Achieving non-linear models for δ Scuti stars

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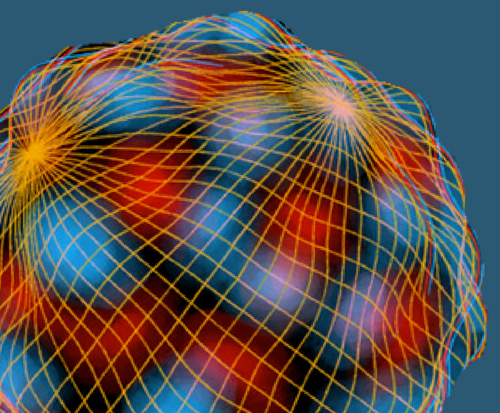


Contents

Introduction

δ Scuti Stars

A2-F2

1.5-2.5 $M_\odot$

30 mins-8 h

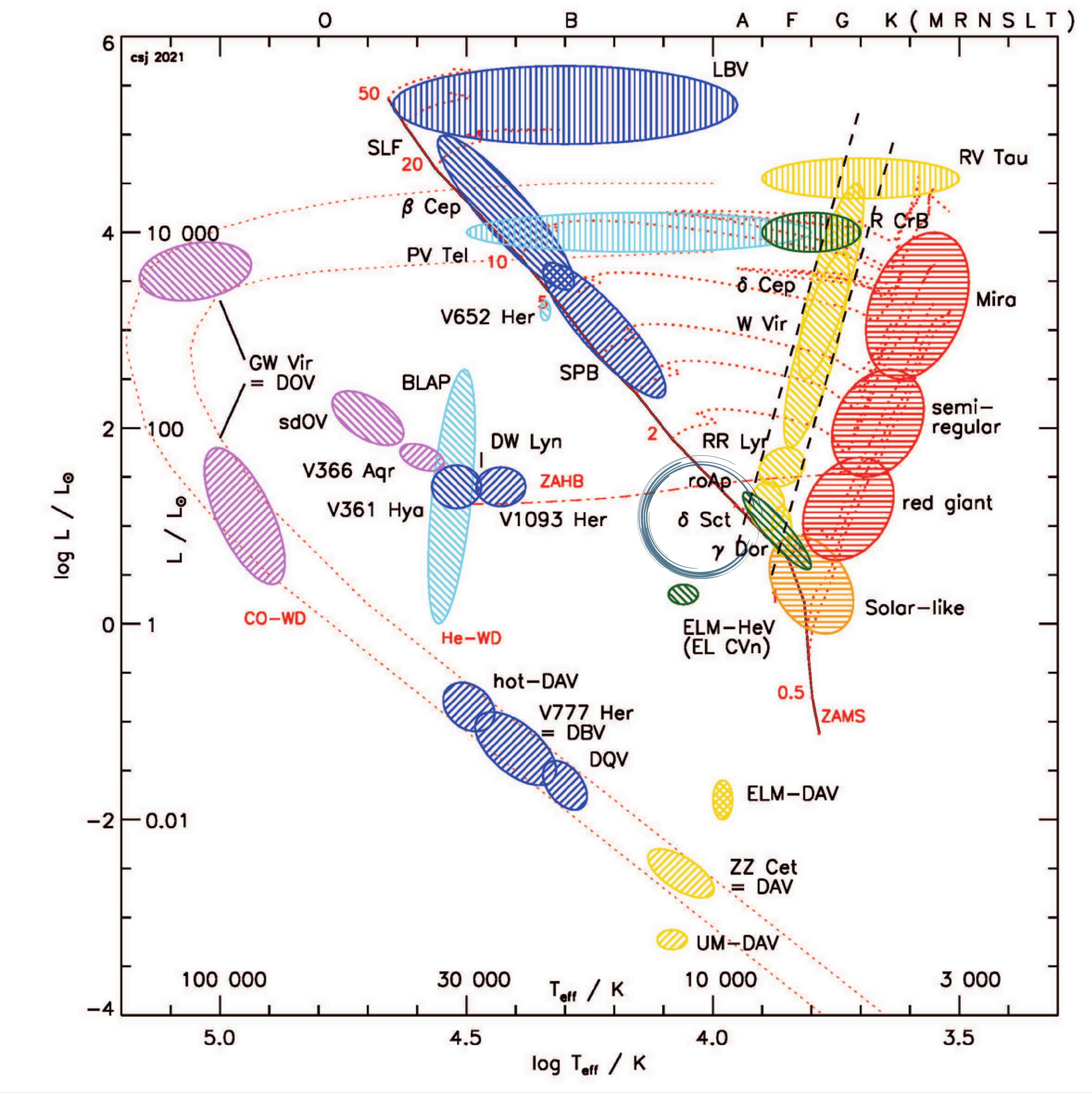
6300 K-8900 K

10^{-3} - 10^{-1} mag

κ mechanism

Radial and nonradial modes

LADS & HADS



Asteroseismic HR Diagram . *Kurtz (2022)*

Non-linear models

Equations of hydrodynamics

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

Conservation of mass

$$\rho \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) \vec{v} = -\nabla p - \rho \nabla \Phi$$

Conservation of momentum

$$\rho \frac{\partial p}{\partial t} + \rho [\vec{v} \cdot (\nabla p)] = \Gamma_1 p \frac{\partial \rho}{\partial t} + \rho \left[\vec{v} \cdot (\nabla \rho) \right]$$

Conservation of energy

$$\nabla^2 \Phi = 4\pi G \rho$$

Equation of Poisson

- ρ density
- p pressure
- $\vec{v}$ velocity
- Φ gravitational potential
- Γ_1 first adiabatic exponent

Equations of hydrodynamics

$$f(\vec{r},t) = f_0(r) + f'(\vec{r},t)$$

small perturbations of the variables

2nd order of perturbations

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho_0 \, \vec{v}\,') + \nabla \cdot (\rho' \, \vec{v}\,') = 0$$

Non-linear conservation of mass

$$(\rho_0 + \rho') \, \frac{\partial \vec{v}\,'}{\partial t} + \rho_0 (\vec{v}\,' \cdot \nabla) \vec{v}\,' = -\nabla p' - (\rho_0 + \rho') \, \nabla \Phi' - \rho' \, \nabla \Phi_0$$

Non-linear conservation of momentum

$$(\rho_0 + \rho') \, \frac{\partial p'}{\partial t} + \rho_0 [\vec{v}\,' \cdot (\nabla p_0) + \vec{v}\,' \cdot (\nabla p')] + \rho' [\vec{v}\,' \cdot (\nabla p_0)] =$$

$$\Gamma_1 (p_0 + p') \frac{\partial \rho'}{\partial t} + \rho_0 [\vec{v}\,' \cdot (\nabla \rho_0) + \vec{v}\,' \cdot (\nabla \rho)] + \rho' [\vec{v} \cdot (\nabla \rho_0)]$$

Non-linear conservation of energy

$$\nabla^2 \Phi' = 4\pi G \rho'$$

Equation of Poisson

Equations of hydrodynamics

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \quad \leftarrow \text{Continuity} \rightarrow \quad \frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho_0 \vec{v}\,') + \boxed{\nabla \cdot (\rho' \vec{v}\,')} = 0$$

$$\rho \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) \vec{v} = -\nabla p - \rho \nabla \Phi \quad \leftarrow \text{Conservation of momentum} \rightarrow \quad (\rho_0 + \boxed{\rho') \frac{\partial \vec{v}\,'}{\partial t}} + \boxed{\rho_0 (\vec{v}\,' \cdot \nabla) \vec{v}\,'} =$$

$$- \nabla p' - (\rho_0 + \boxed{\rho') \nabla \Phi'} - \rho' \nabla \Phi_0$$

$$\rho \frac{\partial p}{\partial t} + \rho \left[\vec{v} \cdot (\nabla p) \right] = \Gamma_1 \, p \frac{\partial \rho}{\partial t} + \rho \left[\vec{v} \cdot (\nabla \rho) \right] \quad \leftarrow \text{Conservation of energy} \rightarrow (\rho_0 + \boxed{\rho') \frac{\partial p'}{\partial t}} + \rho_0 [\vec{v}\,' \cdot (\nabla p_0) + \boxed{\vec{v}\,' \cdot (\nabla p')}] + \boxed{\rho' [\vec{v}\,' \cdot (\nabla p_0)]} =$$

$$\Gamma_1 (p_0 + \boxed{p') \frac{\partial \rho'}{\partial t}} + \rho_0 [\vec{v}\,' \cdot (\nabla \rho_0) + \boxed{\vec{v}\,' \cdot (\nabla \rho')}] + \boxed{\rho' [\vec{v}\,' \cdot (\nabla \rho_0)]}$$

$$\nabla^2 \Phi = 4 \pi G \rho \quad \leftarrow \text{Equation of Poisson} \rightarrow \quad \nabla^2 \Phi' = 4 \pi G \rho'$$

Equations of hydrodynamics

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$ ← Continuity → $\nabla \cdot (\rho' \vec{v}\,')$

$$\rho \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) \vec{v} = -\nabla p - \rho \nabla \Phi$$ ← Conservation of momentum → $\rho')\frac{\partial \vec{v}\,'}{\partial t}$ $\rho_0(\vec{v}\,' \cdot \nabla)\vec{v}\,'$ $\rho')\nabla\Phi'$

$$\rho \frac{\partial p}{\partial t} + \rho\,[\vec{v} \cdot (\nabla p)] = \Gamma_1\, p \frac{\partial \rho}{\partial t} + \rho\,[\vec{v} \cdot (\nabla \rho)]$$ ← Conservation of energy → $\rho')\frac{\partial p'}{\partial t}$ $\vec{v}\,' \cdot (\nabla p')$ $\rho'[\vec{v}\,' \cdot (\nabla p_0)]$ $p')\frac{\partial \rho'}{\partial t}$ $\vec{v}\,' \cdot (\nabla \rho')$ $\rho'[\vec{v}\,' \cdot (\nabla \rho_0)]$

$$\nabla^2 \Phi = 4\pi G \rho$$ ← Equation of Poisson →

Perturbed equations

Approximations

Spherical symmetry

Inviscid flow

Frozen convection

No rotation

Adiabaticity

No magnetic fields

Evaluation of non-linear terms

Evaluation of non-linear terms

- Scheme from *Unno et al. (1989)* and *Aerts et al. (2010)*

Spherical symmetry + **Radial modes**

$$\vec{\xi} = \vec{r} - \vec{r_0} = \vec{\xi_r} + \vec{\xi_h} \quad \text{displacement vector}$$

$$f'(t, r, \theta, \phi) = f'(r)\, Y_l^m(\theta, \phi)\, e^{i\sigma t} \quad \text{perturbed variable}$$

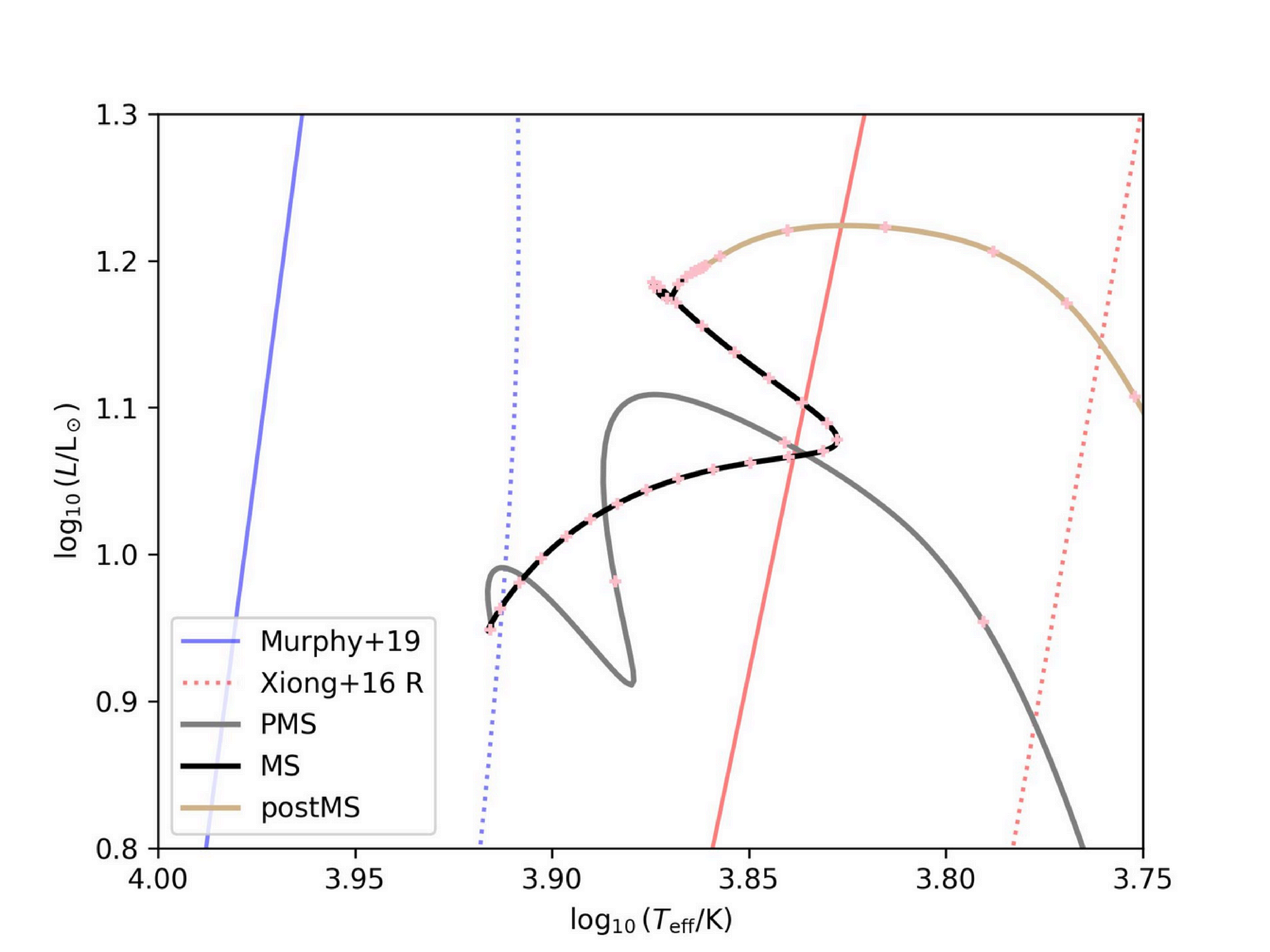
$$l = 0 \rightarrow Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}}$$

- 1.5 $M_\odot$ equilibrium model obtained with CESAM
- 1.7 $M_\odot$ equilibrium model obtained with MESA, PMS to post MS

Linear adiabatic system of equations

1.7 $M_\odot$ evolutionary track

- 1.5 $M_\odot$ equilibrium model obtained with CESAM
- 1.7 $M_\odot$ equilibrium model obtained with MESA, PMS to post MS



Radial modes

Conservation of mass

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho_0 \vec{v}\,') + \boxed{\nabla \cdot (\rho' \vec{v}\,')} = 0$$

$$\rho' + \rho_0 \frac{2}{r}\xi_r + \frac{\partial \rho_0}{\partial r}\xi_r + \rho_0 \frac{\partial \xi_r}{\partial r} = -\boxed{\frac{1}{\sqrt{4\pi}}\frac{\partial \rho'}{\partial r}\xi_r} - \boxed{\frac{1}{\sqrt{4\pi}}\frac{2}{r}\rho'\xi_r} - \boxed{\frac{1}{\sqrt{4\pi}}\rho'\frac{\partial \xi_r}{\partial r}}$$

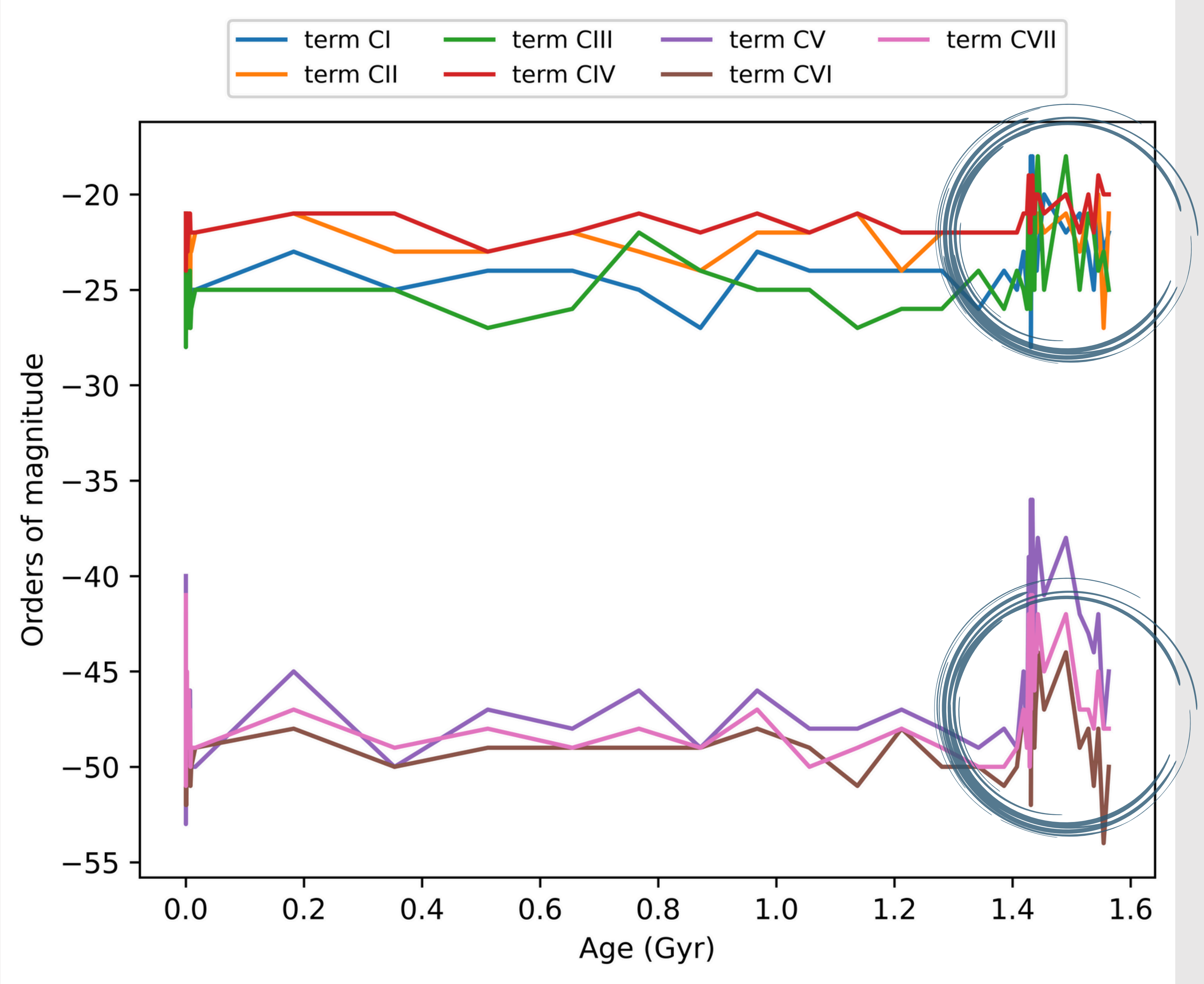
Radial modes

Conservation of mass

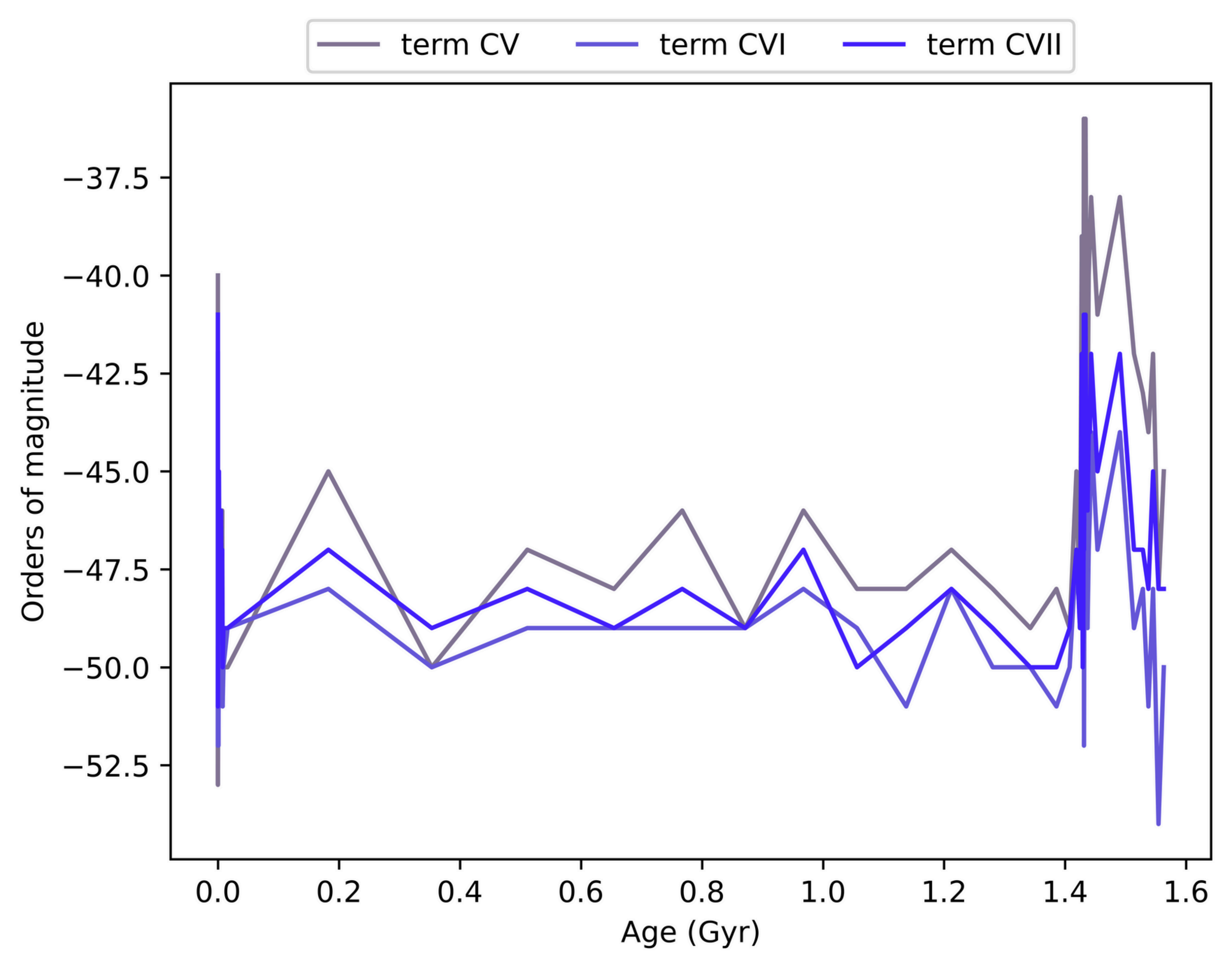
$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho_0 \vec{v}\,') + \boxed{\nabla \cdot (\rho' \vec{v}\,')} = 0$$

$$\rho' + \rho_0 \frac{2}{r}\xi_r + \frac{\partial \rho_0}{\partial r}\xi_r + \rho_0 \frac{\partial \xi_r}{\partial r} = -\boxed{\frac{1}{\sqrt{4\pi}}\frac{\partial \rho'}{\partial r}\xi_r} - \boxed{\frac{1}{\sqrt{4\pi}}\frac{2}{r}\rho'\xi_r} - \boxed{\frac{1}{\sqrt{4\pi}}\rho'\frac{\partial \xi_r}{\partial r}}$$

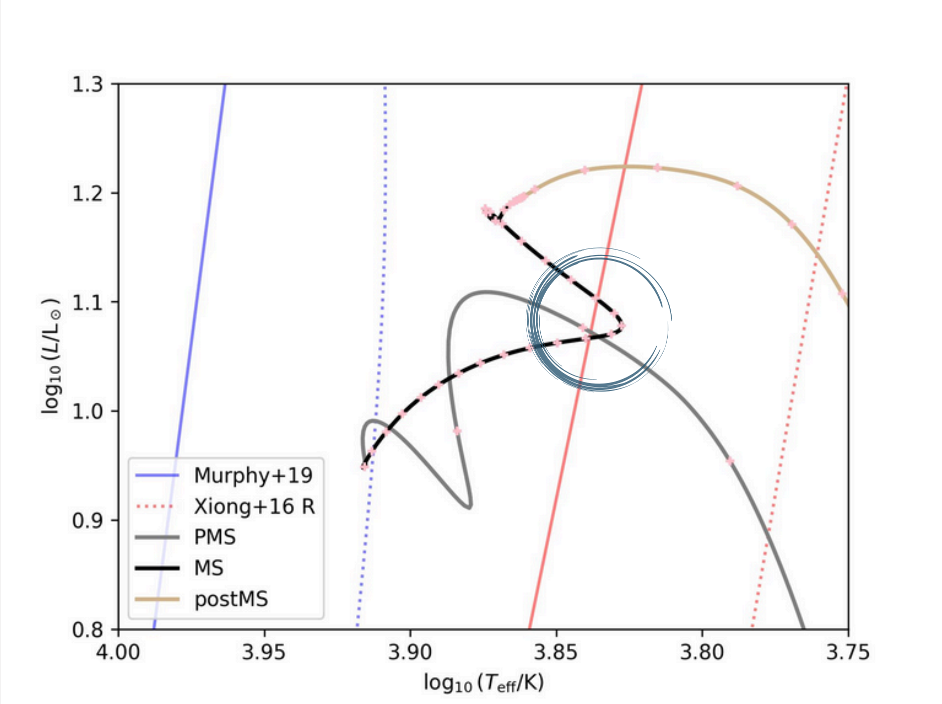
$\boxed{e^{-25}}$ $\boxed{e^{-23}}$ $\boxed{e^{-24}}$ $\boxed{e^{-23}}$ $\boxed{e^{-48}}$ $\boxed{e^{-50}}$ $\boxed{e^{-50}}$



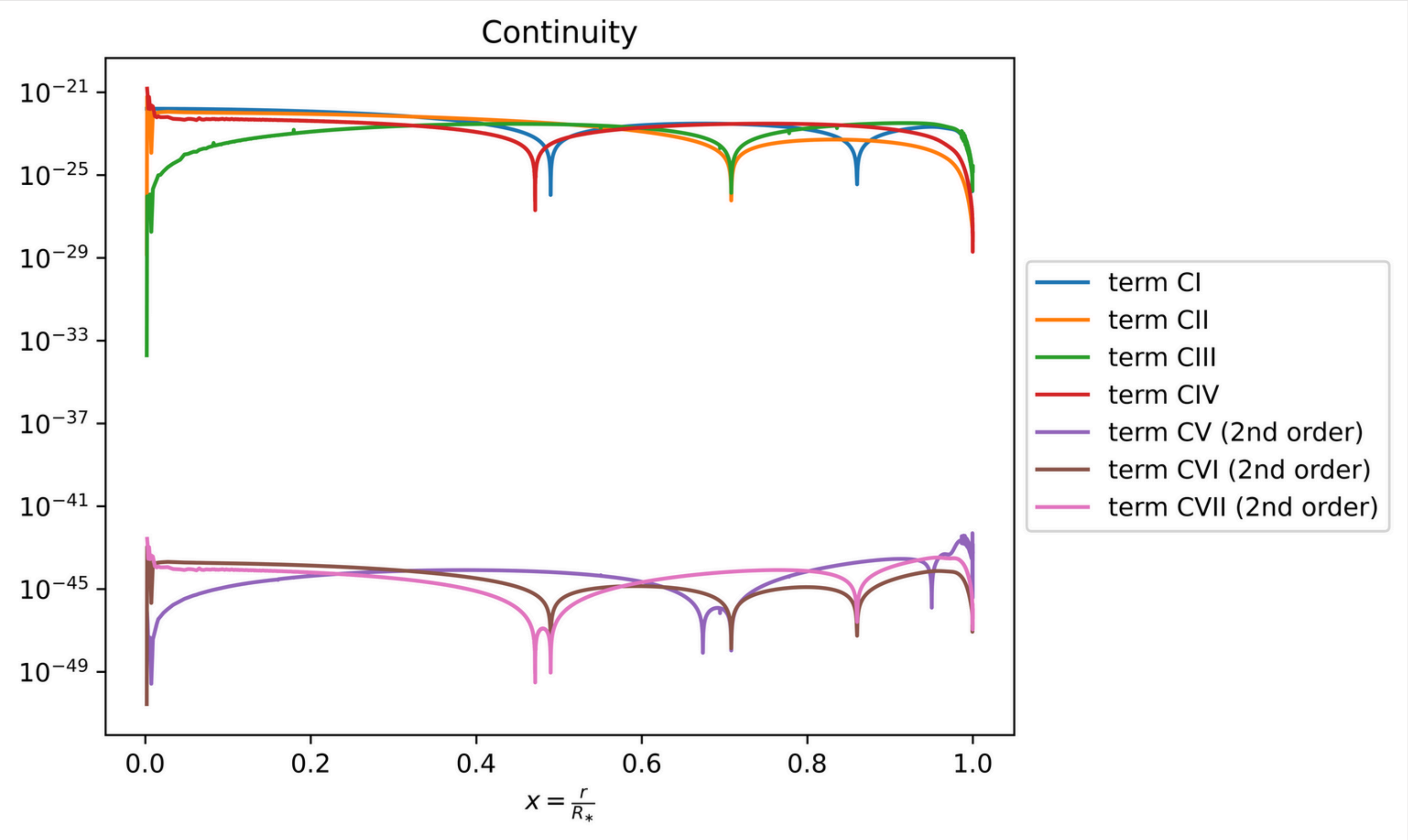
$$\frac{1}{\sqrt{4\pi}}\frac{\partial\rho'}{\partial r}\xi_r \qquad \frac{1}{\sqrt{4\pi}}\frac{2}{r}\rho'\xi_r \qquad \frac{1}{\sqrt{4\pi}}\rho'\frac{\partial\xi_r}{\partial r}$$



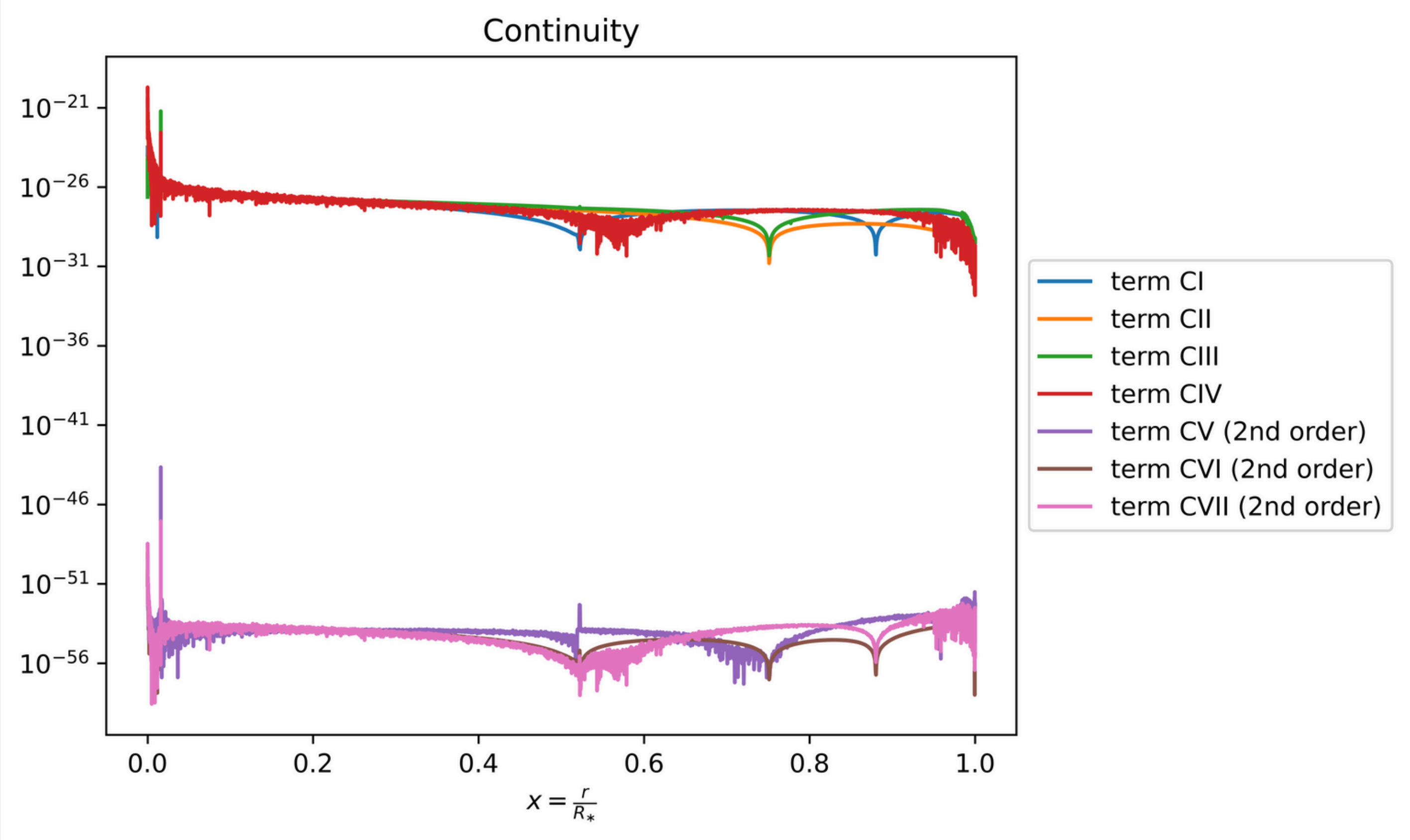
$$\frac{\text{1st}}{\text{2nd}} = \frac{e^{-23}}{e^{-48}}$$



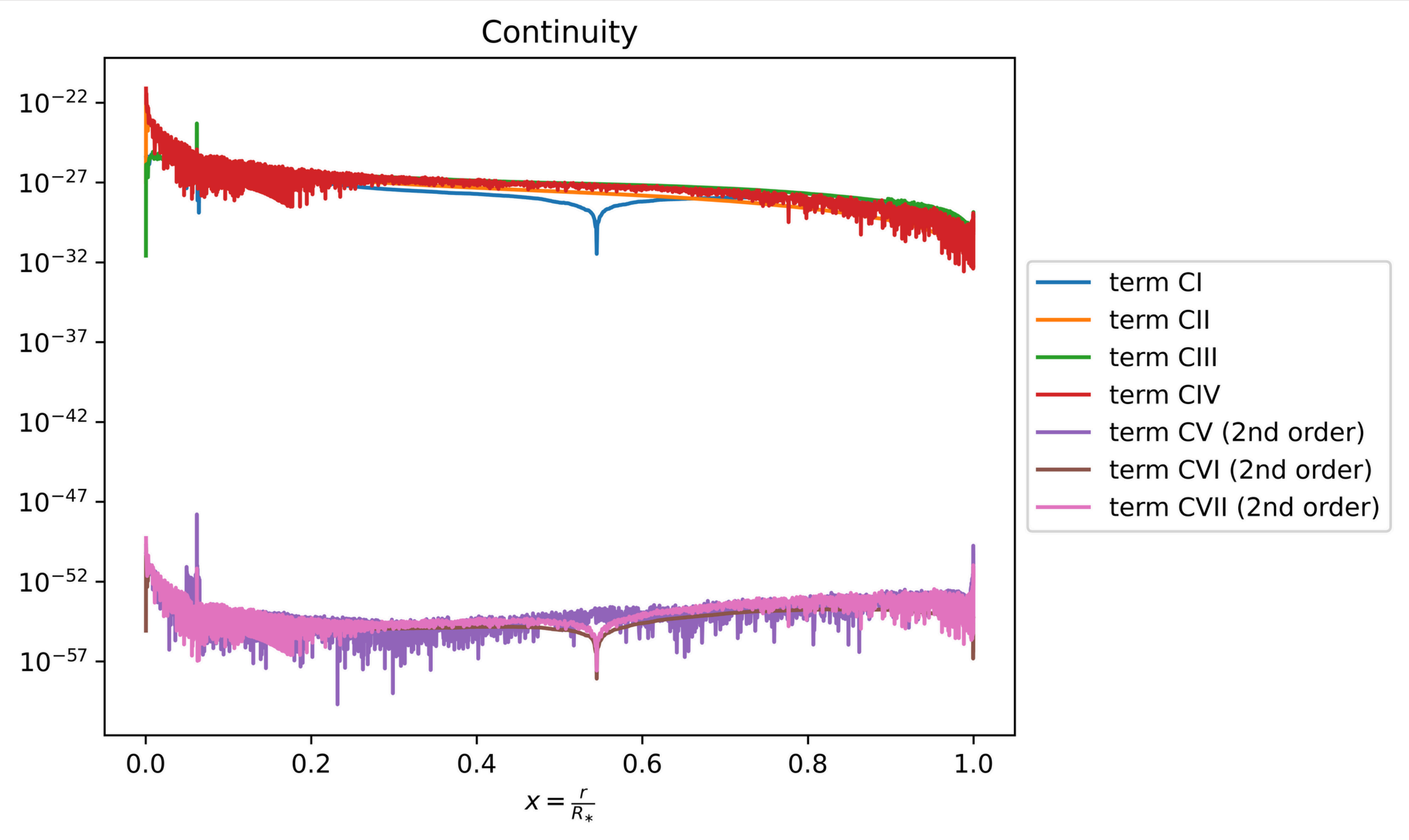
0 Gyr



1.56 Gyr



1.34 Gyr



Radial modes

Conservation of momentum

$$(\rho_0 + \boxed{\rho')\frac{\partial \vec{v}\,'}{\partial t}} + \boxed{\rho_0(\vec{v}\,'\cdot\nabla)\vec{v}\,'} = -\nabla p' - (\rho_0 + \boxed{\rho')\nabla\Phi'} - \rho'\nabla\Phi_0$$

$$\sigma^2\rho_0\,\xi_r - \frac{\partial p'}{\partial r} - \rho_0\frac{\partial \Phi'}{\partial r} - \rho'\frac{\partial \Phi_0}{\partial r} = -\boxed{\frac{\sigma^2}{\sqrt{4\pi}}\frac{2}{r}\rho_0\,\xi_r^2} - \boxed{\frac{\sigma^2}{\sqrt{4\pi}}\rho_0\,\xi_r\frac{\partial \xi_r}{\partial r}} - \boxed{\frac{\sigma^2}{\sqrt{4\pi}}\rho'\xi_r} + \boxed{\frac{1}{\sqrt{4\pi}}\rho'\,\frac{\partial \Phi'}{\partial r}}$$

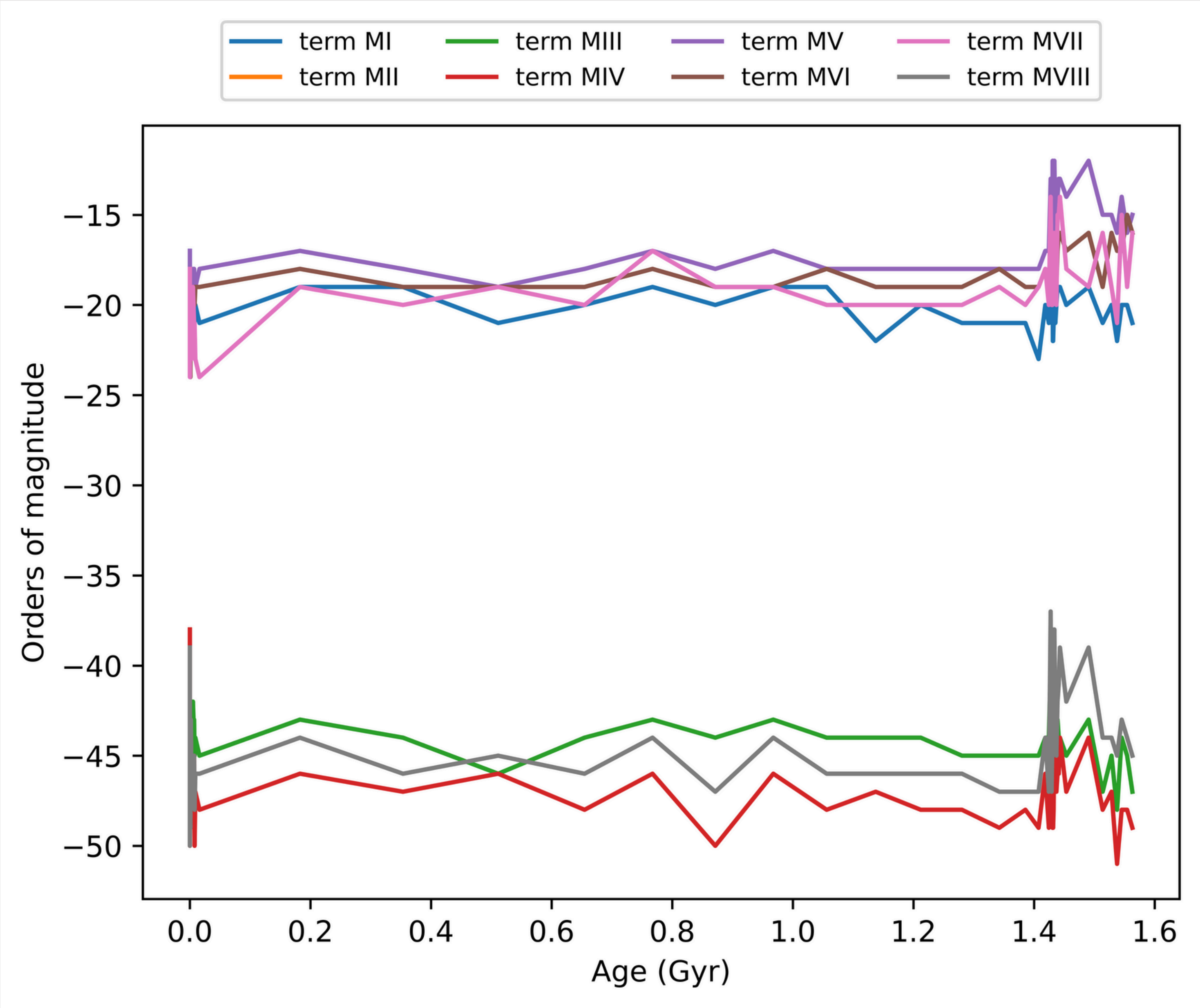
Radial modes

Conservation of momentum

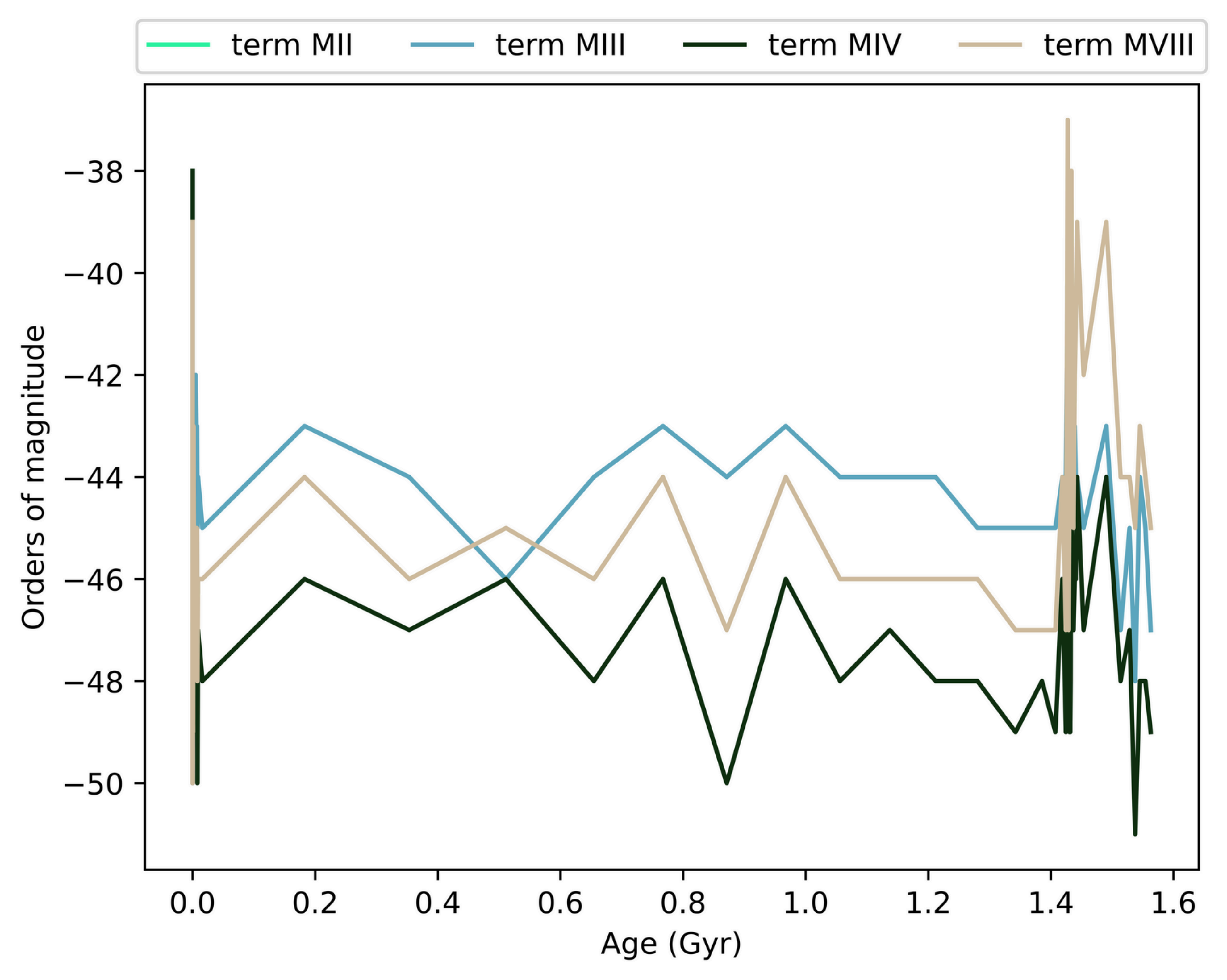
$$(\rho_0 + \boxed{\rho')\frac{\partial \vec{v}\,'}{\partial t}} + \boxed{\rho_0(\vec{v}\,' \cdot \nabla)\vec{v}\,'} = -\nabla p' - (\rho_0 + \boxed{\rho')\nabla\Phi'} - \rho'\nabla\Phi_0$$

$$\sigma^2 \rho_0\, \xi_r - \frac{\partial p'}{\partial r} - \rho_0 \frac{\partial \Phi'}{\partial r} - \rho' \frac{\partial \Phi_0}{\partial r} = -\boxed{\frac{\sigma^2}{\sqrt{4\pi}}\frac{2}{r}\rho_0\, \xi_r^2} - \boxed{\frac{\sigma^2}{\sqrt{4\pi}}\rho_0\, \xi_r \frac{\partial \xi_r}{\partial r}} - \boxed{\frac{\sigma^2}{\sqrt{4\pi}}\rho' \xi_r} + \boxed{\frac{1}{\sqrt{4\pi}}\rho' \frac{\partial \Phi'}{\partial r}}$$

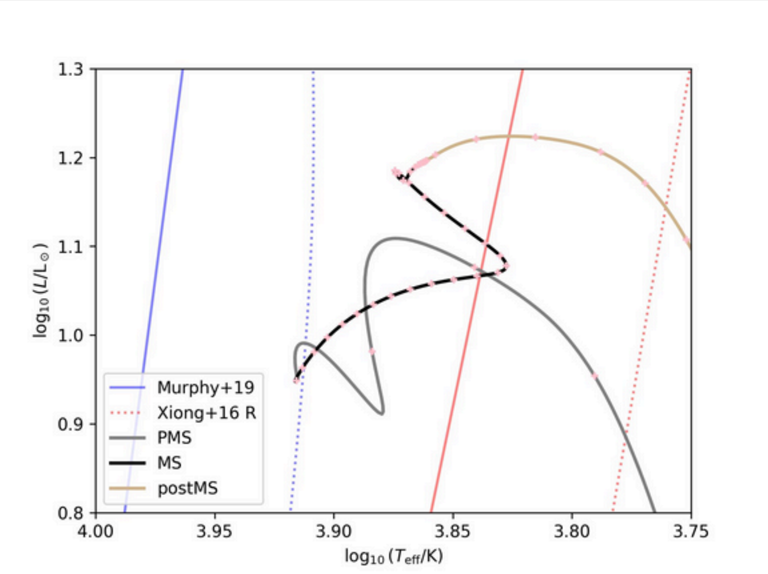
$\boxed{e^{-20}}$ $\boxed{e^{-19}}$ $\boxed{e^{-19}}$ $\boxed{e^{-19}}$ $\boxed{e^{-45}}$ $\boxed{e^{-46}}$ $\boxed{e^{-47}}$ $\boxed{e^{-46}}$



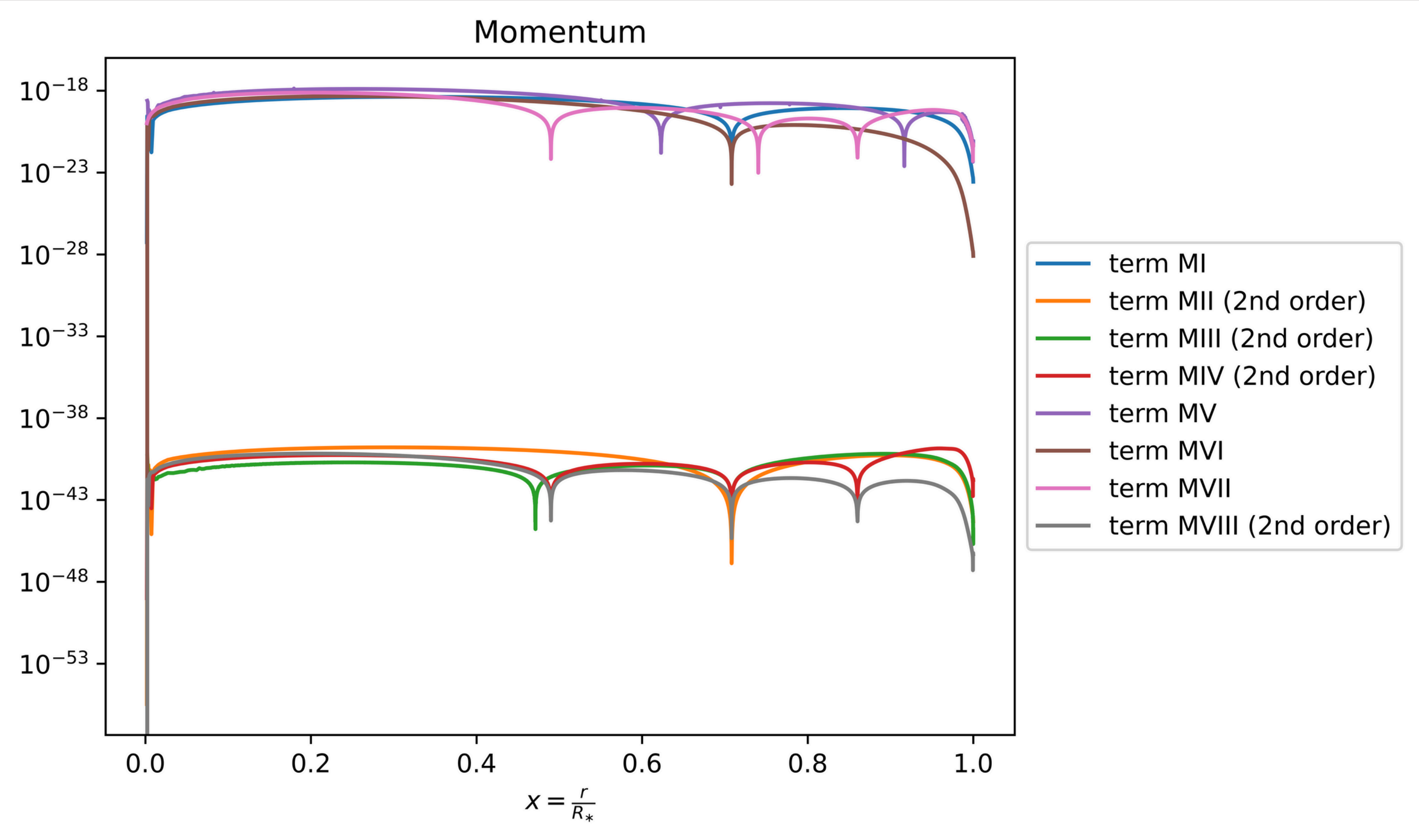
$$\frac{\sigma^2}{\sqrt{4\pi}}\frac{2}{r}\rho_0\,\xi_r^2 \qquad \frac{\sigma^2}{\sqrt{4\pi}}\rho_0\,\xi_r\frac{\partial \xi_r}{\partial r} \qquad \frac{\sigma^2}{\sqrt{4\pi}}\rho'\xi_r \qquad \frac{1}{\sqrt{4\pi}}\rho'\,\frac{\partial \Phi'}{\partial r}$$



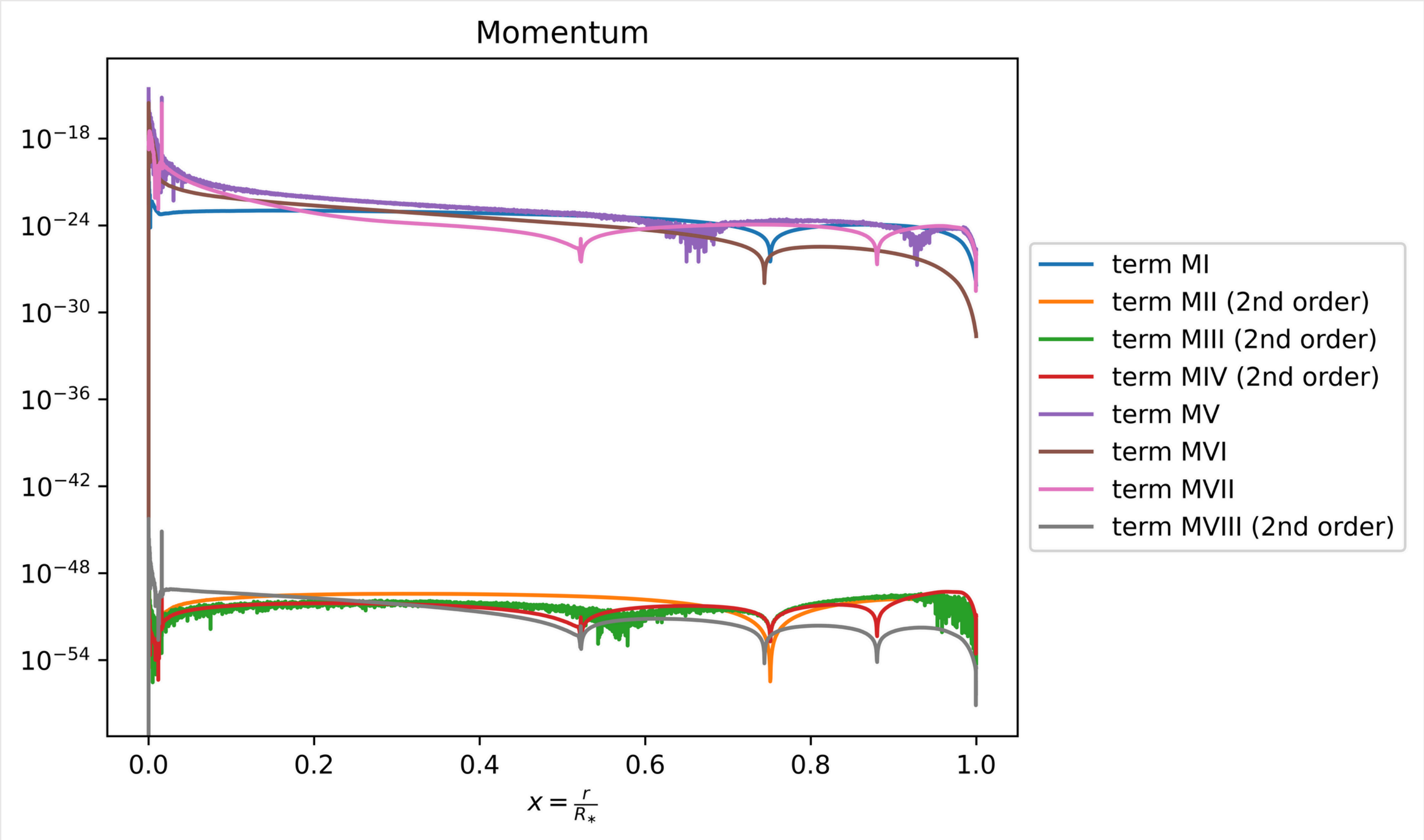
$$\frac{\text{1st}}{\text{2nd}} = \frac{e^{-19}}{e^{-46}}$$

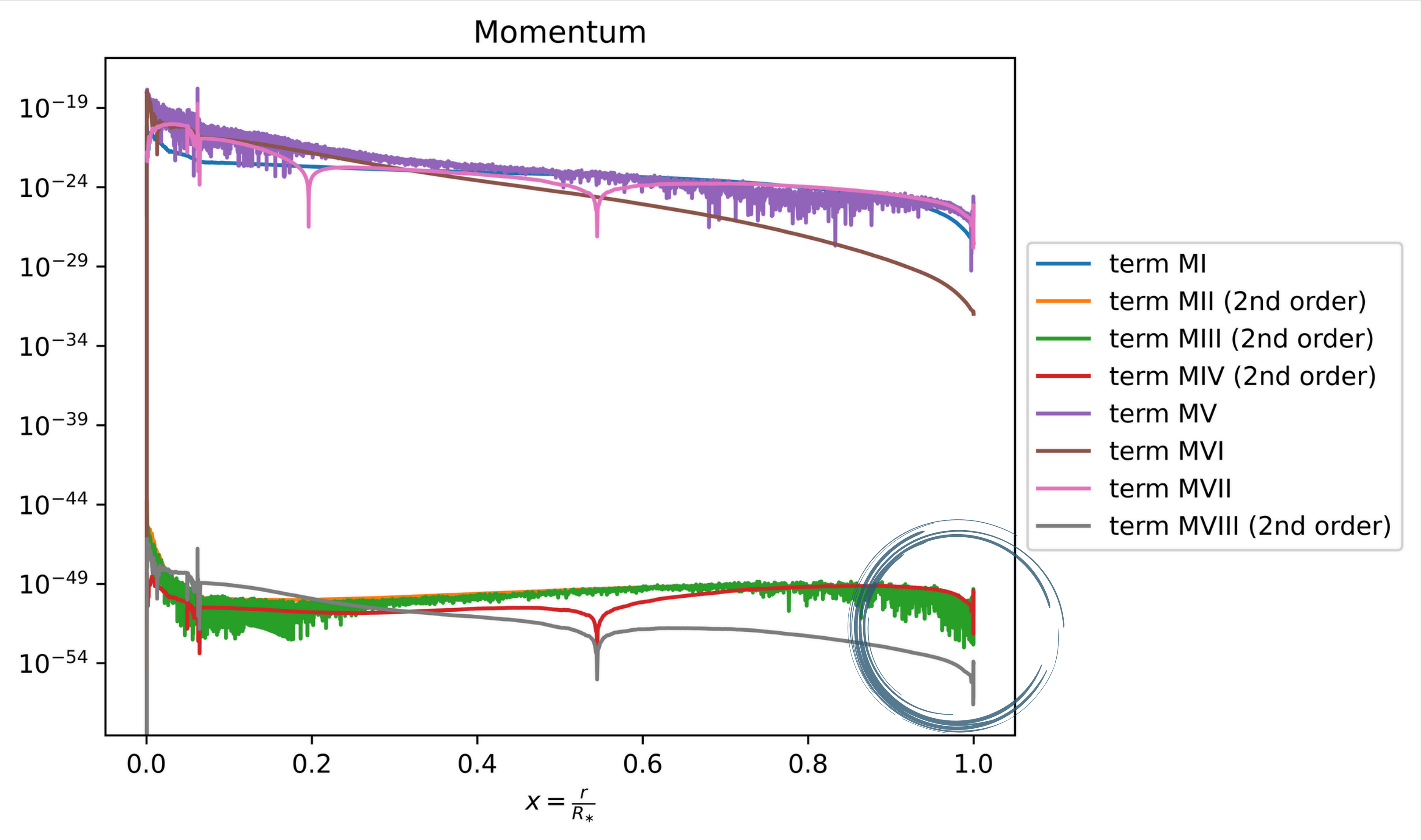


0 Gyr



1.56 Gyr





1.34 Gyr

Radial modes

Conservation of energy

$$(\rho_0 + \rho')\frac{\partial p'}{\partial t} + \rho_0[\vec{v}\,' \cdot (\nabla p_0) + \vec{v}\,' \cdot (\nabla p')] + \rho'[\vec{v}\,' \cdot (\nabla p_0)] = \Gamma_1 (p_0 + p')\frac{\partial \rho'}{\partial t} + \rho_0[\vec{v}\,' \cdot (\nabla \rho_0) + \vec{v}\,' \cdot (\nabla \rho')] + \rho'[\vec{v}\,' \cdot (\nabla \rho_0)]$$

$$\rho_0\, p' + \rho_0\, \xi_r \frac{\partial p_0}{\partial r} - \Gamma_1 p_0 \rho' - \rho_0\, \xi_r \frac{\partial \rho_0}{\partial r} = -\frac{\rho'}{\sqrt{4\pi}} p' - \frac{\rho_0}{\sqrt{4\pi}}\, \xi_r \frac{\partial p'}{\partial r} - \frac{\rho'}{\sqrt{4\pi}}\, \xi_r \frac{\partial p_0}{\partial r} + \frac{\Gamma_1}{\sqrt{4\pi}} p' \rho' + \frac{\rho_0}{\sqrt{4\pi}}\, \xi_r \frac{\partial \rho'}{\partial r} + \frac{\rho'}{\sqrt{4\pi}}\, \xi_r \frac{\partial \rho_0}{\partial r}$$

Radial modes

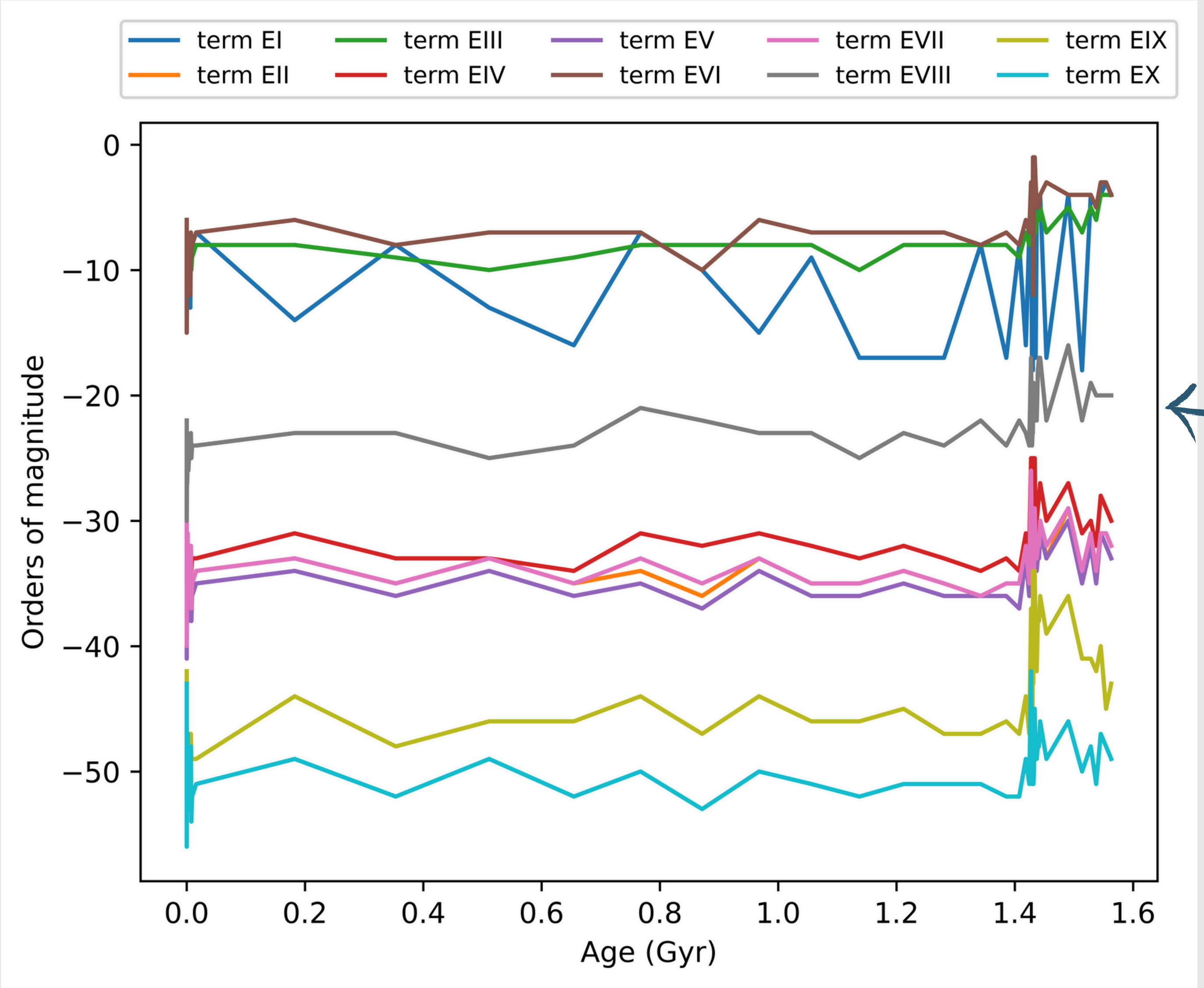
Conservation of energy

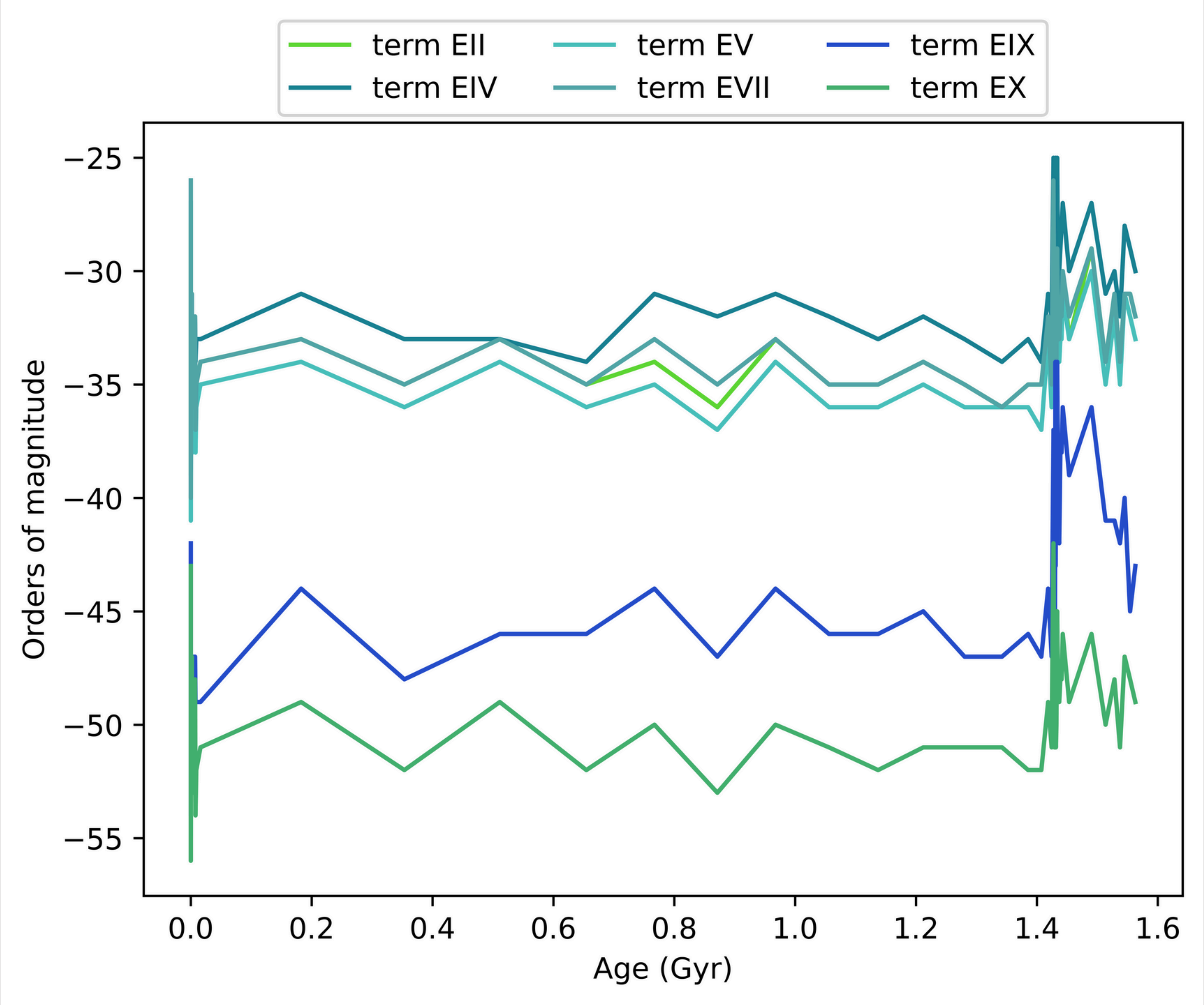
$$(\rho_0 + \boxed{\rho')\frac{\partial p'}{\partial t}} + \rho_0[\vec{v}\,' \cdot (\nabla p_0) + \boxed{\vec{v}\,' \cdot (\nabla p')}] + \boxed{\rho'[\vec{v}\,' \cdot (\nabla p_0)]} = \Gamma_1(p_0 + \boxed{p')\frac{\partial \rho'}{\partial t}} + \rho_0[\vec{v}\,' \cdot (\nabla \rho_0) + \boxed{\vec{v}\,' \cdot (\nabla \rho')}] + \boxed{\rho'[\vec{v}\,' \cdot (\nabla \rho_0)]}$$

$$\rho_0\, p' + \rho_0\, \xi_r \frac{\partial p_0}{\partial r} - \Gamma_1 p_0 \rho' - \rho_0\, \xi_r \frac{\partial \rho_0}{\partial r} = -\boxed{\frac{\rho'}{\sqrt{4\pi}} p'} - \boxed{\frac{\rho_0}{\sqrt{4\pi}}\, \xi_r \frac{\partial p'}{\partial r}} - \boxed{\frac{\rho'}{\sqrt{4\pi}}\, \xi_r \frac{\partial p_0}{\partial r}} + \boxed{\frac{\Gamma_1}{\sqrt{4\pi}} p' \rho'} + \boxed{\frac{\rho_0}{\sqrt{4\pi}}\, \xi_r \frac{\partial \rho'}{\partial r}} + \boxed{\frac{\rho'}{\sqrt{4\pi}}\, \xi_r \frac{\partial \rho_0}{\partial r}}$$

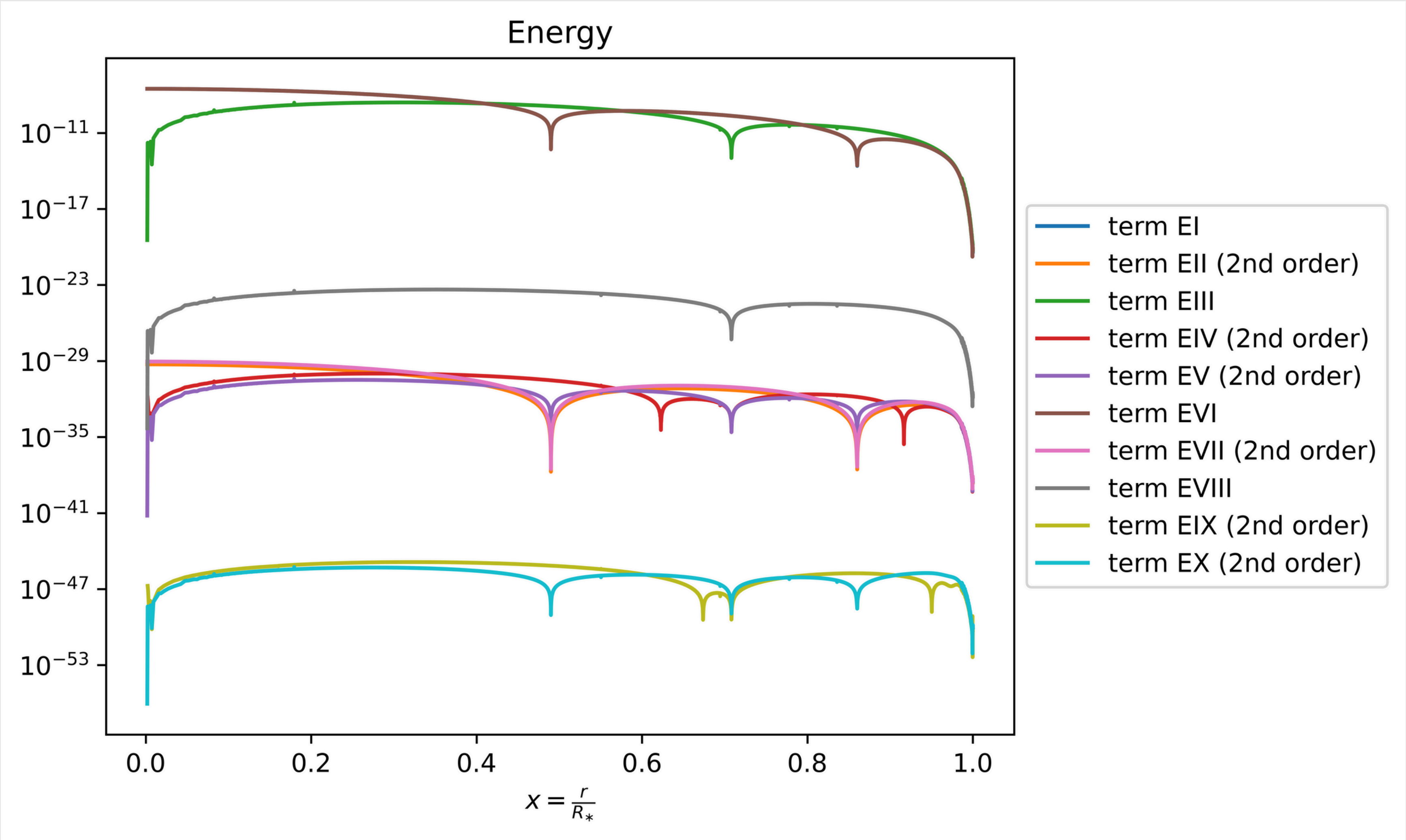
$\boxed{e^{-7}}$ $\boxed{e^{-8}}$ $\boxed{e^{-7}}$ $\boxed{e^{-22}}$ $\boxed{e^{-34}}$ $\boxed{e^{-34}}$ $\boxed{e^{-35}}$ $\boxed{e^{-34}}$ $\boxed{e^{-47}}$ $\boxed{e^{-49}}$

$$\frac{\rho'}{\sqrt{4\pi}}p' \quad \frac{\rho_0}{\sqrt{4\pi}}\xi_r\frac{\partial p'}{\partial r} \quad \frac{\rho'}{\sqrt{4\pi}}\xi_r\frac{\partial p_0}{\partial r} \quad \frac{\Gamma_1}{\sqrt{4\pi}}p'\rho' \quad \frac{\rho_0}{\sqrt{4\pi}}\xi_r\frac{\partial \rho'}{\partial r} \quad \frac{\rho'}{\sqrt{4\pi}}\xi_r\frac{\partial \rho_0}{\partial r}$$

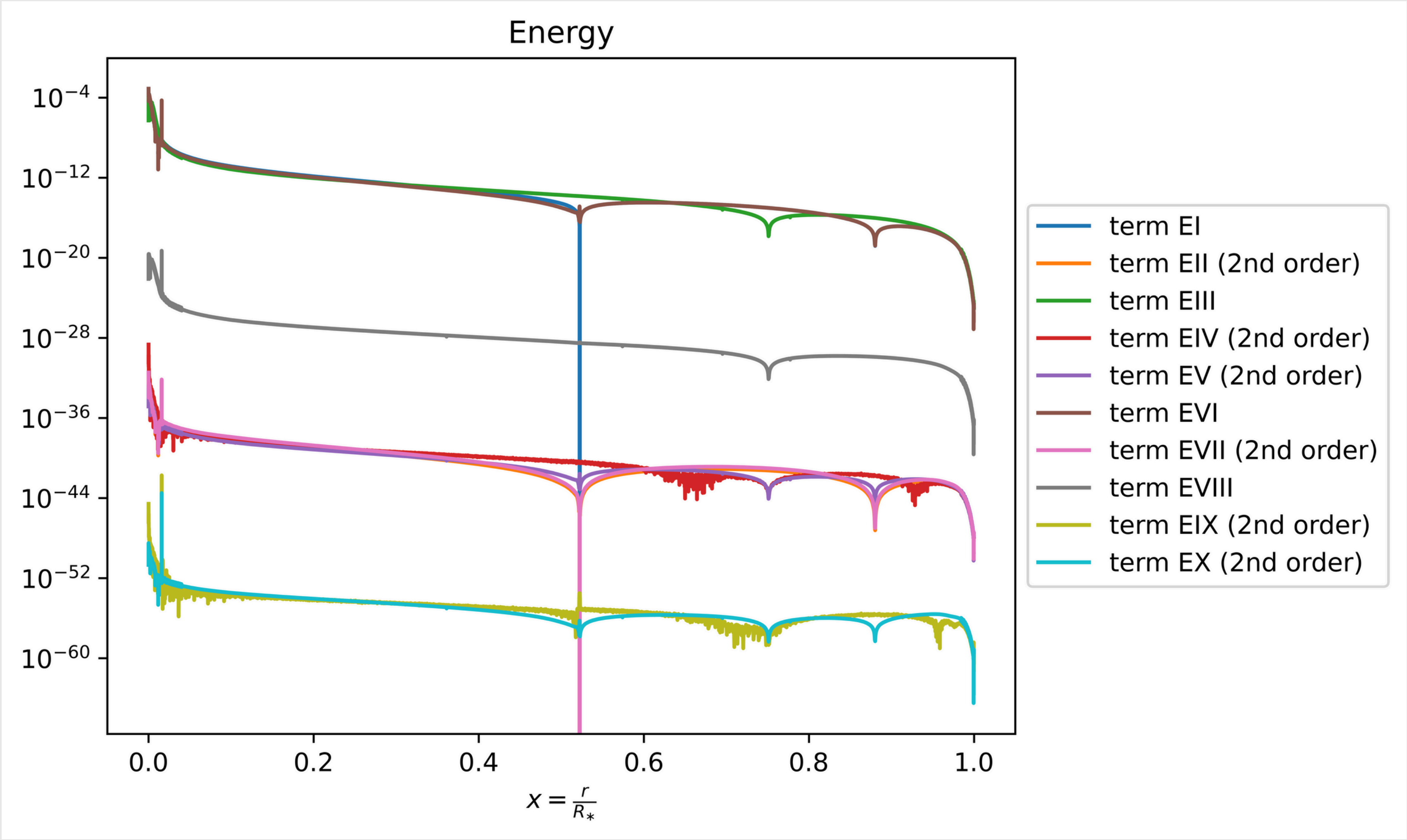




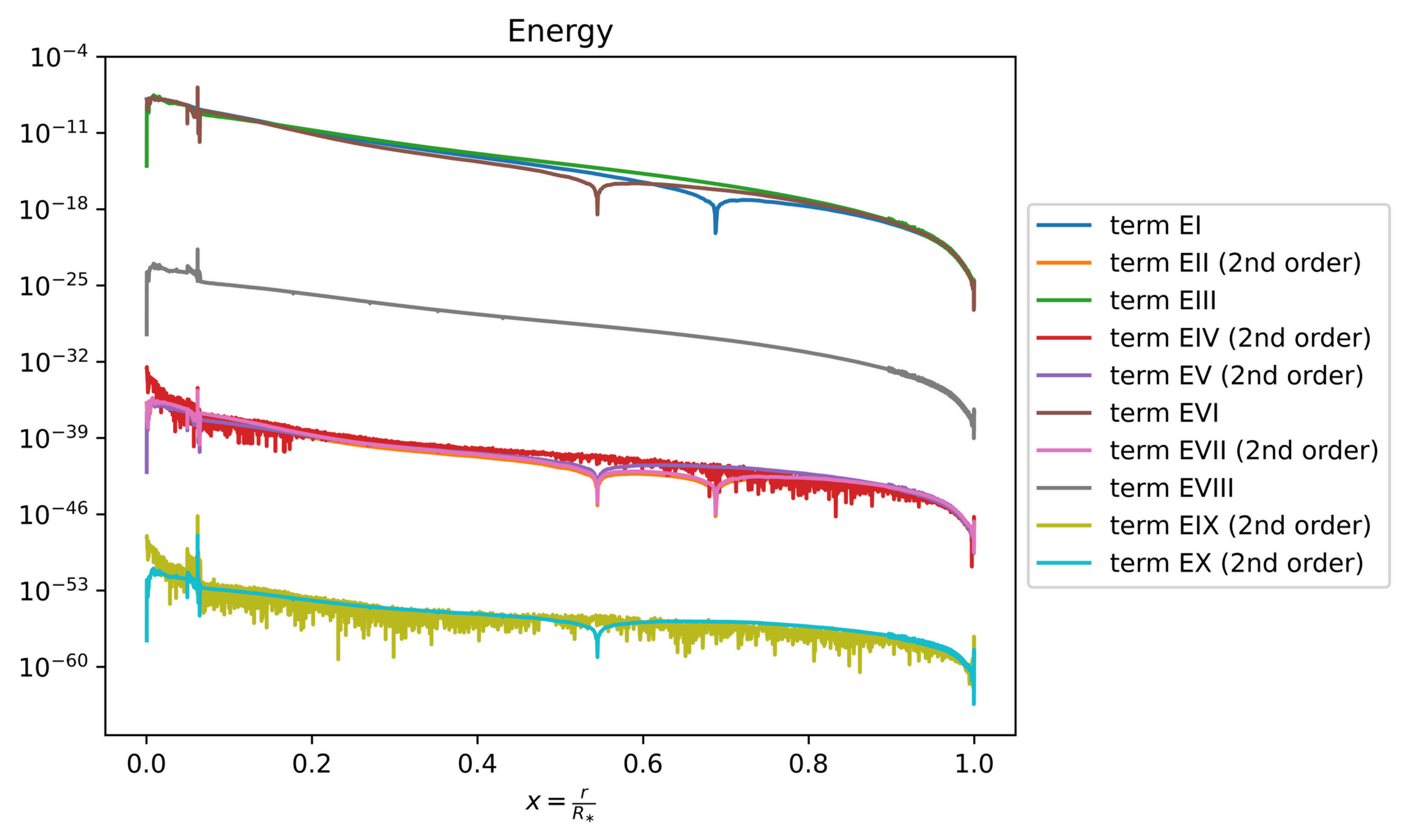
$$\frac{\text{1st}}{\text{2nd}} = \frac{e^{-7}}{e^{-49}}$$



0 Gyr

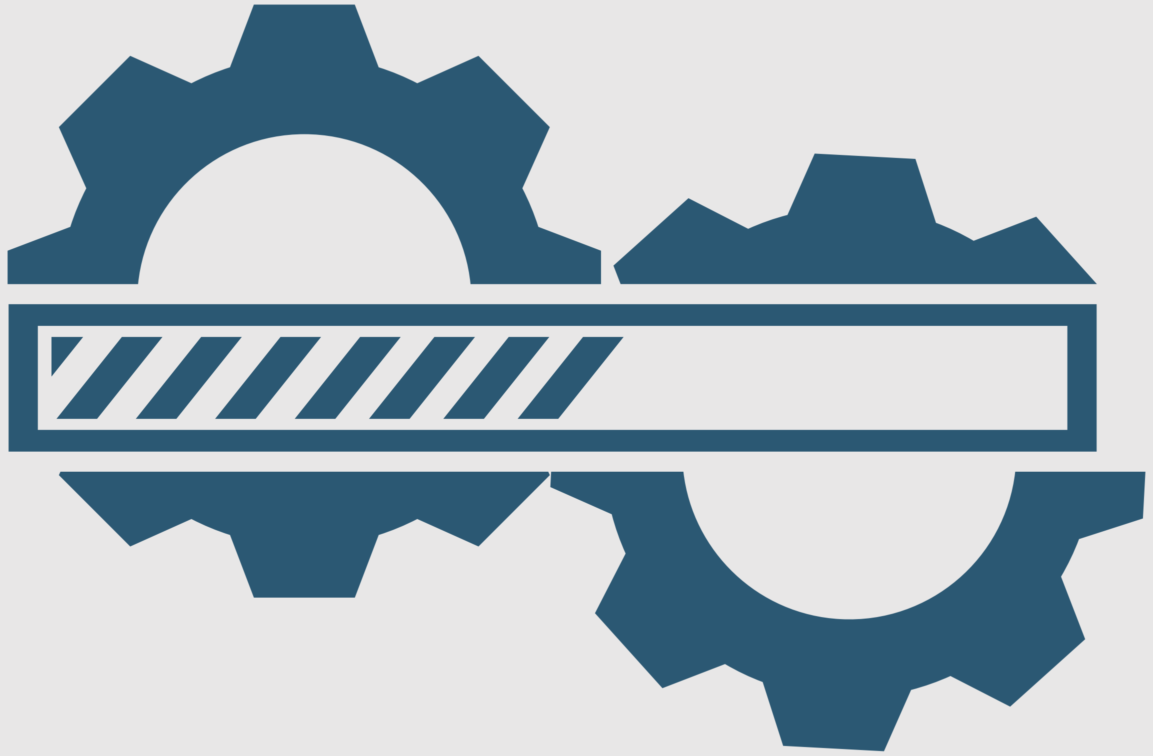


1.56 Gyr



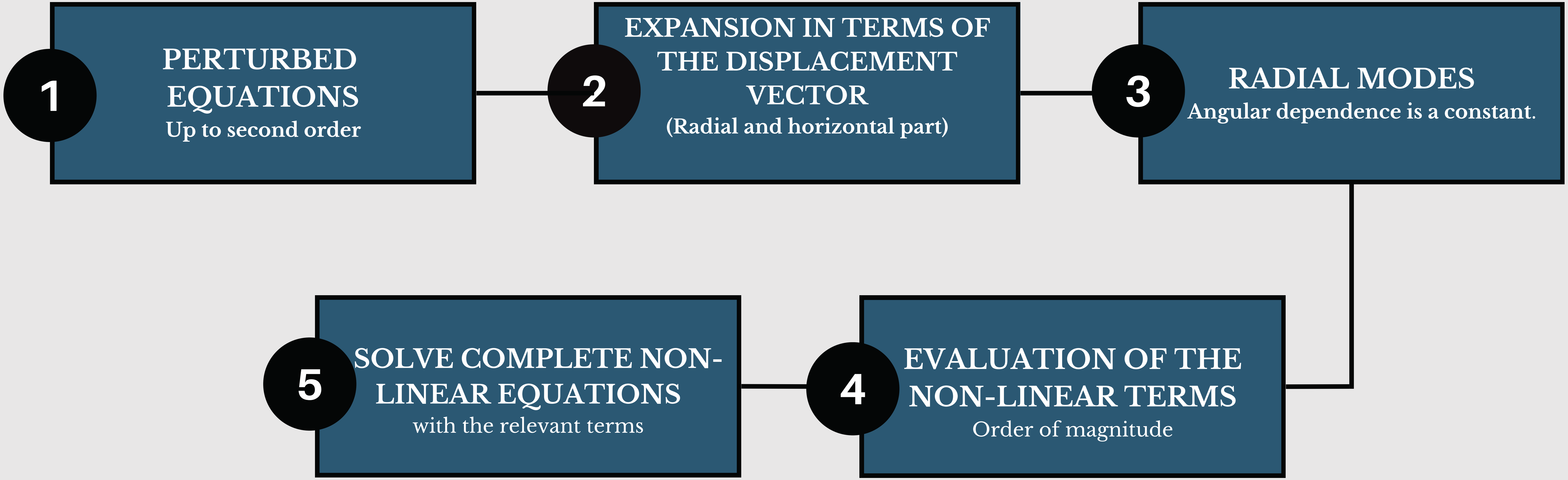
1.34 Gyr

Dimensionless equations



Development of a non-linear pulsational code for radial modes.

Non-radial modes


1
PERTURBED EQUATIONS
Up to second order
2
EXPANSION IN TERMS OF THE DISPLACEMENT VECTOR
(Radial and horizontal part)
3
RADIAL MODES
Angular dependence is a constant.
4
EVALUATION OF THE NON-LINEAR TERMS
Order of magnitude
5
SOLVE COMPLETE NON-LINEAR EQUATIONS
with the relevant terms

Final remarks

- We have expanded the basic equations of hydrodynamics up to the second order of perturbations.
- In order to study and evaluate the non-linear terms, we have studied radial modes.
- In the continuity and momentum conservation equations, we found a difference of two orders of magnitude between the terms. This difference is not enough to ignore those terms.
- In contrast, in the conservation of energy, there are two clearly negligible terms ($\sim e^{-50}$) , which simplify the non-linear expression.
- Next step: analyze the differences in certain regions of the star between specific terms of similar orders. Development of a non-linear pulsational code for radial modes.

Thank You

References

[1] **Nonradial oscillations of stars.** *Unno et al. (1989)*

[2] **Asteroseismology.** *Aerts et al. (2010)*

[3] **Theory of stellar pulsation.** *Cox, J.P. (1980)*

[4] **Numerical Treatment of Linear and Nonlinear Stellar Pulsations.** *Glatzel (2021)*

[4] **Asteroseismology across the HR diagram.** *Kurtz (2022)*

Back up slides

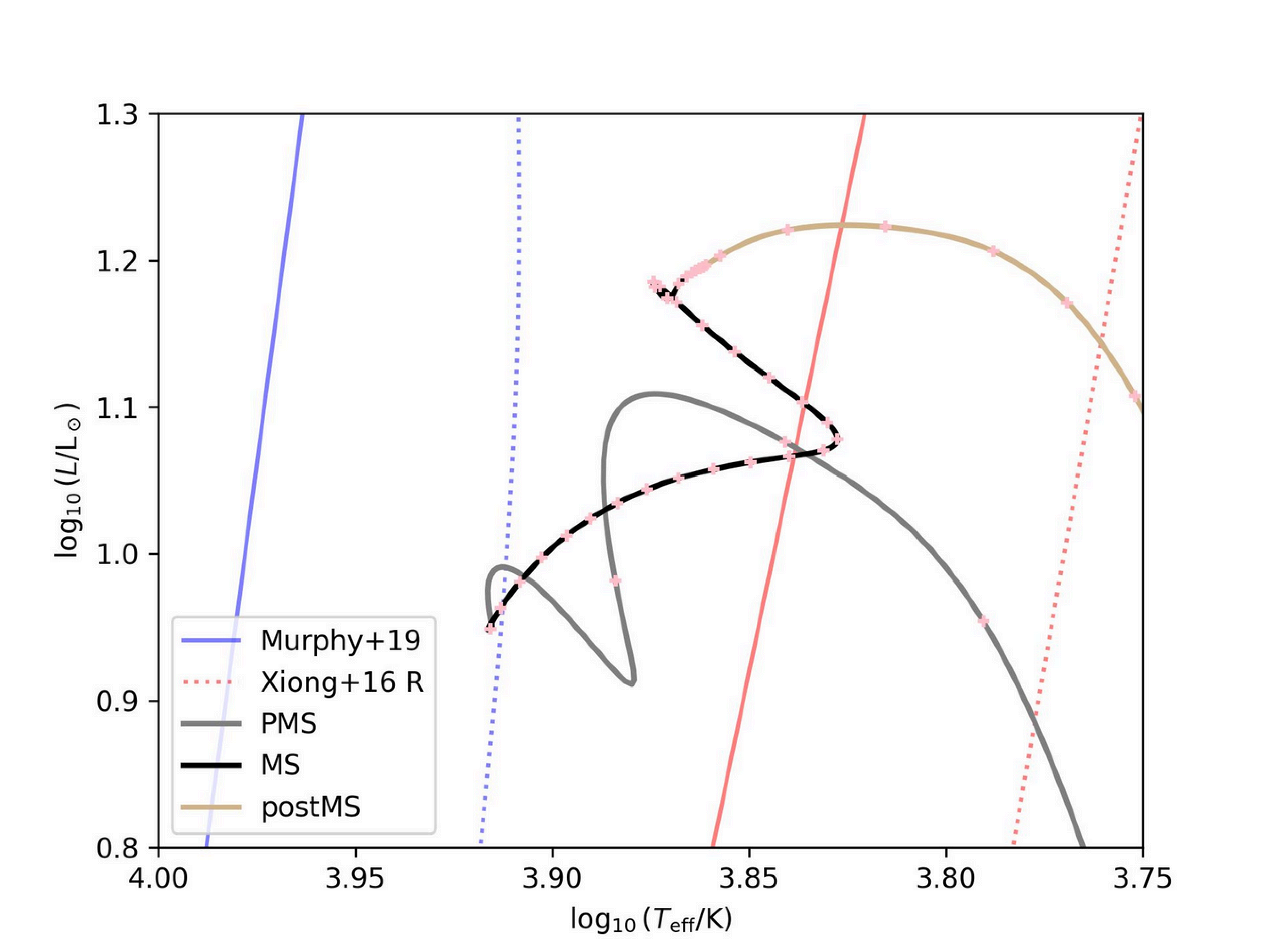
MESA model 1.7 $M_\odot$

23.05.1 version MESA

M=1.7 $M_\odot$
Z=0.0142 *Asplund et al. 2009*
Y=0.02703

No overshooting

Red and blue lines are two different estimates of the location of the instability strip limits.
Grey-black-tan line ->evolutionary track
pink crosses -> available profiles.



CESAM model 1.5 $M_\odot$

- CoRoT ESTA exercise.
- Opacity: OPAL
- Atmosphere: Eddington
- Solar mixture: Greveese & Noels
- Nuclear reaction: NACRE
- Mixing length α=1.8
- Overshooting: No

Boundary conditions

Boundary conditions :
Unno et al. (1989)

$$y_1 = \frac{\xi_r}{r}$$

$$y_2 = \frac{1}{gr}\left(\frac{p'}{\rho_0} + \Phi'\right)$$

$$y_3 = \frac{1}{gr}\ \Phi'$$

$$y_4 = \frac{1}{g}\frac{d\Phi}{dr}$$

- Inner:

$$l y_3 - y_4 = 0$$

$$\frac{c_1\omega^2}{l}\ y_1 - y_2 = 0$$

- Outer:

$$(l+1)\ y_3 + y_4 = 0$$

$$y_1 - y_2 + y_3 = 0$$

- Normalization:

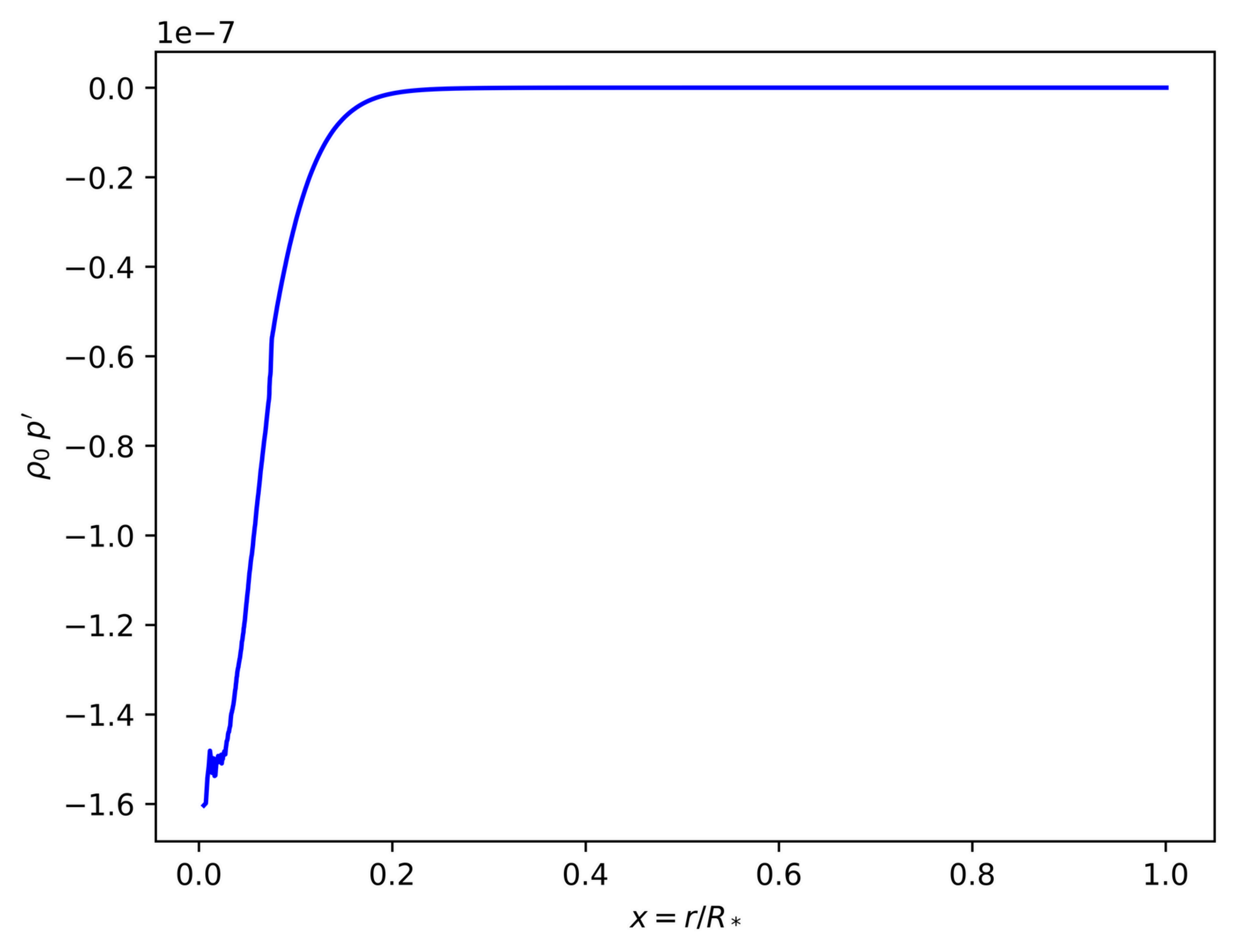
$$y_1(R_*) = 1$$

Python package

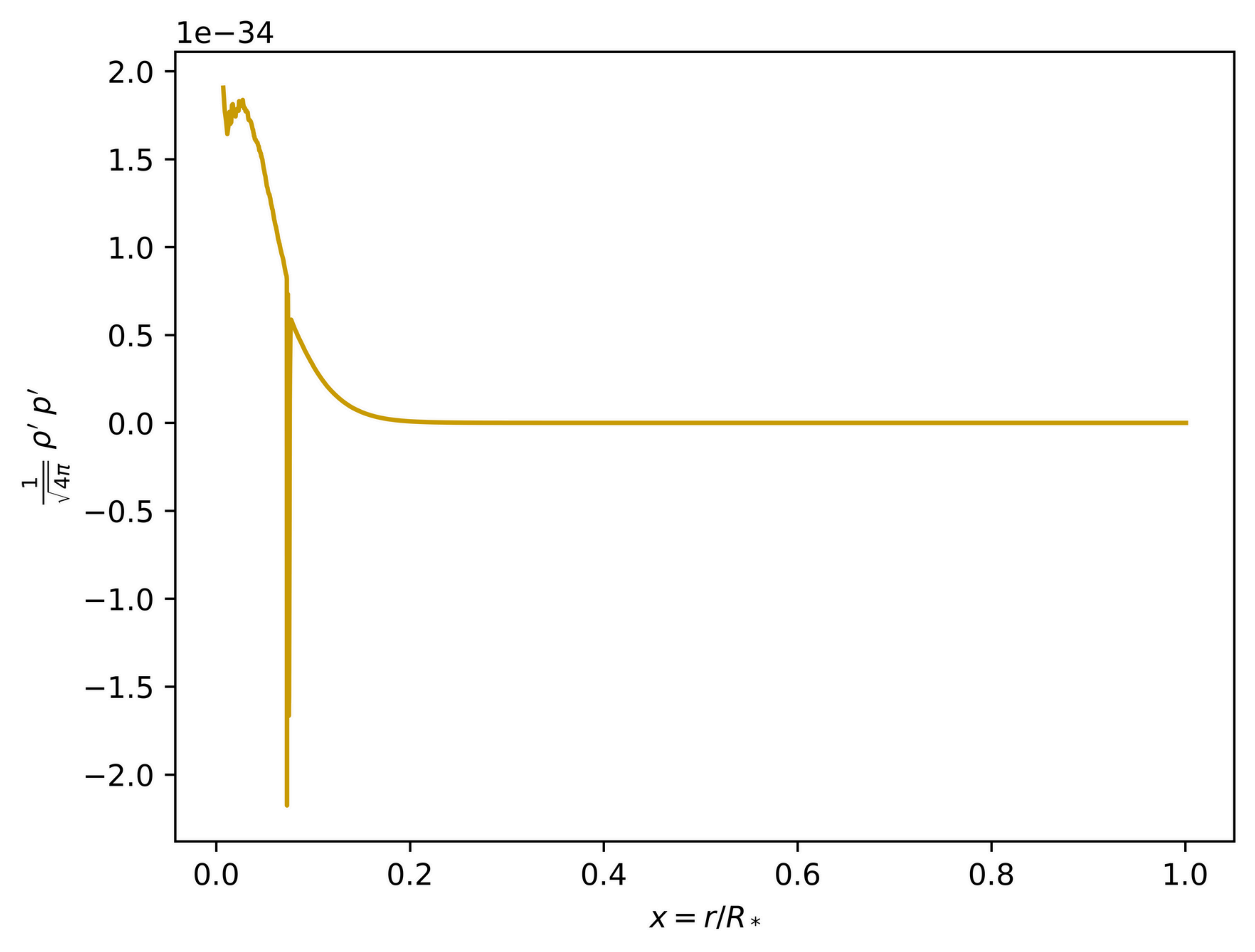
Python package: scipy

- Boundary condition function
- System of linear adiabatic equations function

Radial modes

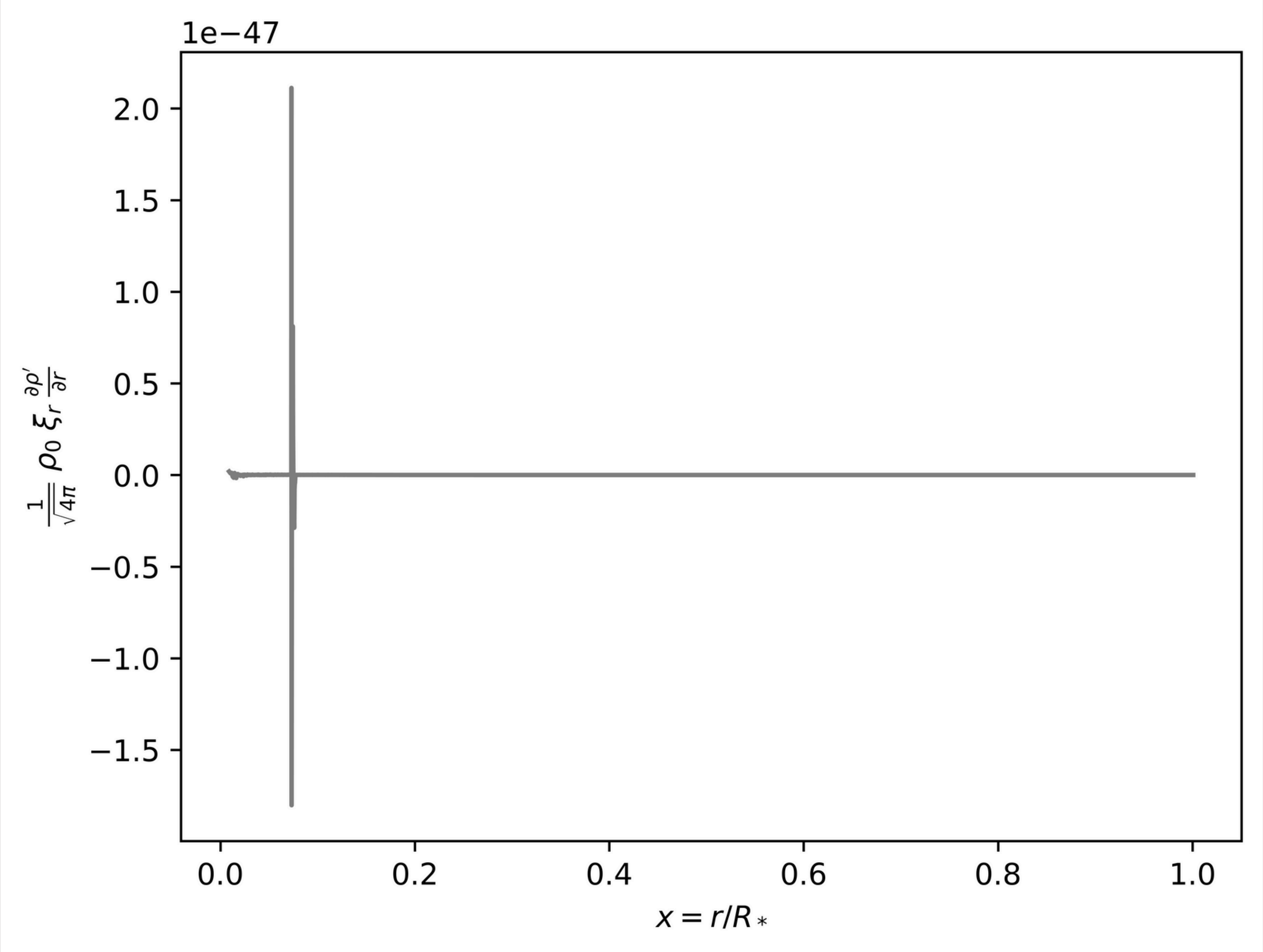


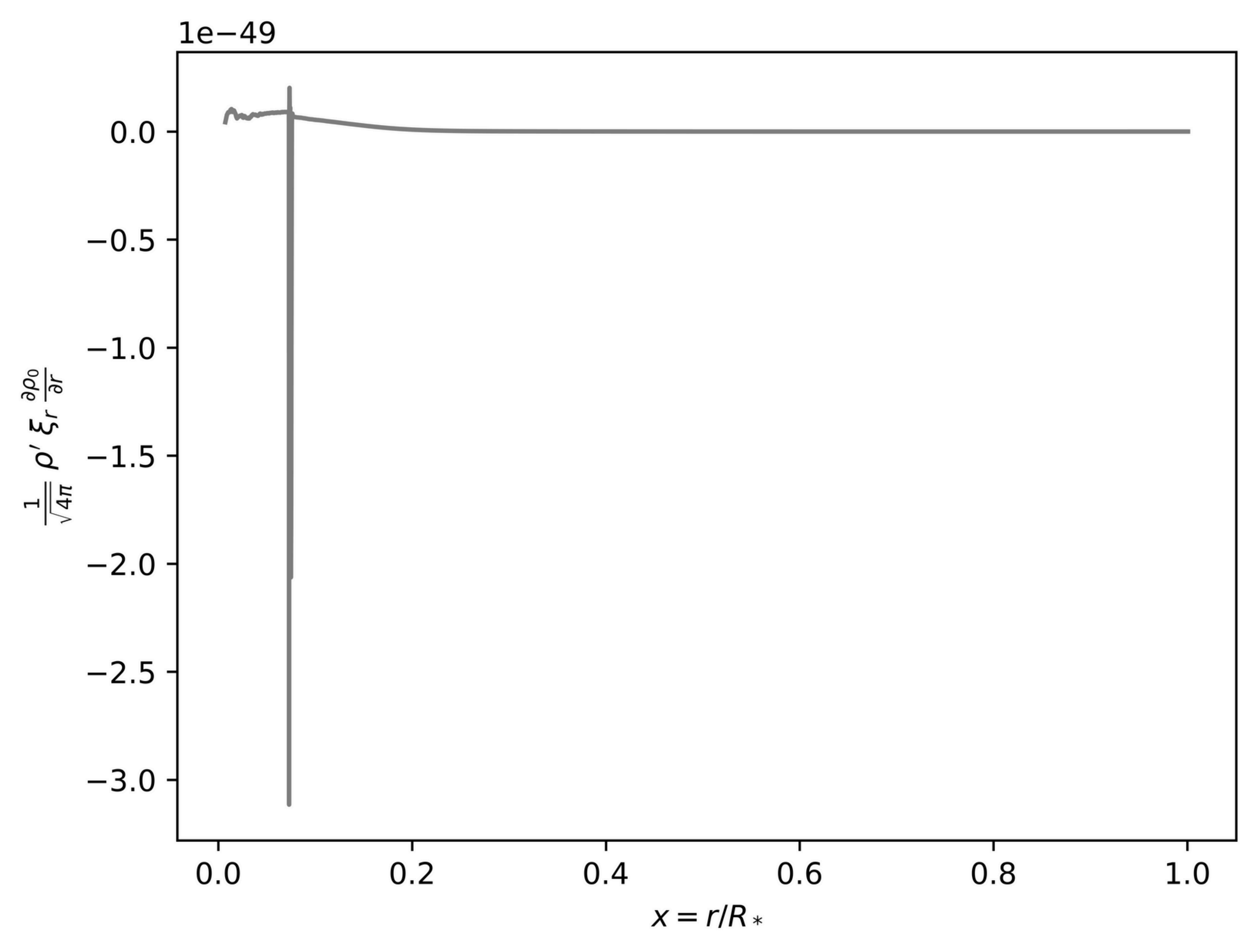
Conservation of energy



Radial modes

Conservation of energy





ρ density

p pressure

$\vec{v}$ velocity

Φ gravitational potential

Γ_1 first adiabatic exponent

$f(\vec{r},t) = f_0(r) + f'(\vec{r},t)$ small perturbations of the variables

$\vec{v}_0 = 0$

$\vec{\xi} = \vec{r} - \vec{r_0}$ displacement vector

$\vec{v}\,' = \dfrac{d\vec{\xi}}{dt}$ perturbed velocity

$l = 0$

$\vec{\xi} = \vec{r} - \vec{r_0} = \vec{\xi_r} + \vec{\xi_h}$ displacement vector

$f'(t,r,\theta,\phi) = f'(r)\,Y_l^m(\theta,\phi)\,e^{i\sigma t}$ perturbed variable

$l = 0 \rightarrow Y_0^0(\theta,\phi) = \dfrac{1}{\sqrt{4\pi}}$

ρ density

p pressure

$\vec{v}$ velocity

Φ gravitational potential

Γ_1 first adiabatic exponent

$f(\vec{r},t) = f_0(r) + f'(\vec{r},t)$ small perturbations of the variables

$\vec{v}_0 = 0$

$\vec{\xi} = \vec{r} - \vec{r_0}$ displacement vector

$\vec{v}\,' = \dfrac{d\vec{\xi}}{dt}$ perturbed velocity

$l = 0$

$\vec{\xi} = \vec{r} - \vec{r_0} = \vec{\xi_r} + \vec{\xi_h}$ displacement vector

$f'(t,r,\theta,\phi) = f'(r)\,Y_l^m(\theta,\phi)\,e^{i\sigma t}$ perturbed variable

$l = 0 \rightarrow Y_0^0(\theta,\phi) = \dfrac{1}{\sqrt{4\pi}}$

Non-linear continuity	$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\, \rho_0\, \vec{v}\,') + \nabla \cdot (\, \rho'\, \vec{v}\,') = 0$
Non-linear conservation of momentum	$(\rho_0+\rho')\, \frac{\partial \vec{v}\,'}{\partial t} + \rho_0(\vec{v}\,'\cdot\nabla)\vec{v}\,' = -\nabla p' - (\rho_0+\rho')\, \nabla\Phi' - \rho'\, \nabla\Phi_0$
Non-linear conservation of energy	$(\rho_0 + \rho')\, \frac{\partial p'}{\partial t} + \rho_0[\vec{v}\,'\cdot(\nabla p_0) + \vec{v}\,'\cdot(\nabla p')] + \rho'[\vec{v}\,'\cdot(\nabla p_0)] = \Gamma_1(p_0 + p')\frac{\partial \rho'}{\partial t} + \rho_0[\vec{v}\,'\cdot(\nabla\rho_0) + \vec{v}\,'\cdot(\nabla\rho)] + \rho'[\vec{v}\cdot(\nabla\rho_0)]$
Non-linear equation of Poisson	$\nabla^2\Phi' = 4\pi G\rho'$

$\vec{\xi} = \vec{r} - \vec{r_0} = \vec{\xi_r} + \vec{\xi_h}$ displacement vector

$f'(t, r, \theta, \phi) = f'(r)\, Y_l^m(\theta, \phi)\, e^{i\sigma t}$ perturbed variable

$$l = 0 \rightarrow Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}}$$