Exceptional holonomy and calibrated submanifolds.

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Exceptional holonomy groups: a review

Motivating question: which subgroups of SO(n) can be the holonomy group of a simply-connected n-manifold M?

- 1955: Berger gives list of 8 families of subgroups of SO(n) that *could* be holonomy groups of a simply-connected irreducible and nonsymmetric Riemannian manifold.
- Includes 3 exceptional cases in dim 7, 8 and 16.

$$G_2 \subset \mathrm{SO}(7), \qquad Spin(7) \subset \mathrm{SO}(8), \qquad Spin(9) \subset \mathrm{SO}(16).$$

- 1962: Simons simplifies Berger's proof: shows Hol(g) must act transitively and effectively on the unit sphere in $\mathbb{R}^n$.
- 1968: Alekseevskii proves any Riemannian metric with holonomy group $Spin(9) \subset \mathrm{SO}(16)$ is symmetric.

Exceptional holonomy question:

Do manifolds with holonomy G_2 and $Spin(7)$ exist?

What is the group G_2?

Unhelpful answer: G_2 is the unique compact 1-connected simple Lie group of dimension 14.

Two geometric characterizations of G_2:

(i) *the automorphism group of the octonions* $\mathbb{O}$

(ii) *the stabilizer of a generic* 3*-form in* $\mathbb{R}^7$

Define a vector cross-product on $\mathbb{R}^7 = \mathrm{Im}(\mathbb{O})$

$$u \times v = \mathrm{Im}(uv)$$

where uv denotes octonionic multiplication.

Cross-product has an associated 3-form

$$\phi_0(u, v, w) := \langle u \times v, w \rangle = \langle uv, w \rangle$$

ϕ_0 is a generic 3-form so

$$G_2 = \{A \in \mathrm{GL}(7, \mathbb{R}) |\ A^* \phi_0 = \phi_0\} \subset \mathrm{SO}(7).$$

$\mathbf{SU}(2) \subset \mathbf{SU}(3) \subset G_2$

$\exists$ *close relations between G_2 holonomy and Calabi-Yau geometries in 2 and 3 dimensions.*

• Write $\mathbb{R}^7 = \mathbb{R} \times \mathbb{C}^3$ with $(\mathbb{C}^3, \omega, \Omega)$ the std SU(3) structure then

$$\phi_0 = dt \wedge \omega + \mathrm{Re}(\Omega)$$

Hence stabilizer of $\mathbb{R}$ factor in G_2 is SU(3) $\subset G_2$.
More generally if (X, g) is a Calabi-Yau 3-fold then product metric on $\mathbb{S}^1 \times X$ has holonomy SU(3) $\subset G_2$.

• Write $\mathbb{R}^7 = \mathbb{R}^3 \times \mathbb{C}^2$ with coords (x_1, x_2, x_3) on $\mathbb{R}^3$, with std SU(2) structure $(\mathbb{C}^2, \omega_I, \Omega = \omega_J + i\omega_K)$ then

$$\phi_0 = dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge \omega_I + dx_2 \wedge \omega_J + dx_3 \wedge \omega_K,$$

Hence subgroup of G_2 fixing $\mathbb{R}^3 \subset \mathbb{R}^3 \times \mathbb{C}^2$ is SU(2) $\subset G_2$.

Exceptional holonomy: some milestones

1984: (Bryant) locally $\exists$ many metrics with holonomy G_2 and $Spin(7)$. Proof uses Exterior Differential Systems.

1989: (Bryant-Salamon) explicit complete metrics with holonomy G_2 and $Spin(7)$ on noncompact manifolds.

1996: (Joyce) Gluing methods used to construct *compact* 7-manifolds with holonomy G_2 and 8-manifolds with holonomy $Spin_7$. Uses a modified Kummer-type construction.

- Start with flat orbifold T^7/Γ for appropriate finite groups $\Gamma \in G_2$.
- Resolve singularities of orbifold to give smooth (but nearly singular) 7-mfd and a metric which is close to G_2 holonomy (small torsion).
- Use analysis (on nearly singular mfd) to perturb 3-form to torsion-free G_2 structure.

Calibrations − Definitions

A *calibrated geometry* is a distinguished class of minimal submanifolds associated with a differential form.

- A *calibrated form* is a closed differential p-form ϕ on a Riemannian manifold (M, g) satisfying $\phi \leq vol_g$.

  $$i.e. \qquad \phi(e_1, \ldots, e_p) \leq 1$$

  for any orthonormal set of p tgt vectors

- For $m \in M$ associate with ϕ the subset $G_m(\phi)$ of oriented p-planes for which equality holds in $(*)$ − the *calibrated* planes.

- A submanifold *calibrated* by ϕ is an oriented p-dim submanifold whose tangent plane at each point m lies in the subset $G_m(\phi)$ of distinguished p-planes.

Lemma: (Harvey–Lawson) Calibrated submanifolds minimize volume in their homology class.

Holonomy, constant tensors & calibrations

Key fact: Parallel tensors on (M, g) determined by holonomy group $G = \mathrm{Hol}\, g$.

$G \subset \mathrm{O}(n)$ also acts on k-forms on $\mathbb{R}^n$.

G-invariant k-forms $\leftrightsquigarrow$ parallel k-forms on (M, g)

If ϕ_0 is G-invariant k-form on $\mathbb{R}^n$, by rescaling can arrange comass 1 property
$\Rightarrow$ ϕ_0 is a calibration on $\mathbb{R}^n$.

Also ϕ_0 G-invariant $\Rightarrow$

ξ a calibrated plane $\Rightarrow$ so is $\gamma.\xi$ for any $\gamma \in G$,
i.e. $\exists$ many ϕ_0-calibrated planes.

ϕ_0 calibration $\Rightarrow$ corresponding parallel k-form ϕ on (M, g) also a calibration with a large set of calibrated k-planes.

Suggests *locally should exist many ϕ-calibrated submfds.*

Associative & coassociative calibrations

3-form ϕ_0 and 4-form $*\phi_0$ on $\mathbb{R}^7$ are G_2-invariant calibrations.

Oriented 3-planes calibrated by ϕ_0 are called *associative* planes.

- $\mathbb{R}^3 \subset \mathbb{R}^3 \times \mathbb{C}^2$ is an associative 3-plane.
- G_2 acts transitively on associative 3-planes.

Oriented 4-planes calibrated by $*\phi_0$ are called *coassociative*.

- 4-plane is coassociative iff its orthogonal complement is associative.

Holonomy/constant tensor correspondence $\Rightarrow$

on any mfd (M, g) with Hol$(g) \subset G_2$ we have parallel 3 and 4-forms ϕ and $*_g\phi$.

$\Rightarrow$ *associative and coassociative calibrations exist on any mfd with holonomy* G_2.

G_2 structures and positive 3-forms

Positive 3-forms $\leftrightsquigarrow$ (oriented) G_2-structures

• A 3-form ϕ on an oriented 7-mfd M is *positive* if $\forall\ p \in M\ \exists$ an oriented isomorphism

$$i : T_pM \to \mathbb{R}^7, \text{ such that } i^*\phi_0 = \phi.$$

• Positive 3-forms on $\mathbb{R}^7 \leftrightsquigarrow \mathrm{GL}_+(7, \mathbb{R})/G_2$.

• $\dim(\mathrm{GL}_+(7, \mathbb{R})/G_2) = 35 = \dim \Lambda^3\mathbb{R}^7$.

$\Rightarrow$ Positive 3-forms on M form an *open* subbundle of Λ^3T^*M *i.e.* small perturbations of a G_2 structure are G_2 structures.

Prop: (S. Salamon) Let (M, ϕ, g) be a G_2 structure on a compact 7-manifold. TFAE

1. $\mathrm{Hol}(g) \subset G_2$ and ϕ is the induced 3-form
2. $\nabla\phi = 0$ where ∇ is Levi-Civita w.r.t g
3. $d\phi = d^*\phi = 0$.

NB (3) is nonlinear in ϕ because metric g depends nonlinearly on ϕ.

The topology of G_2 manifolds

Prop:

(a). A compact 7-manifold M admits a G_2-structure iff M is orientable and spinnable.

(b). A compact 7-manifolds M with a torsion-free G_2 structure (ϕ, g) has $\mathrm{Hol}(g) = G_2$ iff $\pi_1 M$ is finite.

(c). A compact 7-manifold (M, g) with $\mathrm{Hol}(g) = G_2$ has nonzero first Pontrjagin class $p_1(M)$.

Compact mfds with holonomy G_2 via neck-stretching

Donaldson suggested constructing compact G_2 manifolds via a neck-stretching argument

- Use noncompact version of Calabi conjecture to construct asymptotically cylindrical Kähler-Ricci-flat (AC KRF) 3-folds X with one end $\sim \mathbb{C}^* \times D$, with D a smooth $K3$
- $M = \mathbb{S}^1 \times X$ is a Riem 7-mfd with $\mathrm{Hol}\, g = \mathrm{SU}(3) \subset G_2$ with end $\sim \mathbb{R}^+ \times T^2 \times K3$.
- Take a *twisted connect sum* of a pair of $M_i = \mathbb{S}^1 \times X_i$
- For $T >> 1$ construct a G_2-structure w/ small torsion (exponentially small in T) and prove it can be corrected to torsion-free.

Hyperkähler rotation (or matching data)

Product G_2 structure on M_i asymptotic to

$$d\theta_1 \wedge d\theta_2 \wedge dt + d\theta_1 \wedge \omega_I^{(i)} + d\theta_2 \wedge \omega_J^{(i)} + dt \wedge \omega_K^{(i)}$$

- $\omega_I^{(i)}$ denotes Ricci-flat Kähler metric on D_i
- $\omega_J^{(i)} + \sqrt{-1}\,\omega_K^{(i)}$ parallel $(2,0)$-form on D_i.

To get a well-defined G_2 structure using

$$F : [T-1, T] \times T^2 \times D_1 \to [T-1, T] \times T^2 \times D_2$$

given by

$$(t, \theta_1, \theta_2, y) \mapsto (2T - 1 - t, \theta_2, \theta_1, f(y))$$

to identify end of M_1 with M_2 we need $f : D_1 \to D_2$ to satisfy

$$f^*\omega_I^{(2)} = \omega_J^{(1)}, \quad f^*\omega_J^{(2)} = \omega_I^{(1)}, \quad f^*\omega_K^{(2)} = -\omega_K^{(1)}$$

Constructing such hyperkähler rotations is nontrival and a major part of the construction.

Kovalev's compact G_2 manifolds

Kovalev carried out Donaldson's proposal.

Main points of Kovalev's approach (2003):

1. Construct asymptotically cylindrical Calabi-Yau 3-folds from smooth *Fano 3-folds*, using work of Tian-Yau

2. Need to find sufficient conditions for existence of a "hyperkähler rotation" between D_1 and D_2.

3. Given a pair of AC KRF 3-folds X_i and a HK-rotation $f : D_1 \to D_2$ can *always* glue M_1 and M_2 to get a 1-parameter family of cpt manifolds M_T with holonomy G_2.

4. Use global Torelli theorems and lattice embedding results (Nikulin) to find hyperkähler rotations from suitable initial pairs of Fano 3-folds

5. When set up in terms of analysis on exponentially weighted Sobolev spaces the gluing /perturbation argument is relatively straightforward (no small eigenvalues)

An asymptotically cylindrical Calabi conjecture

Tian-Yau I (JAMS 1990): The Calabi conjecture on fibred quasiprojective manifolds.

Setup:

- $\overline{X}$ is a projective manifold
- $D \subset \overline{X}$ a divisor
- $\pi : \overline{X} \to \overline{S}$ is a fibre space over a smooth algebraic curve $\overline{S}$ with connected fibres
- $D = \pi^{-1}(D_{\overline{S}})$, $D_{\overline{S}} \subset \overline{S}$ consists of finitely many smooth reduced fibres.

Thm: Let $X = \overline{X} \setminus D$. Given any $(1,1)$-form Ω representing $c_1(K_{\overline{X}}^{-1} \otimes [D]^{-1})$, there is a complete Kähler metric with Ω as its Ricci form and this metric has linear volume growth.

$\Rightarrow$ *Get complete Calabi-Yau metrics with linear volume growth from anti-canonical divisors in fibred quasiprojective varieties.*

Q: How do we find such $K3$ fibred projective 3-folds?

$K3$-fibred 3-folds from Fano 3-folds

X a smooth Fano 3-fold: a nonsingular cx 3-fold X with K_X^{-1} *ample*.

A generic anticanonical divisor $D_1 \in |K_X^{-1}|$ is a smooth $K3$ surface. BUT, normal bundle of D_1 in X is not trivial.

If $D_1, D_1' \in |K_X^{-1}|$ are generic then $C = D_1 \cap D_1'$ is a smooth curve of genus g.
g is the *genus* of the Fano 3-fold and satisfies

$$(K_X^{-1})^3 = 2g - 2.$$

Blowing up C yields a new 3-fold $\overline{X}$ and a map

$$\pi : \overline{X} \to \mathbb{P}^1$$

whose fibres are the proper transforms of the surfaces in the pencil defined by D_1 and D_1'. Proper transform of D_1 is an anticanonical divisor on $\overline{X}$.

$M = \overline{X} \setminus \overline{D}_1$ is a quasiprojective 3-fold with trivial canonical bundle which fibres over $\mathbb{C}$ with generic fibre a smooth $K3$;
M admits an asymptotically cylindrical Calabi-Yau metric

Q: *how can we find more K3 fibred quasiprojective 3-folds?*

Weak Fano 3-folds

Basic idea: replace condition K_X^{-1} is positive, with K_X^{-1} sufficiently "non-negative"; replace ample with *nef* and *big*.

Definition: A smooth cx 3-fold X is a *weak Fano manifold* if K_X^{-1} is big and nef.

- A holomorphic line bundle L on X is *nef* if

$$c_1(L).C = \int_C c_1(L) \geq 0$$

for every irreducible (holo) curve $C \subset X$.

- A holomorphic line bundle L on X is *big* if

$$h^0(L^{\otimes m}) \geq Cm^n, \text{ for } m \gg 1, \quad n = \dim_{\mathbb{C}} X.$$

There exist many more weak Fano 3-folds than Fano 3-folds (thousands versus around 100 deformation families of Fanos)

Classification of smooth weak Fano 3-folds ongoing

Weak Fano 3-folds and G_2 manifolds

Main points:

1. Generic elements of $|K_X^{-1}|$ smooth K3s for weak Fano 3-folds.
$\Rightarrow$ can still construct asymptotically cylindrical Calabi-Yau 3-folds from weak Fanos.

2. Need *more* than weak Fano to construct hyperkahler rotation $f : D_1 \to D_2$. Need a sufficiently good deformation/moduli theory for anticanonical K3 divisors in deformation family of the 3-fold

Definition: A weak Fano 3-fold is *weak-*$*$ if the natural morphism to its anti-canonical model is *small*.

Also useful to allow intermediate class of weak Fano 3-folds where AC model is only *semismall*.

For weak-$*$ Fano 3-folds can still construct HK rotations.
$\Rightarrow$ can use them to construct compact G_2 manifolds.

Simple examples of weak-* Fano 3-folds

Example 1: start with a (singular) Fano 3-fold Y containing a plane Π and resolve.

If $\Pi = (x_0 = x_1 = 0)$ then eqn of Y is

$$Y = (x_0 a_3 + x_1 b_3 = 0) \subset \mathbb{P}^4$$

where a_3 and b_3 are homogeneous cubic forms in $(x_0, \ldots, x_4)$. Generically the plane cubics

$$(a_3(0, 0, x_2, x_3, x_4) = 0) \subset \Pi,$$

$$(b_3(0, 0, x_2, x_3, x_4) = 0) \subset \Pi$$

intersect in 9 distinct points, where Y has 9 ordinary double points.

Simultaneous resolution of these ODPs by blowing-up $\Pi \subset Y$ gives a weak-* Fano 3-fold X such that:

X contains 9 smooth rigid rational curves with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ with genus 3 and Picard rank 2.

Examples of weak-* Fano 3-folds

Example 2: A quartic 3-fold in $\mathbb{P}^4$ with only ordinary double points has at most 45 singular points. Up to coordinate change, there is a unique such 3-fold, the *Burkhardt quartic* Y

$$(x_0^4 - x_0(x_1^3 + x_2^3 + x_3^3 + x_4^3 + 3x_1x_2x_3x_4)) = 0) \subset \mathbb{P}^4.$$

Y admits a small projective resolution X

X is a weak-* Fano 3-fold w/ genus 3, Picard rank 16 and 45 smooth rigid $\mathbb{P}^1$s with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Example 2 shows weak-* Fano 3-folds can have larger Picard rank. $\Rightarrow$ can get G_2 manifolds with larger Betti numbers.

Deformation classes of Fano 3-folds classified in 1980s via minimal model techniques. Classification results $\Rightarrow$ any Fano 3-fold has Picard rank ≤ 10. In fact, Picard rank ≥ 6 forces X to be $\mathbb{P}^1 \times$a 2d Fano (del Pezzo surface).

Toric weak-* Fano 3-folds

Any terminal toric Fano 3-fold has only ODPs as singularities

Toric terminal Fano 3-folds classified in terms of reflexive polytopes $\Rightarrow \exists$ 82 terminal toric Fano 3-folds

Every terminal Fano 3-folds admits at least one projective small resolution; most admit many such resolutions.

$\Rightarrow$ lots of smooth toric weak-* Fano 3-folds

Advantages of weak-* Fano vs. Fano

1. *Many* more weak-* Fano than Fano 3-folds
   $\Rightarrow$ get more topological types of G_2 mfds

2. In a Fano 3-fold K_X^{-1} is ample:
   $\Rightarrow$ any compact holo curve $C \subset X$ must intersect any anti-canonical divisor

   A *weak-* Fano* 3-fold can contain holo curves C that do not meet anticanonical divisors.

3. For each smooth rigid $\mathbb{P}^1$ in a weak-* Fano 3-fold X any G_2 manifold built from X contains a *rigid associative submanifold* w/ topology $S^1 \times S^2$.

Theorem: (Corti-Haskins-Norstrom-Pacini)
There exist many topological types of compact G_2 manifold which contain *rigid* associative submanifolds diffeomorphic to $S^1 \times S^2$.

Why do we get rigid associatives?

Let C be a cpt holo curve in X not meeting anticanonical divisor D_1
$\rightsquigarrow$ cpt holo curve $C \subset M = \overline{X} \setminus D_1$
$\rightsquigarrow \mathbb{S}^1 \times C$ is cpt associative submfd in $\mathbb{S}^1 \times M$.

• C rigid as a holo curve in M iff $\mathbb{S}^1 \times C$ rigid as associative submfd of $\mathbb{S}^1 \times M$

• Since $\mathbb{S}^1 \times C$ is rigid in $\mathbb{S}^1 \times M$, easy to perturb $\mathbb{S}^1 \times C$ to rigid associative submfd in glued G_2 structure for $T \gg 1$.

Remarks:

• First examples of *rigid* associative submanifolds in compact G_2 manifolds.

• Infinitesimal deformations of associative submfds $\leftrightsquigarrow$ twisted harmonic spinors.

$\Rightarrow$ deformation theory can be obstructed (unlike special Lagrangians & coassociatives)

• Index of twisted Dirac operator is zero since in odd dimension, but hard to control kernel or cokernel seperately.

• Can attempt to build invariants of G_2 manifolds by counting associative submfds in a given homology class. Generically expect only 0-diml moduli spaces of associative submfds.

Existence of HK-rotations

Basic strategy:

1. Understand which $K3$ surfaces D arise in $|K_X^{-1}|$ for X in a deformation class of Fano or weak Fano 3-folds

2. Use understanding from 1, together with global Torelli/surjectivity of periods for $K3$ surfaces to reduce to problem about embedding certain types of lattice in the $K3$ lattice.

3. Apply Nikulin's results on existence of lattice embeddings to construct the HK-rotation.

4. Some subtleties from 1: only get Zariski open (so dense) subset of natural K3 moduli spaces.

Question: Which $K3$ surfaces D arise in $|K_X^{-1}|$ for X in a deformation class of Fano 3-folds?

A. Any such $K3$ is projective

B. $H^2(X, \mathbb{Z})$ inherits a lattice structure via

$$(L, M) = L \cdot M \cdot K_X^{-1}$$

satisfying $(K_X^{-1}, K_X^{-1}) = 2g - 2$.

Lefschetz Hyperplane Theorem $\Rightarrow$ lattice of any such $K3$ contains a certain type of sublattice P (the Picard lattice of the Fano 3-fold)

Get special class of $K3$ surfaces called *ample P-polarized $K3$ surfaces*. Studied by e.g. Dolgachev.

Need to study the forgetful map $(X, D) \mapsto D$ between moduli of pairs and moduli of P-polarized $K3$ surfaces and the two moduli spaces.

Beauville studied this problem in *Fano context*.

Lefschetz Hyperplane Theorem and *Nakano Vanishing Theorem* are crucial ingredients.

Vanishing results for weak and weak-* Fano 3-folds

Kodaira vanishing for ample line bundles:

$$H^i(X, K_X \otimes L) = 0 \text{ for all } i > 0.$$

Kawamata-Viehweg: Kodaira vanishing still holds if L is only big and nef.

I. Akizuki-Nakano vanishing for ample L:

$$H^q(X, \Omega_X^p \otimes L) = 0, \text{ for } p + q > n.$$

($\Leftrightarrow H^q(X, \Omega_X^p \otimes L^{-1}) = 0$ for $p + q < n$.)

Nakano vanishing *fails* in general for weak Fanos.

II. Lefschetz Hyperplane Theorem (LHT)

X Fano: K_X^{-1} ample so Lefschetz Hyperplane Theorem applies to $D \in |K_X^{-1}|$.

$\Rightarrow \pi_1 X = \pi_1 D = (0)$ and

$i^* : H^2(X, \mathbb{Z}) \to H^2(D, \mathbb{Z})$ is injective.

$\Rightarrow$ Picard lattice of Fano 3-fold embeds as (primitive) sublattice in $K3$ lattice.

Akizuki-Nakano type vanishing and Lefschetz hyperplane theorem on weak* Fanos

I. *Sommese-Esnault-Viehweg vanishing* for k-*ample* line bundles gives us an analogue of Akizuki-Nakano vanishing

$\Rightarrow$ deformation theory used in Beauville's work in the Fano context still goes through for weak-* Fanos

II. Can apply Goresky-MacPherson's version of Lefschetz Hyperplane Theorem for lef line bundles L to prove:

Picard lattice of a weak-* Fano 3-fold still embeds as (primitive) sublattice in $K3$ lattice.

Sommese-Esnault-Viehweg vanishing for k-ample bundles

Definition: A line bundle L is k-*ample* if for some $m > 0$

1. $L^{\otimes m}$ is globally generated i.e. $H^0(X, L^{\otimes m})$ separates points of X.

2. the corresponding morphism

$$\phi_{L^{\otimes m}} : X \to \mathbb{P}(H^0(X, L^{\otimes m}))$$

has at most k dimensional fibres.

Remark: L is 0-ample iff L is ample.

Theorem (Sommese-Esnault-Viehweg) If L is a k-ample line bundle of Iataka dimension $\kappa(L)$ on a compact Kähler manifold then

$$H^q(X, \Omega_X^p \otimes L^{-1}) = 0,$$

for $p + q < \min(\kappa(L), n - k + 1)$.

Remark: L big iff Iataka dimension of L is $n = \dim_{\mathbb{C}} X$.

Vanishing for K_X^{-1} of weak-* Fano 3-folds

Proof is application of Esnault-Viehweg's logarithmic de Rham complexes machinery (Asterisque 1989).

Corollary If L is 1-ample and big then Nakano vanishing holds for L.

In particular . . .

If X is a smooth weak-* Fano 3-fold then Nakano vanishing holds for the line bundle K_X^{-1}.

Main Application: Beauville's results about the moduli of pairs (X,D) and the image of map $(X, D) \mapsto D$ for Fano 3-folds still hold on any smooth weak-* Fano 3-fold.

Gives enough control to use Global Torelli Theorem for K3 surfaces to construct HK rotations associated to pairs of weak-* Fano 3-folds in similar way to Fano case.