Pointwise symmetrization inequalities for Sobolev functions on Probability Metric Spaces

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Introduction

(Ω, d, μ) Metric space. μ Borel probability measure.
$f : \Omega \to \mathbb{R}$, $f \in Lip(\Omega)$

Problem:

Obtain pointwise inequalities that can be applied to provide a unified treatment of Sobolev-Poincaré inequalities

$$\left\| f - \int_\Omega f d\mu \right\|_Y \preceq \|\nabla f\|_X , \quad f \in Lip(\Omega),$$

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- What does mean $|\nabla f|$, (f Lip.)
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- What kind of function spaces on Ω ?
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Basic definitions: Function spaces

To measure the integrability
Only the "size of the function" is needed
$\Rrightarrow$ r.i spaces.

distribution function

$$\mu_u(t) = \mu\left\{x \in \Omega : |u(x)| > t\right\}, \ (t \geq 0).$$

decreasing rearrangement u_μ^* of u:

$$u_\mu^*(s) = \inf\left\{t : \mu_u(t) \leq s\right\}, \ (s \geq 0).$$

maximal function u_μ^{**} of u:

$$u_\mu^{**}(t) = \frac{1}{t}\int_0^t u_\mu^*(s)ds. \qquad (f+g)_\mu^{**}(t) \leq f_\mu^{**}(t) + g_\mu^{**}(t).$$

$X = X(\Omega)$ Banach function space is a r.i. space if:

$$f \in X, \ g_\mu^* = f_\mu^* \Rightarrow g \in X \text{ and } \|g\|_X = \|f\|_X .$$

$$f_\mu^{**}(t) \leq g_\mu^{**}(t) \Rightarrow \|f\|_X \leq \|g\|_X .$$

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Basic definitions: Function spaces

Examples:

- L^p-spaces

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \, d\mu(x)\right)^{1/p} = \left(\int_0^\infty \mu_u(t) d(t^p)\right)^{1/p}$$
$$= \left(\int_0^1 f_\mu^*(t)^p dt\right)^{1/p}.$$

- Lorentz spaces $L^{p,q}$

$$\|f\|_{p,q} = \left(\int_0^1 \left(t^{1/p} f_\mu^*(t)\right)^q \frac{dt}{t}\right)^{1/q}. \qquad L^{p,1} \subset L^{p,p} = L^p \subset L^{p,\infty}.$$

- Others:

$$H_n(\Omega) = \left(\int_0^1 \left(\frac{f_\mu^*(t)}{\log(\frac{e}{t})}\right)^n \frac{dt}{t}\right)^{1/n}.$$

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$$\|f\|_X = \|f_\mu^*\|_{\bar{X}},$$

for every $f \in X$.

Hardy's Operators:

$$Pf(t) = \frac{1}{t}\int_0^t f(s)ds; \qquad Qf(t) = \int_t^\infty f(s)\frac{ds}{s}.$$

$$P : \bar{X} \to \bar{X} \Leftrightarrow \overline{\alpha}_{\bar{X}} < 1. \qquad Q : \bar{X} \to \bar{X} \Leftrightarrow \underline{\alpha}_{\bar{X}} > 0.$$

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Classical inequalities: $\Omega \subset \mathbb{R}^n$, "nice". $\left(\int_\Omega f = 0\right)$

Gagliardo-Nirenberg-Sobolev-Petre: $1 \leq p < n,\ q = \frac{pn}{n-p}$

$$\int_0^1 \left(t^{1/q} f^*(t)\right)^p \frac{dt}{t} \leq C \int_\Omega |\nabla f(x)|^p \, dx.$$

$$q = \frac{nn}{n-n} = \infty, \qquad \int_0^1 f^*(t)^n \frac{dt}{t} \leq C \int_\Omega |\nabla f(x)|^n \, dx.$$

Maz'ya, Hansson, Brézis-Wainger: Lorentz norms are replaced in order to accommodate exponential integrability.

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Classical inequalities

(Bennett-DeVore-Sharpley) Oscillation of f: $f^{**}(t) - f^{*}(t)$

If $1 < q < \infty$, $0 < p \le \infty$, then

$$\|f\|_{L^{q,p}} \simeq \left\{ \int_0^\infty \left[\left(f^{**}(t) - f^{*}(t)\right) t^{1/q} \right]^p \frac{dt}{t} \right\}^{1/p}$$

Sobolev embedding $p = n$: (Bastero-Milman-Ruiz; Milman-Pustylnik)

$$W_0^{1,n}(\Omega) \subset L(\infty, n)(\Omega) \subsetneq H_n(\Omega)$$

$$L(\infty, n)(\Omega) = \left\{ f : \|f\|_{L(\infty,n)}^n = \int_0^{|\Omega|} \left(f^{**}(t) - f^{*}(t)\right)^n \frac{ds}{s} < \infty \right\}.$$

In particular

$$\|f\|_{L(\infty,n)} < \infty \Rightarrow f \in e^{L^{n'}}.$$

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$1 \leq p < \infty, \; q = \frac{pn}{n-p}$

$$\int_0^1 \left(t^{1/q} (f^{**}(t) - f^*(t)) \right)^p \frac{dt}{t} \leq C \int_\Omega |\nabla f(x)|^p \, dx.$$

If $p > n$

$$\|f\|_\infty \preceq \int_0^1 \left(t^{1/q} (f^{**}(t) - f^*(t)) \right)^p \frac{dt}{t}.$$

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Euclidean setting: Characterization and Optimality

M-Milman-Pustylnik: Let $X(\Omega)$, $Y(\Omega)$ r.i spaces. There are equiv.

1. $$\|f\|_Y \preceq \|\nabla f\|_X$$
2. $$\left\| \int_t^{1/2} f(s) s^{1/n} \frac{ds}{s} \right\|_{\bar{Y}} \preceq \|f\|_{\bar{X}}$$

Moreover:

1. If $\underline{\alpha}_X > \frac{1}{n}$
$$\|f\|_Y \preceq \left\| f^*(t) t^{-1/n} \right\|_{\bar{X}} \preceq \|\nabla f\|_X$$
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Gaussian setting: Why?

Gross' inequality:

$$\int_{\mathbb{R}^n} |f(x)|^2 \ln |f(x)| \, d\gamma_n(x) \leq \int_{\mathbb{R}^n} |\nabla f(x)|^2 \, d\gamma_n(x),$$

This inequality does not depend on n!!!

In terms of r.i. spaces, we can write

$$\left\| \overbrace{\left(\frac{f^{**}(t) - f^*(t)}{t} \right)}^{(-f^{**})'(t)} t \sqrt{\log \frac{1}{t}} \right\|_{L^2[0,1]} \preceq \|\nabla f\|_{L^2}$$

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$$\left\| \overbrace{\left(\frac{f^{**}(t) - f^{*}(t)}{t} \right)}^{(-f^{**})'(t)} t \sqrt{\log \frac{1}{t}} \right\|_{L^2[0,1]} \preceq \|\nabla f\|_{L^2}$$

Gaussian setting: Characterization

M-Milman: X, Y r.i. There are equivalent:

(i) $f \in Lip(\mathbb{R}^n)$ $\quad \left(\int_{\mathbb{R}^n} f d\gamma_n = 0\right)$.

$$\|f\|_Y \preceq \|\nabla f\|_X .$$

(ii) $f \in X$ positive,

$$\left\| \int_t^{1/2} f(s) \frac{ds}{s\sqrt{\log \frac{1}{s}}} \right\|_{\bar{Y}} \preceq \|f\|_{\bar{X}} .$$

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Gaussian setting: Optimality

(iii) If $\underline{\alpha}_X > 0$,

$$\|f\|_Y \leq \left\| f_\mu^*(t)\sqrt{\log\frac{1}{t}} \right\|_{\bar{X}} \leq \|\nabla f\|_X.$$

(iv) If $0 = \underline{\alpha}_X < \overline{\alpha}_X < 1$,

$$\begin{aligned} \|f\|_Y &\preceq \left\| \left(f_\mu^{**}(t) - f_\mu^*(t)\right)\sqrt{\log\frac{1}{t}} \right\|_{\bar{X}} + \|f\|_{L^1} \\ &\preceq \|\nabla f\|_X. \end{aligned}$$

$X = L^2$, $\underline{\alpha}_{L^2} = 1/2$

$$\int |f(x)|^2 \ln|f(x)|\, d\gamma_n(x) \preceq \int |\nabla f(x)|^2\, d\gamma_n(x), \quad (Gross)$$

$X = L^\infty$, $\underline{\alpha}_{L^\infty} = 0$ we obtain the concentration result

$$\left\| \left(f_\mu^{**}(t) - f_\mu^*(t)\right)\sqrt{\log\frac{1}{t}} \right\|_{L^\infty} < \infty \Rightarrow f \in e^{L^2}.$$

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Symmetrization by truncation: Isoperimetry

(Ω, d, μ) Metric spaces, μ Borel probability measure.
$A \subset \Omega$, Borelian set

$$\mu^+(A) = \liminf_{h \to 0} \frac{\mu(A_h) - \mu(A)}{h},$$

$A_h = \{x \in \Omega : d(x, A) < h\}$.

The boundary measure is a natural generalization of the notion of surface area to the metric probability space setting.
An isoperimetric inequality measures the relation between $\mu^+(A)$ and $\mu(A)$ by means of the isoperimetric profile $I = I_{(\Omega,d,\mu)}$ defined as the pointwise maximal function $I_{(\Omega,d,\mu)} : [0,1] \to [0,\infty)$ such that

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Example: Isoperimetric Inequality on $\mathbb{R}^2$
Among all regions in the plane, enclosed by a piecewise C^1 boundary curve, with area A and perimeter L,

$$4\pi A \leq L^2.$$

If equality holds, then the region is a circle.

Symmetrization by truncation: Isoperimetry

$I_{(\Omega,d,\mu)}$ isoperimetric profile.

$J : [0,1] \to [0,\infty)$ continuous, concave function, symmetric about $1/2$ with $J(0) = 0$ st.

$$I_{(\Omega,d,\mu)}(t) \geq J(t), \ (t \in [0,1/2])$$

will be called an **isoperimetric estimator**

$\Omega \subset \mathbb{R}^n$ ("nice") $J(t) \simeq t^{(n-1)/n}$

$\mathbb{R}^n, d\gamma_n(x) = (2\pi)^{-n/2} e^{-|x|^2/2} dx, \ J(t) \simeq t \left(\log \frac{1}{t}\right)^{1/2}$

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Symmetrization by truncation: The gradient

Metric spaces: Rademacher theorem is not true in general.

Modulus of the gradient:

$$|\nabla f(x)| = \limsup_{d(x,y)\to 0} \frac{|f(x) - f(y)|}{d(x,y)},$$

Condition 2. We assume that (Ω, μ) is such that for every $f \in Lip(\Omega)$, and every $c \in R$, we have that $|\nabla f(x)| = 0$, a.e. on the set $\{x : f(x) = c\}$.

Condition 1 and 2 are verified in all the classical cases: Euclidean, Gaussian, Riemannian manifolds as well as for doubling measures (homogeneous spaces).

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1. $$\forall A \subset \Omega,\ \text{Borel set},\ \mu^+(A) \geq I(\mu(A)).\ \text{Isoperimetric}$$

2. $$\int_0^\infty I(\mu_f(s))ds \leq \int_\Omega |\nabla f(x)|\, d\mu(x).\ \text{Ledoux}$$

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Applying a r.i. norm X on Ω

$$\|f\|_{LS(X)} := \left\| \left(f_\mu^{**}(t) - f_\mu^{*}(t)\right) \frac{I(t)}{t} \right\|_{\bar{X}} \leq \left\| |\nabla f|_\mu^{**} \right\|_{\bar{X}}.$$

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$(\mathbf{1}) \Rightarrow (\mathbf{2})$. (Coarea formula in metric spaces (Bobkov-Houndre)).

Idea others implications: $0 < t_1 < t_2 < \infty$. The smooth truncations of f are defined by

$$f_{t_1}^{t_2}(x) = \begin{cases} t_2 - t_1 & \text{if } |f(x)| \geq t_2, \\ |f(x)| - t_1 & \text{if } t_1 < |f(x)| < t_2, \\ 0 & \text{if } |f(x)| \leq t_1. \end{cases}$$

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For $s > 0$ and $h > 0$, pick $t_1 = f_\mu^*(s+h)$, $t_2 = f_\mu^*(s)$,

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I Isoperimetric estimator

$$Q_I f(t) = \int_t^{1/2} f(s) \frac{ds}{I(s)},$$

$\Omega \subset \mathbb{R}^n$ ("nice") $\quad I_\Omega(t) = c_n t^{(n-1)/n}$

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Hardy isoperimetric type

X, Y r.i. spaces on Ω. Assume that $0 \le f \in \bar{X}$, supp$f \subset (0, 1/2)$,

$$\|Q_I f\|_{\bar{Y}} \le C \|f\|_{\bar{X}} .$$

Then, $\forall g \in Lip(\Omega)$

$$\left\| g - \int_\Omega g d\mu \right\|_Y \preceq \|\nabla g\|_X .$$

Moreover

(a) Si $\tilde{Q}_I f(t) = \frac{I(t)}{t} \int_t^{1/2} f(s) \frac{ds}{I(s)}$ is bounded on $\bar{X}$. $\forall g \in Lip(\Omega)$,

$$\left\| g - \int_\Omega g d\mu \right\|_Y \preceq \left\| \left(g - \int_\Omega g d\mu \right)_\mu^* (t) \frac{I(t)}{t} \right\|_{\bar{X}} \preceq \|\nabla g\|_X .$$

(b) If $\overline{\alpha}_X < 1$,

$$\left\| g - \int_\Omega g d\mu \right\|_Y \preceq \left\| g - \int_\Omega g d\mu \right\|_{LS(X)} \preceq \|\nabla g\|_X .$$

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X, Y r.i. spaces on Ω. Assume that $0 \leq f \in \bar{X}$, supp$f \subset (0, 1/2)$,

$$\|Q_I f\|_{\bar{Y}} \leq C \|f\|_{\bar{X}} .$$

Then, $\forall g \in Lip(\Omega)$

$$\left\| g - \int_\Omega g d\mu \right\|_Y \preceq \|\nabla g\|_X .$$

Moreover

(a) Si $\tilde{Q}_I f(t) = \frac{I(t)}{t} \int_t^{1/2} f(s) \frac{ds}{I(s)}$ is bounded on $\bar{X}$. $\forall g \in Lip(\Omega)$,

$$\left\| g - \int_\Omega g d\mu \right\|_Y \preceq \left\| \left(g - \int_\Omega g d\mu \right)_\mu^* (t) \frac{I(t)}{t} \right\|_{\bar{X}} \preceq \|\nabla g\|_X .$$

(b) If $\overline{\alpha}_X < 1$,

$$\left\| g - \int_\Omega g d\mu \right\|_Y \preceq \left\| g - \int_\Omega g d\mu \right\|_{LS(X)} \preceq \|\nabla g\|_X .$$

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Hardy isoperimetric type

Definition

(Ω, d, μ) is of isoperimetric Hardy type if for any given isoperimetric estimator I, the following are equivalent for all r.i. spaces $X = X(\Omega)$, $Y = Y(\Omega)$: there exists a constant $c = c(X, Y)$ such that

1. $$\left\| f - \int_\Omega f d\mu \right\|_Y \leq c \left\| \nabla f \right\|_X , \quad \forall f \in Lip(\Omega).$$

2. There exists a constant $c_1 = c_1(X, Y) > 0$ such that
$$\left\| Q_I f \right\|_{\bar{Y}} \leq c_1 \left\| f \right\|_{\bar{X}} , \quad 0 \leq f \in \bar{X}, \text{ with } supp(f) \subset (0, 1/2),$$
where Q_I is the isoperimetric Hardy operator
$$Q_I f(t) = \int_t^{1/2} f(s) \frac{ds}{I(s)}. \tag{1}$$

Hardy isoperimetric type

Examples:

1. $\Omega \subset \mathbb{R}^n$ "nice"
2. $(\mathbb{R}^n, \gamma_n)$ Gaussian
3. $(\mathbb{R}^n, d\mu_r^{\otimes n})$, $\mu_\Phi^{\otimes n} = \varphi(x_1) \cdots \varphi(x_n) dx$

$$\mu_\Phi(x) = Z_\Phi^{-1} \exp\left(-\Phi(|x|)\right) dx = \varphi(x) dx, \qquad x \in \mathbb{R}$$

   Φ convex, $\sqrt{\Phi}$ concave, Z_Φ^{-1} tq. $\mu_\Phi(\mathbb{R}) = 1$.
4. $\mathbb{S}^n$ the the $n-$sphere.
5. Model Riemannian manifolds (Ros).
6. Surfaces of revolution ("controlled" curvature).

Proof: "There is a special symmetrization that does not increase the norm of the gradient"

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Gaussian case: $H(r)=\int_{-\infty}^{r}\phi_1(t)dt$,

$$f\in Lip(\mathbb{R}^n),\qquad f^{\circ}(x)=f_{\mu}^{*}(H(x_1))\qquad x\in\mathbb{R}^n.$$

A Young's function. $s=H(x_1)$.

$$\int_0^1 A\left((-f_\mu^*)'(s)I(s)\right)ds=\int_{\mathbb{R}}A\left((-f_\mu^*)'\left(H(x_1)\right)I(H(x_1))\left|H'(x_1)\right|dx_1\right.$$
$$=\int_{\mathbb{R}^n}A(|\nabla f^{\circ}(x)|)d\mu(x),\ (I(t)=\phi_1(H^{-1}(t)))$$

$$\int_0^t|\nabla f^{\circ}|_\mu^*\,ds=\int_0^t((-f_\mu^*)'(\cdot)I(\cdot))^*(s)ds\leq\int_0^t|\nabla f|_\mu^*(s)ds.$$

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$(\mathbb{R}^n, \gamma_n)$

$$\|u\|_Y \preceq \|\nabla u\|_X \,,$$

$0 \leq f$, $suppf \subset (0, 1/2)$,

$$F(t) = \int_t^1 f(s) \frac{ds}{I(s)}$$

$$u(x) = F(H(x_1))$$

$$|\nabla u|_\mu^*(t) = f^*(t), \quad u_\mu^*(t) = \int_t^1 f(s) \frac{ds}{I(s)}$$

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Hardy isoperimetric type

Let $0 < \beta < 1/2$. Consider

$$I(s) = s^{1-\beta}, \qquad 0 \leq s \leq 1/2.$$

1. There is a metric space (Ω_0, d_0, μ_0) st. $I(s) \simeq I_{(\Omega_0, d_0, \mu_0)}(s)$,

$$\left\| g - \int_{\Omega_0} g d\mu_0 \right\|_Y \leq c \left\| \nabla g \right\|_X, \quad g \in Lip(\Omega_0),$$

   but $Q_I : \bar{X} \to \bar{Y}$ is not bounded.

2. There is a metric spaces (Ω_1, d_1, μ_1) st. $I(s) \simeq I_{(\Omega_1, d_1, \mu_1)}(s)$,

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Transference Principle

Theorem

(Ω, d, μ) isoperimetric Hardy type. Suppose that (Ω_1, d_1, μ_1) probability metric space st.

$$I_{(\Omega_1,d_1,\mu_1)}(t) \geq cI_{(\Omega,d,\mu)}(t), \quad t \in (0, 1/2].$$

$X(\Omega)$, $Y(\Omega)$ r.i. spaces st.

$$\left\| g - \int_\Omega g d\mu \right\|_{Y(\Omega)} \leq c \left\| |\nabla g| \right\|_{X(\Omega)}, \quad \textit{for all } g \in Lip(\Omega).$$

Then,

$$\left\| g - \int_{\Omega_1} g d\mu_1 \right\|_{Y(\Omega_1)} \leq c \left\| |\nabla g| \right\|_{X(\Omega_1)}, \quad \textit{for all } g \in Lip(\Omega_1).$$

Corollary

Let M be a (compact) connected Riemannian manifold of dimension $n \geq 2$, with Ricci curvature bounded from below by $\rho > 0$. Let σ be the normalized volume on M. Let $\bar{X}(0,1)$, $\bar{Y}(0,1)$ be two r.i. spaces for which the following Poincaré inequality holds in the probability space $(\mathbb{S}^n, d, \sigma_n)$

$$\left\| g - \int_{\mathbb{S}^n} g d\sigma_n \right\|_{Y(\mathbb{S}^n)} \preceq \| |\nabla g| \|_{X(\mathbb{S}^n)}, \quad g \in Lip(\mathbb{S}^n).$$

holds. Then,

$$\left\| g - \int_M g d\sigma \right\|_{Y(M)} \preceq \| |\nabla g| \|_{X(M)}, \quad g \in Lip(M).$$

Gromov's theorem:

$$I_M \geq \sqrt{\frac{\rho}{n-1}} I_n.$$

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A question of Triebel

Dimension free Sobolev inequalities for the space

$$\mathring{W}_1^1(Q^n) = \overline{C_0^\infty(Q^n)}^{W_1^1(Q^n)} \qquad Q^n = (0,1)^n.$$

In our notation:

$$\left(\int_0^1 [f^*(t)]^q (1+\log\frac{1}{t})^\alpha dt\right)^{1/q} \preceq \|\nabla f\|_{L^q(Q^n)} + \|f\|_{L^q(Q^n)},$$

for a suitable power $\alpha =$? of the logarithm.

$$I_{Q^n} \geq I_{\gamma_n}. \ (Ros)$$

$(\mathbb{R}^n, \gamma_n)$ of Hardy isoperimetric type, we can transfer to Q^n the Gaussian Poincaré inequalities.

By the asymptotic of I_{γ_n}

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$$B_p^n = \left\{ x = (x_1, \cdots, x_n) : \|x\|_p^p = |x_1|^p + \cdots + |x_n|^p \le 1 \right\}, \qquad 1 \le p \le 2,$$

$$V_p^n = \frac{vol\ |_{B_p^n}}{vol(B_p^n)}.$$

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Weak assumptions

Condition 1: The isoperimetric profile $I_{(\Omega,d,\mu)}$ is a concave continuous function, increasing on $(0,1/2)$, symmetric about the point $1/2$ such that, moreover, vanishes at zero.

Condition 2: For every $f \in Lip(\Omega)$, and every $c \in R$, we have that $|\nabla f(x)| = 0$, $\mu-$a.e. on the set $\{x : f(x) = c\}$.

Condition 1': The isoperimetric profile $I_{(\Omega,d,\mu)}$ is a positive continuous function that vanishes at zero.

$I = I_{(\Omega,d,\mu)}$ isoperimetric. $1 \leq q < \infty$.

$$w_q(t) = \begin{cases} \left(\frac{1}{t} \int_0^t \left(\frac{s}{I(s)} \right)^{\frac{q}{q-1}} ds \right)^{\frac{1-q}{q}} & \text{if } q > 1 \\ \inf_{0<s<t} \frac{I(s)}{s} & \text{if } q = 1. \end{cases}$$

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$$w_q(t) = \begin{cases} \left(\frac{1}{t} \int_0^t \left(\frac{s}{I(s)} \right)^{\frac{q}{q-1}} ds \right)^{\frac{1-q}{q}} & \text{if } q > 1 \\ \inf_{0<s<t} \frac{I(s)}{s} & \text{if } q = 1. \end{cases}$$

Weak assumptions

Condition 1: The isoperimetric profile $I_{(\Omega,d,\mu)}$ is a concave continuous function, increasing on $(0, 1/2)$, symmetric about the point $1/2$ such that, moreover, vanishes at zero.

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Theorem

Let (Ω, d, μ) be a metric probability space that satisfies Conditions 1' and 2, and let $1 \leq q < \infty$. Then for $f \in Lip(\Omega) \cap L^1(\Omega)$, and for all $t \in (0,1)$, we have

1.
$$\int_0^t \left[\left(\left(-f_\mu^*\right)'(\cdot) I(\cdot) \right)^* (s) \right]^q ds \leq \int_0^t \left(|\nabla f|_\mu^* \right)^q (s) ds. \quad (2)$$

2.
$$(f_\mu^{**}(t) - f_\mu^*(t)) w_q(t) \leq \left(\frac{1}{t} \int_0^t \left(|\nabla f|_\mu^* \right)^q (s) ds \right)^{1/q}. \quad (3)$$

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Computing Isoperimetric estimators. Semigrup approach

(Ideas comes from: Ledoux-E.Milman)

- (M, g) n-dimensional ($n \geq 2$) smooth complete oriented connected Riemannian manifold or $(M, g) = (\mathbb{R}^n, |\cdot|)$, and $\Omega = M$.
- d induced geodesic distance on (M, g)
- $d\mu = exp(\psi)d_{volM}$, $\psi \in C^2(M)$, and as tensor fields on M:

$$Ric_g + Hess\psi \geq 0$$

associated Laplacian:

$$\Delta_{(\Omega,\mu)} = \Delta_\Omega - \nabla\psi \cdot \nabla$$

(Δ_Ω Laplace-Beltrami operator on Ω).
$(P_t)_{t\geq 0}$ semi-group associated to the diffusion process with infinitesimal generator $\Delta_{(\Omega,\mu)}$ characterized by

$$\frac{\partial}{\partial t} P_t(f) = \Delta_{(\Omega,\mu)}(P_t(f)), \quad P_0(f) = f,$$

$(P_t)_{t\geq 0}$ heat semig. infinitesimal generator $L = \triangle - \nabla\psi \cdot \nabla$

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- $P_t 1 = 1$.
- $f \geq 0 \Rightarrow P_t f \geq 0$.
- $\int (P_t f)\, g d\mu = \int f\, (P_t g)\, d\mu$.
- $(P_t f)^\alpha \leq P_t f^\alpha, \forall \alpha \geq 1$.
- $P_t \circ P_s = P_{s+t}$.
- $P_t : X(\Omega) \to X(\Omega)$ is bounded for any r.i. space $X(\Omega)$.
- $$2t \left|\nabla P_t f\right|^2 \leq P_t f^2 - (P_t f)^2 .$$

Theorem

$\Omega = (M, g)$ as before
(i) X is q concave ($q \geq 2$);
or
(ii) $\bar{\alpha}_X < 1/2$.
(iii)

$$\left\| g - \int_\Omega g d\mu \right\|_Y \leq c \left\| \nabla g \right\|_X, \qquad \forall g \in Lip(\Omega).$$

Then

$$I_{(M,g,\mu)}(t) \geq c_1 t(1-t) \frac{\varphi_Y(t(1-t))}{\varphi_X(t(1-t))},$$

1) Approximation

$$\sqrt{2t}\mu^+(A) \geq \int |\chi_A - P_t\chi_A| \, d\mu.$$

2)

$$\begin{aligned}\int |\chi_A - P_t\chi_A| \, d\mu &= \int_A (1 - P_t\chi_A) \, d\mu + \int_{\Omega\setminus A} P_t\chi_A d\mu \\ &= 2\left(\mu(A) - \int_A P_t\chi_A d\mu\right) \\ &= 2\left(\begin{array}{c} \mu(A)\,(1-\mu(A)) \\ -\int_\Omega (P_t\chi_A - \mu(A))\,(\chi_A - \mu(A))\, d\mu \end{array}\right).\end{aligned}$$

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\end{aligned}$$

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3) From hypothesis

$$J(t) = \int_\Omega \left(P_t\left(\chi_A - \mu(A)\right)\right)\left(\chi_A - \mu(A)\right) d\mu \le \frac{c}{\sqrt{2t}}\varphi_X(\mu(A))\frac{\mu(A)}{\varphi_Y(\mu(A))}. \tag{5}$$

4) For $0 < \mu(A) \le 1/2$

$$\begin{aligned} \mu^+(A) &\ge \frac{\mu(A)\left(1-\mu(A)\right)}{\sqrt{2t}} - J(t) \\ &\ge \frac{\mu(A)\left(1-\mu(A)\right)}{\sqrt{2t}} - \frac{c}{t}\varphi_X(\mu(A))\frac{\mu(A)}{\varphi_Y(\mu(A))} \\ &\ge \mu(A)\left(\frac{(1-\mu(A))}{\sqrt{2t}} - \frac{c}{t}\frac{\varphi_X(\mu(A))}{\varphi_Y(\mu(A))}\right) \\ &\ge \mu(A)\left(\frac{1}{2\sqrt{2t}} - \frac{c}{t}\frac{\varphi_X(\mu(A))}{\varphi_Y(\mu(A))}\right). \end{aligned}$$

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&\geq \mu(A)\left(\frac{(1 - \mu(A))}{\sqrt{2t}} - \frac{c}{t}\frac{\varphi_X(\mu(A))}{\varphi_Y(\mu(A))}\right) \\
&\geq \mu(A)\left(\frac{1}{2\sqrt{2t}} - \frac{c}{t}\frac{\varphi_X(\mu(A))}{\varphi_Y(\mu(A))}\right).
\end{aligned}$$