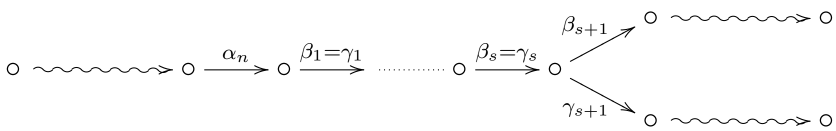
*Proof.* Since $\Omega$ is generated by paths, $KQ = C \oplus \Omega$ as $K$-vector space. Then $KQ_e \cong e(KQ)e = eCe \oplus e\Omega e$ and therefore $KQ^e = KQ_e \cap KQ^e \cong eCe \oplus (e\Omega e \cap KQ^e)$. It is easy to see that $\Omega^e$ is generated by paths in $Q^e$ of length greater than one.

Let us suppose that there is a vertex $i \in (Q^e)_0$ which is the source of three different arrows $\alpha, \beta, \gamma \in (Q^e)_1$. Then there exist three different paths $p_\alpha = \alpha_n \cdots \alpha_1, p_\beta = \beta_m \cdots \beta_1, p_\gamma = \gamma_r \cdots \gamma_1 \in \mathcal{C}ell_X^Q \cap C$ such that their cellular decompositions are $\alpha$, $\beta$ and $\gamma$, respectively. We have that $\alpha_1, \beta_1$ and $\gamma_1$ are three arrows in $Q$ starting at $i$ and, since $C$ is string, at least two of them are the same. For instance, suppose that $\alpha_1 = \beta_1$. Furthermore, $p_\alpha \neq p_\beta$ so there exists an integer $s$ such that $\alpha_s \cdots \alpha_1 = \beta_s \cdots \beta_1$ and $\alpha_{s+1}\alpha_s \cdots \alpha_1 \neq \beta_{s+1}\beta_s \cdots \beta_1$.




By the condition $(c)$ of the definition of string coalgebra, $\beta_{s+1}\beta_s \notin C$ or $\alpha_{s+1}\alpha_s \notin C$ and then $p_\alpha \notin C$ or $p_\beta \notin C$. We get a contradiction. We may proceed analogously if there are three different arrows ending at $i$.

Let $j \xrightarrow{\alpha} i$, $i \xrightarrow{\beta} k$ and $i \xrightarrow{\gamma} l$ be three arrows in $Q^e$ such that $\beta\alpha \in eCe$ and $\gamma\alpha \in eCe$. As above, there exist three different paths $p_\alpha = \alpha_n \cdots \alpha_1, p_\beta = \beta_m \cdots \beta_1, p_\gamma = \gamma_r \cdots \gamma_1 \in \mathcal{C}ell_X^Q \cap C$ such that their cellular decompositions are $\alpha$, $\beta$ and $\gamma$, respectively. Since $p_\beta \neq p_\gamma$, there exists an integer $s$ such that $\gamma_s \cdots \gamma_1 = \beta_s \cdots \beta_1$ and $\gamma_{s+1}\gamma_s \cdots \gamma_1 \neq \beta_{s+1}\beta_s \cdots \beta_1$.




If $s \geq 1$ then $\beta_{s+1}\beta_s \notin C$ or $\gamma_{s+1}\gamma_s \notin C$ and then $p_\beta \notin C$ or $p_\gamma \notin C$. This is a contradiction, so $s = 0$. But in that case, since $C$ is string, $\beta_1\alpha_n \notin C$ or $\gamma_1\alpha_n \notin C$ and then $\beta\alpha \notin eCe$ or $\gamma\alpha \notin C$. The dual case is similar and the proof follows. $\square$

**Definition 5.4.2.** *A coalgebra is said to be **gentle** if it is a string coalgebra $C(Q, \Omega)$ which satisfies the following extra statement:*

*(d) Given an arrow $i \xrightarrow{\beta} j$ in $Q$, there is at most one arrow $j \xrightarrow{\alpha} k$ in $Q$ an at most one arrow $l \xrightarrow{\gamma} i$ in $Q$ such that $\alpha\beta \notin C$ and $\beta\gamma \notin C$*

Unlike it happens with string coalgebras, the localized coalgebra of a gentle coalgebra does not have to be gentle.

**Example 5.4.3.** *Let $Q$ be the quiver*

$$\begin{tikzcd}
& & \circ \arrow[r, "\beta_2"] & \circ \\
\circ \arrow[r, "\alpha"] & \circ \arrow[ur, "\beta_1"] \arrow[dr, "\gamma"'] & & \\
& & \circ &
\end{tikzcd}$$

*and $C$ the gentle subcoalgebra of $KQ$ generated by all arrows and all vertices, and the paths $\beta_2\beta_1$ and $\beta_1\alpha$. Let $e$ be the idempotent element associated to the subset of vertices $X = Q_0 \backslash s(\beta_2)$. Then $eCe$ is the admissible subcoalgebra of the path coalgebra of the quiver*

$$\begin{tikzcd}
& & \circ \\
\circ \arrow[r, "\alpha"] & \circ \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & \\
& & \circ
\end{tikzcd}$$

*generated by the set of vertices and the set of arrows. Obviously $eCe$ is not a gentle coalgebra.*

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