Lorentzian manifolds isometrically embeddable in Lorentz-Minkowski space $\mathbb{L}^N$

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0. Introduction

Theorem

(Nash, Ann. Math.'56.) *Any smooth Riemannian manifold* (M, g) *can be isometrically embedded in [an arbitrarily small open subset* U *of]* $\mathbb{R}^N$, *for some* N.

Notes:

- Smooth: C^∞ —but C^3 is enough
- Value of N: if $m =$dim (M), Nash's bound was $N = (m+1)(3m(m+1)/2 + 4m)$
- Günther's '89 bound: $\max \{2m + m(m+1)/2, m + 5 + m(m+1)/2\}$ (optimal?, it can be lowered in many particular cases)

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  max $\{2m + m(m+1)/2, m+5+m(m+1)/2\}$
  (optimal?, it can be lowered in many particular cases)

We do not worry about these bounds but for the following:

Is a Lorentzian version available?

0. Introduction

Independent simple arguments by Greene (Memoirs AMS'70) and Clarke (Proc. London'70) show:

Theorem

Any smooth manifold M endowed with a pseudo-metric (or, equivalently a possibly degenerate quadratic form) can be isometrically embedded in semi-Euclidean space $\mathbb{R}^N_\nu$ for sufficiently large dimension N and index ν.

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(just by reducing the problem to the Riemannian one).
However, the question is not so trivial if the index ν is not allowed to be arbitrary. That is, we focus in:

> *Which Lorentzian manifolds can be isometrically embedded in some $\mathbb{L}^N$?*

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About this problem:

1. Not all the Lorentzian manifolds (M, g) are isometrically embeddable in $\mathbb{L}^N$ (consider some (M, g) with closed timelike curves)

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2. The most important class of Lorentzian manifolds (from the viewpoint of hyperbolic equations, General Relativity, etc.) are the *globally hyperbolic spacetimes*
3. Clarke '70 claimed that any globally hyperbolic spacetime can be isometrically embedded in $\mathbb{L}^N$.
4. However, his proof was affected by the so-called *folk problems of smoothability* of causally-constructed functions.

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A Lorentzian manifold (M, g) is isometrically embeddable in some $\mathbb{L}^N$ iff: it admits a steep temporal function τ *(i.e. $g(\nabla\tau, \nabla\tau) \leq -1$).*

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Theorem

Any globally hyperbolic s-t. admits a steep temporal function τ

by using techniques which are not affected by the folk problems of smoothability. Moreover, τ will be *Cauchy*, recall:

0. Introduction

Folk problems and structure of globally hyperbolic (M, g):

1. Cornerstone: $\exists$ a Cauchy time function (R. Geroch, JMP'70)
   $\Longrightarrow$ Topological splitting $M \cong_{\mathrm{top}} \mathbb{R} \times S$
   (with levels acausal Cauchy hypersurfaces)

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   $(M, g) \equiv (\mathbb{R} \times S, g = -\beta dt^2 + g_t), \beta < \mathbf{1}$
   $\longrightarrow$ **Isometric embedding in $\mathbb{L}^N$ by means of a reduction to Nash' theorem, and t with interest in its own right!**

0. Introduction

Contents:

1. Reduction to Nash Riemannian theorem
   (a) Greene's result for arbitrary metrics
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2. Background: causal volume functions and "folk problems"
   (a) Future/past volume functions
   (b) Relation with the causal ladder of spacetimes
   (c) Geroch's topological construction
   (d) Folk problems related to smoothability
3. Steep temporal functions on globally hyperbolic spacetimes

1. Reduction to Nash: (a) Greene's result

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Theorem

(Greene '70). *Assume that any Riemannian metric on $M = M^m$ admits an isometric embedding in $\mathbb{R}^N$. Then, any pseudo-metric g on M admits an isometric embedding ψ_g in $\mathbb{R}^{N+2m+1}_{2m+1}$.*

(For M compact as well as isometric immersions, the dimension and index can be reduced in 1.)

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Proof. By Whitney theorem there exists a closed (proper) embedding $\psi : M \to \mathbb{R}^{2m+1}$ and by the claim below we can assume that [notation: g_0 natural metric in Euclidean space]

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is Riemannian. By assumption there exists an isometric embedding $\psi_R : (M, g_R) \to \mathbb{R}^N$ and the required one is

$$\psi_g(x) = (\psi(x), \psi_R(x)) \in \mathbb{R}^{2m+1}_{2m+1} \times \mathbb{R}^N.$$

□

1. Reduction to Nash: (a) Greene's result

Claim

Let $\phi : M \to \mathbb{R}^{N'}$ be a (smooth) proper embedding, and g be a pseudo-metric on M. Then, there exists a positive function $f : \mathbb{R}^{N'} \to \mathbb{R}$ such that the embedding $\phi_f := (f \circ \phi) \cdot \phi$ satisfies:

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is a Riemannian metric.

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Proof. Obvious if M is compact: choose f as a (big enough) constant c.
Otherwise, for each compact $K_r = \phi^{-1}(\overline{B_0(r)}), r = 1, 2, \ldots$ take the constant c_r corresponding to K_r, and choose f radial and monotone with $f|_{K_r \setminus K_{r-1}} \geq c_r$. $\square$

(b) Lorentzian preliminary conventions

- Causal character of a tangent vector $v \in T_pM$ in a Lorentzian manifold $(-, + \dots, +)$ (analogously for curves, hypersurfaces):
  - v is causal when timelike $g(v, v) < 0$, or lightlike $g(v, v) = 0, \mathbf{v} \neq \mathbf{0}$
  - Otherwise, spacelike: $g(v, v) > 0$, or $v = 0$.
- (M, g) spacetime: time-orientable (and time-oriented, when necessary) connected smooth Lorentzian $n-$manifold.
  (No restrictive: any Lorentzian manifold isometrically embeddable in $\mathbb{L}^N$ must be time-orientable).
  One can speak on future and past directed causal vectors.

1. Reduction to Nash: (b) Lorentzian conventions

- The (piecewise) smooth timelike (resp. causal) curves define the chronological $\ll$ (resp. causal $\leq$) relation.
- Future and past of points (analogously subsets)
  - Chronological fut. $I^+(p) = \{q \in M : p \ll q\}$ (future-directed timelike curve from p to q)
  - Causal future $J^+(p) = \{q \in M : p \leq q\}$ (fut.-dir. causal curve from p to q, or $p = q$)
  - Analogously $I^-(p), J^-(p)$.
  - For an open subset $U \subset M$ regarded as spacetime: $I^+(p, U), J^-(p, U)$...
  - $\mathbf{J(p, q) = J^+(p) \cap J^-(q)}$ $(J(p, S) = J^+(p) \cap J^-(S))$

1. Reduction to Nash: (b) Lorentzian conventions

- Time-separation (or Lorentzian distance):
  $d : M \times M \to [0, +\infty]$

$$d(p,q) = \begin{cases} 0, \text{ if } \mathcal{C}_{p,q} = \emptyset \\ \sup\{L(\alpha), \alpha \in \mathcal{C}_{p,q}\}, \text{ if } \mathcal{C}_{p,q} \neq \emptyset \end{cases}$$

  $\mathcal{C}_{p,q}$ space of future-directed causal curves from p to q }

- Temporal function: smooth function τ with $\nabla\tau$ timelike and past-directed
  —in particular, it is a time-function: continuous function which increases on any future-directed causal curve

(c) Characterization of embeddability in $\mathbb{L}^N$

Theorem

For a Lorentzian manifold (M, g), it is equivalent:

(i) (M, g) *admits a isometric embedding in $\mathbb{L}^N$ for some $N \in \mathbb{N}$.*

(ii) (M, g) *(is a stably causal spacetime which) admits a* steep temporal function τ *($g(\nabla\tau, \nabla\tau) \leq -1$).*

In this case, d is finite.

1. Reduction to Nash: (c) embeddability in $\mathbb{L}^N$

Lemma

If $i : M \to \mathbb{L}^N$ is an isometric embedding, then:
(a) the natural time coordinate $t = x^0$ of $\mathbb{L}^N$ induces a steep temporal function on M, and
(b) the time-separation d of (M, g) is finite-valued.

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Proof. (a) $x^0 \circ i$ is clearly temporal, and it is steep because $1 \equiv |\nabla^0 x^0|(i(M)) \leq |\nabla(x^0 \circ i)|$ the latter as $\nabla(x^0 \circ i)_p$ is the projection of $\nabla^0 x^0_{i(p)}$ onto the tangent space $di(T_pM)$ (and its orthogonal $di(T_pM)^\perp$ in $T_{i(p)}\mathbb{L}^N$ is spacelike).

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(b) The finiteness of d is consequence of the finiteness of the time-separation d_0 on $\mathbb{L}^N$ and the inequality
$d(p, q) \leq d_0(i(p), i(q))$ for all $p, q \in M$.
$\square$

Theorem

For a Lorentzian manifold (M, g), it is equivalent:

(i) (M, g) *admits a isometric embedding in $\mathbb{L}^N$ for some $N \in \mathbb{N}$.*

(ii) (M, g) *(is a stably causal spacetime which) admits a* steep temporal function τ *$(g(\nabla\tau, \nabla\tau) \leq -1)$.*

In this case, d is finite.

Proof. From the lemma , only (ii) $\Rightarrow$ (i) needs to be proved.

1. Reduction to Nash: (c) embeddability in $\mathbb{L}^N$

Lemma

If (M, g) admits a temporal function τ then the metric g admits a decomposition

$$g = -\beta d\tau^2 + g_\tau,$$

(g_{τ_0}: Riemannian metric on the slice $S_{\tau_0} = \tau^{-1}(\tau_0)$ varying locally smoothly with τ_0 –globally the topology of S_{τ_0} may change), where $\beta = |\nabla\tau|^{-2}$.
In particular, if τ is steep then $\beta \leq 1$.

Proof. Decomposition: restrict g.
Value of β: $d\tau(\nabla\tau) = g(\nabla\tau, \nabla\tau) = -\beta\,(d\tau(\nabla\tau))^2$. $\Box$

1. Reduction to Nash: (c) embeddability in $\mathbb{L}^N$

Proof of Th. (steep temporal function $\Rightarrow$ embeddability in $\mathbb{L}^N$).
Using the decomposition of previous lemma, the auxiliary Riemannian metric

$$g_R := (4-\beta)d\tau^2 + g_\tau$$

admits a Nash isometric embedding

$$i_{\text{nash}} : (M, g_R) \hookrightarrow \mathbb{R}^{N_0}$$

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The required isometric embedding $i : (M, g) \hookrightarrow \mathbb{L}^{N_0+1}$ is just:

$$i(\tau, x) = (2\tau, i_{\text{nash}}(\tau, x)).$$

1. Reduction to Nash: (c) embeddability in $\mathbb{L}^N$

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However, even this question is harmless for conformal embeddings.

(d) Consequences for conformal embeddings

(1) A Lorentzian manifold is a stably causal spacetime if and only if it admits a conformal embedding in some $\mathbb{L}^N$.

Proof. (1) Let τ be any temporal function. Then τ is temporal for any conformal metric, and steep for $g^* = \sqrt{|\nabla t|}g$ $(|\nabla^* \tau|^* \equiv 1)$.

1. Reduction to Nash: (d) conformal embeddings

(d) Consequences for conformal embeddings

(1) A Lorentzian manifold is a stably causal spacetime if and only if it admits a conformal embedding in some $\mathbb{L}^N$.

(2) In this case, there is a representative of its conformal class whose time-separation (Lorentzian distance) function is finite-valued.

Proof. (2) For $g^* = \sqrt{|\nabla t|}g$ as above, (M, g^*) is isometrically embeddable, and, then, its time-separation d^* is finite.

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(1) A Lorentzian manifold is a stably causal spacetime if and only if it admits a conformal embedding in some $\mathbb{L}^N$.

(2) In this case, there is a representative of its conformal class whose time-separation (Lorentzian distance) function is finite-valued.

(3) A stably causal spacetime is conformal to a spacetime non-isometrically embeddable in $\mathbb{L}^N$ if [*and only if*] it is not globally hyperbolic .

Proof. (3) Such spacetimes are conformal to a spacetime with infinite-valued d and, thus, non-isometrically embeddable. □

1. Reduction to Nash: (d) conformal embeddings

Summing up, we will conclude:

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Remark. As a difference with the Riemannian case, there is a (very neat) obstruction to the existence of isometric and conformal embeddings.

(a) Future/past volume functions

Definition

Admissible Borel measure on M for Geroch-type construction:

1. $m(M) < \infty$,
2. $m(U) > 0$ if $U \neq \emptyset$ is open
3. $m(\partial I^{\pm}(z)) = 0, \forall z \in M$.

The one associated to a Riemannian metric with finite volume suffices

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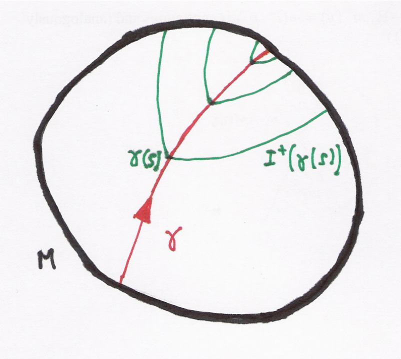
Definition

Associated past/future *volume* functions on M:

- $t^- : M \to \mathbb{R}$, $t^-(p) = m(I^-(p))$
- $t^+ : M \to \mathbb{R}$, $t^+(p) = -m(I^+(p))$

2. Background: (a) volume functions

Let $\gamma : (a, b) \to M$ fut.-pointing causal: $s \to t^{\pm}(\gamma(s))$ is non-decreasing.



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Remark: in general, $t^{\pm}$ are not generalized time functions (for ex.: when there exist closed causal curves). To understand this well...

2. Background: (b) causal ladder

(b) Relation with the causal ladder of spacetimes

- To obtain spacetimes both, physically realistic and mathematical interesting, it is useful to impose conditions on the global causality of the spacetime.
- Such conditions are always conformally invariant
- This yields a causal ladder or hierarchy of spacetimes.
- The steps directly related to volume functions are:

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$\Downarrow$

$\boxed{\text{Causally simple}}$

$\Downarrow$

$\boxed{\text{Causally continuous}}$

$\Downarrow$

$\boxed{\text{Stably causal}}$

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- **Characterization**: (M, g) is chronological $\iff t^-$ (resp. t^+) is strictly increasing on any future-directed **timelike** curve.

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- **Characterization**: (M, g) distinguishing $\iff t^{-}, t^{+}$ are strictly increasing on any future-directed causal curv, i.e. $t^{\pm}$ generalized time functions (non-necessarily continuous).

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$\Downarrow$
Causal
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Chronological

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- A spacetime is stably causal if it remains causal under C^0 perturbations of the metric (or if it remains causal when its cones are opened slightly)

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- (1) $\Rightarrow$ (2) (Hawking) Even though volume functions are only generalized time functions, a true (continuous) time function can be obtained by integrating the t^- of close metrics $g_\lambda, \lambda \in [0, 1]$ with wider lightcones: $t(p) = \int_0^1 t_\lambda^-(p) d\lambda.$

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- (3) $\Rightarrow$ (1) Not difficult

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- (M, g) is globally hyperbolic if it is causal and it does not contain naked singularities: $J(p,q) := J^+(p) \cap J^-(p)$ compact for all p, q.

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- (M, g) is globally hyperbolic if it is causal and it does not contain naked singularities: $J(p, q) := J^+(p) \cap J^-(p)$ compact for all p, q.
- Natural strengthening of the causal requirements...
- ... but implies spectacular properties for the spacetime!

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(i) (M, g) is globally hyperbolic.
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(iii) (M, g) admits a Cauchy time function, i.e., an onto time function $t : M \to \mathbb{R}$ such that all its levels $S_{t_0} = t^{-1}(t_0)$, $t_0 \in \mathbb{R}$, are (acausal) Cauchy hypersurfaces.

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(iv) (M, g) admits a spacelike Cauchy hypersurface (a smooth hypersurface which is spacelike and Cauchy).
(v) (M, g) admits a Cauchy temporal function, i.e., an onto temporal function $t : M \to \mathbb{R}$ such that all its levels S_{t_0}, $t_0 \in \mathbb{R}$, are Cauchy hypersurfaces ($\Rightarrow$ orthogonal splitting)

2. Background: (c) Geroch's construction

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- Necessarily, S is then an *embedded topological hypersurface*
- Cauchy hyp. yield an alternative definition of glob. hyp. as: ((M, g) is globally hyperbolic $\Leftrightarrow$ it admits a Cauchy hypersurface)
- We will focus on one of the implications by Geroch:

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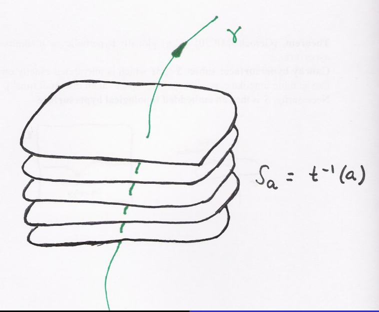
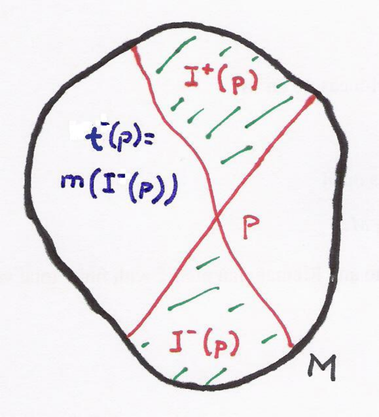
Theorem

If M is glob. hyp., there exists a ("Cauchy time function")
$t : M \to \mathbb{R}$ continuous and onto such that:
(1) t is strictly increasing on any future-directed causal curve (and then a time function).
(2) $S_a := t^{-1}(a)$ Cauchy hyp. $\forall a \in \mathbb{R}$.
(As a consequence, M is homeomorphic to $\mathbb{R} \times S$).



2. Background: (c) Geroch's construction

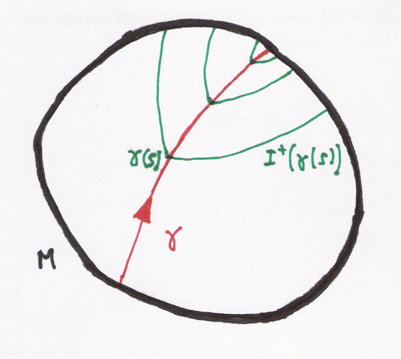
Idea of the proof. Consider the volume functions $t^{\pm}$



2. Background: (c) Geroch's construction

If $\gamma : (a, b) \to M$ is causal, fut.-directed and inextensible:

1. $s \to t^+(\gamma(s))$ (resp. $t^-(\gamma(s))$) is strictly increasing [t^-, t^+ were time functions]
2. $\lim_{s\to b} t^+(\gamma(s)) = 0 = \lim_{s\to a} t^-(\gamma(s))$
3. $\lim_{s\to a}(-t^+(\gamma(s))), \lim_{s\to b} t^-(\gamma(s)) > 0$

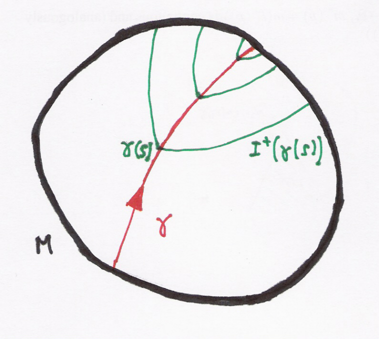


2. Background: (c) Geroch's construction

Required "Cauchy time" function:

$$t(z) = \log\left(-t^{-}(z)/t^{+}(z)\right)$$

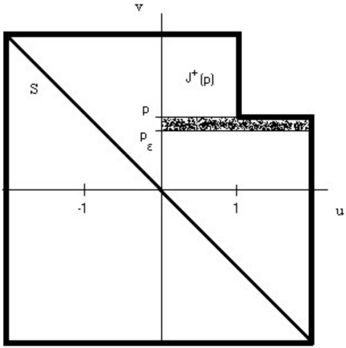
$$\left.\begin{array}{l} \lim_{s\to b} t(\gamma(s)) = \infty \\ \lim_{s\to a} t(\gamma(s)) = -\infty \end{array}\right\} \Longrightarrow t = \text{const. is Cauchy}$$



2. Background: (d) folk problems

(d) Folk problems related to smoothability

Remark. From the constructive proof, $t, t^{\pm}$ is not always smooth:



$M \subset \mathbb{L}^2$, (null coord. u, v) Diagonal S Cauchy hyp, t^+

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(1) Find a **(smooth) spacelike** Cauchy hyp. (Sachs & Wu, Bulletin AMS '77)
■ Even more (Brunetti/Ruzzi): can any smooth spacelike compact submanifold with boundary be extended to a spacelike Cauchy hyp.?

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(2) Find a Cauchy **temporal** function (i.e. additionally t smooth with past-pointing timelike gradient and Cauchy hyp, as levels)
$\rightsquigarrow$ such a function would yield the structural orthogonal splitting
$(M, g) \equiv (\mathbb{R} \times S, g = -\beta dt^2 + g_t)$,
■ Even more (Bär/Ginoux/Pfäffle): given a spacelike Cauchy hyp. S, find a Cauchy temporal function with one of the levels equal to S

(3) Prove that functions such as

$$f(x) = \int_{H^+(\tau(x),\sigma(x))} \mu$$

where μ is some (admissible) measure. Here, τ is a Cauchy temporal function which splits the spacetime, σ a sort of spacelike radial coordinate and

$$H^+(t,s) = J^+(\tau^{-1}(0)) \cap J^-(\tau^{-1}(t) \cap \sigma^{-1}([0,s])).$$

■ A proof of the smoothness of such functions would complete Clarke's proof on embeddability and would have interest in its own right.

2. Background: (d) folk problems

Difficulties to solve them with the strictly involved tools:

1. Try to approximate Geroch's time function by smooth ones (Seifert '77):
   BUT even a smooth one may have degenerate Cauchy hypersurfaces.
2. Try to use a different admissible measure for the job (Dieckmann '88):
   BUT for the related problem of smoothability of time functions, no admissible measure can make $t^{\pm}$ be a time function (this happened iff the spacetime was causally continuous).

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- The procedure developed in AN Bernal, MS. '03, '05 '06 yields a Cauchy temporal function (and a temporal function in the stably causal case, as well as solve the other refined problems)

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- The procedure developed in AN Bernal, MS. '03, '05 '06 yields a Cauchy temporal function (and a temporal function in the stably causal case, as well as solve the other refined problems)
- Next, we will construct a steep Cauchy temporal function by a modification of this procedure (Müller & MS, '11)
  This re-proves and simplifies widely (even though only in the globally hyperbolic case) the proof of the existence of a Cauchy temporal function.

(a) Technical tools

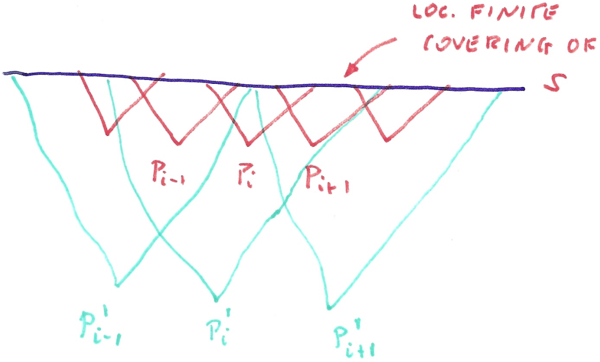
(1) We assume the existence of a Cauchy time function t as in Geroch's, each $S_a = t^{-1}(a)$ Cauchy.

(2) Function

$$j_p : M \to \mathbb{R}, \quad j_p(q) = \exp(-1/d(p,q)^2).$$

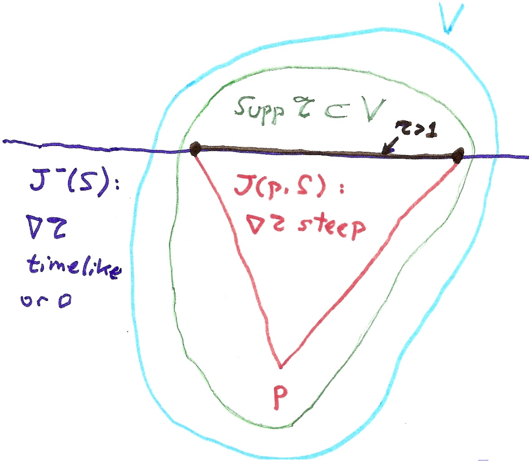
Restricted to a convex neighborhood of p, this is a smoothed version of the Lorentzian distance to p (smooth even at 0).

(3) For any Cauchy S, a fat cone covering:
sequence of pairs of points $p_i' \ll p_i, i \in \mathbb{N}$ such that both, $\mathcal{C}' = \{I^+(p_i') : i \in \mathbb{N}\}$ and $\mathcal{C} = \{I^+(p_i) : i \in \mathbb{N}\}$ yield a locally finite covering of S.



3. Steep temporal functions on glob. hyp. s.-t.

(4) For any Cauchy $S = S_a$, $p \in J^-(S)$ and $V \supset J(p, S)$, a smoooth function τ steep temporal on $J(p, S)$ with support in V (“τ steep on the forward cone $J(p, S)$”).



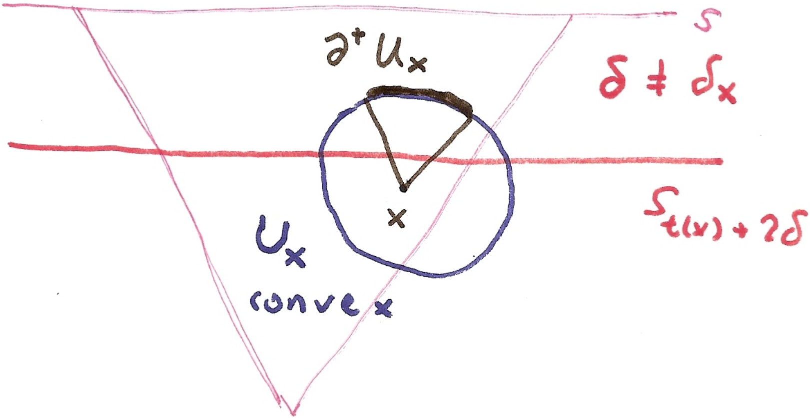
3. Steep temporal functions on glob. hyp. s.-t.

Proposition

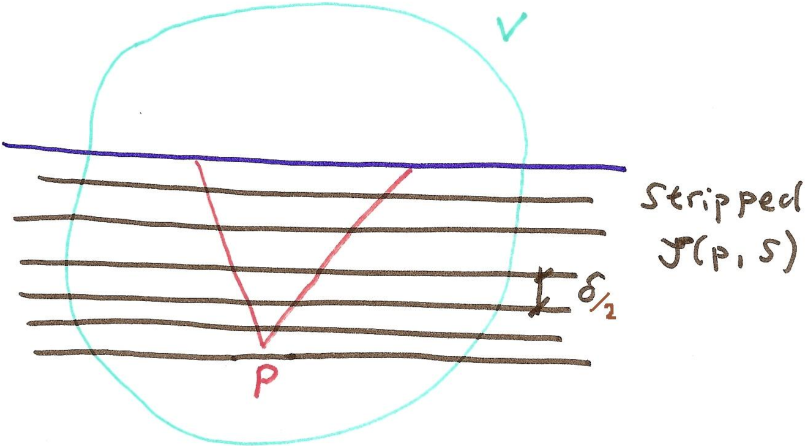
Let S be a Cauchy hypersurface, $p \in J^-(S)$. For all neighborhood V of $J(p,S)$ there exists a smooth function $\tau \geq 0$ such that:
(i) $\operatorname{supp} \tau \subset V$
(ii) $\tau > 1$ *on* $S \cap J^+(p)$.
(iii) $\nabla\tau$ *is timelike and past-directed in* $\mathrm{Int}(\mathrm{Supp}\ (\tau) \cap J^-(S))$.
(iv) $g(\nabla\tau, \nabla\tau) < -1$ *on* $J(p,S)$.

3. Steep temporal functions on glob. hyp. s.-t.

Sketch of proof. Choose K compact, $J(p, S) \subset \mathrm{Int}(K)$, $K \subset V$ and $\delta > 0$ s.t.: $\forall x \in K$, $\exists U_x \subset V$ convex with $\partial^+ U_x \subset J^+(S_{t(x)+2\delta})$ (where $\partial^+ U_x := \partial U_x \cap J^+(x)$).



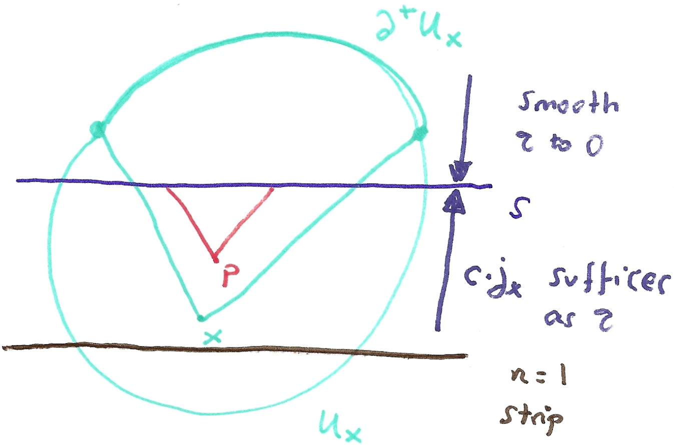
Slice $J(p, S)$ in n strips, $a_0 < t(p) < a_1 < \cdots < a_n = a$ with $a_{i+1} - a_i < \delta/2$



3. Steep temporal functions on glob. hyp. s.-t.

If $n = 1$ strip suffices (otherwise, careful induction!):

- $\tau = cj_x$ on $J^-(S)$ for c large and some close $x \ll p$
- τ is smoothed to 0 on $V \cap J^+(S)$.

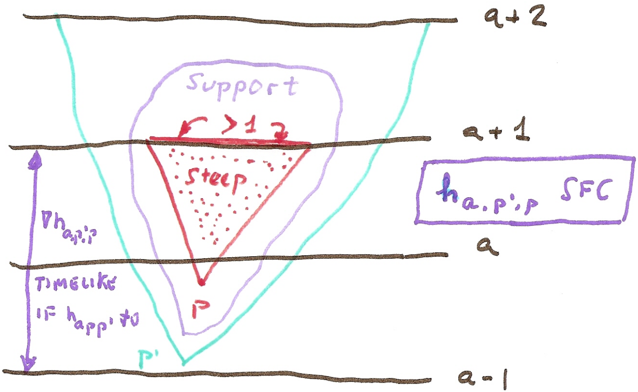


(b) Steps of the proof

- **Step 1**: for any $a \in \mathbb{R}$ and $p' \ll p$, $p, p' \in J(S_{a-1}, S_a)$, construct a *steep forward cone function* (SFC) $h^+_{a,p',p} : M \to [0, \infty)$ which satisfies:

1. $supp(h^+_{a,p',p}) \subset J^+(p', S_{a+2})$,
2. $h^+_{a,p',p} > 1$ on $S_{a+1} \cap J^+(p)$,
3. If $x \in J^-(S_{a+1})$ and $h^+_{a,p',p}(x) \neq 0$ then $\nabla h^+_{a,p',p}(x)$ is timelike and past-directed, and
4. $g(\nabla h^+_{a,p',p}, \nabla h^+_{a,p',p}) < -1$ on $J(p, S_{a+1})$.

3. Steep temporal functions on glob. hyp. s.-t.

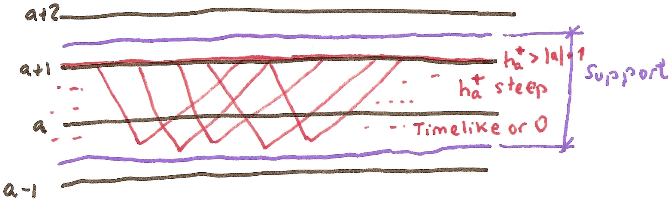


(this is straightforward from the above constructed steep funtions on $J(p, S)$).

3. Steep temporal functions on glob. hyp. s.-t.

- **Step 2**: by using a fat cone covering $\{p'_i \ll p_i | i \in \mathbb{N}\}$ for $S = S_a$, adjust a locally finite sum of SFC functions to obtain some $h_a^+ > 0$ which satisfies:

1. $supp(h_a^+) \subset J(S_{a-1}, S_{a+2})$,
2. $h_a^+ > |a| + 1$ S_{a+1}, [this will ensure that the finally obtained temporal function is Cauchy]
3. If $x \in J^-(S_{a+1})$ and $h_a^+(x) \neq 0$ then $\nabla h_a^+(x)$ is timelike and past-directed, and
4. $g(\nabla h_a^+, \nabla h_a^+) < -1$ on $J(S_a, S_{a+1})$.

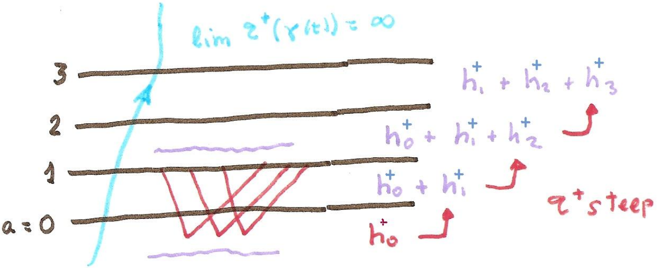


3. Steep temporal functions on glob. hyp. s.-t.

- **Step 3:**
  1. given $h_a^+ \geq 0$ ensure that h_{a+1}^+ can be chosen such that

     $$g(\nabla(h_a^+ + h_{a+1}^+), \nabla(h_a^+ + h_{a+1}^+)) < -1 \qquad \text{on } J(S_{a+1}, S_{a+2}) \tag{1}$$

     So, $h_a^+ + h_{a+1}^+$ is steep on all $J(S_a, S_{a+2})$.
  2. Inductively, construct a Cauchy steep function $\mathcal{T}^+ \geq 0$ on $J^+(S_0)$ with $\mathcal{T}^+(S_a) \geq a$ for $a = 1, 2, \ldots$.
  3. By reversing the time orientation and working on $J^-(S_0)$, obtain the Cauchy steep function $\mathcal{T} = \mathcal{T}^+ - \mathcal{T}^-$ on all M.



By construction this function not only is smooth, temporal and steep, but also satisfies the abstract properties in Geroch's proof which ensure that the levels are Cauchy.